Good deals in markets with frictions

Alejandro Balbás[a], Beatriz Balbás[b], Raquel Balbás[c]

[a] *University Carlos III of Madrid. C/ Madrid, 126. 28903 Getafe (Madrid, Spain).* alejandro.balbas@uc3m.es

[b] *University of Castilla La Mancha. Avda. Real Fábrica de Seda, s/n. 45600 Talavera (Toledo, Spain). beatriz.balbas@uclm.es*

[c] *University Complutense of Madrid. Department of Actuarial and Financial Economics. Somosaguas Campus. 28223 Pozuelo de Alarcón (Madrid, Spain). raquel.balbas@ccee.ucm.es*

Abstract

This paper studies a portfolio choice problem such that the pricing rule may incorporate transaction costs and the risk measure is coherent and expectation bounded. We will prove the necessity of dealing with pricing rules such that there exists an essentially bounded stochastic discount factor, which must be also bounded from below by a strictly positive value. Otherwise good deals will be available to traders, *i.e.*, depending on the selected risk measure, investors can build portfolios whose ($risk$, $return$) will be as close as desired to $(-\infty, \infty)$ or $(0, \infty)$. This pathologic property still holds for vector risk measures (*i.e.*, if we minimize a vector valued function whose components are risk measures). It is worthwhile to point out that essentially bounded stochastic discount factors are not usual in financial literature. In particular, the most famous frictionless, complete and arbitrage free pricing models imply the existence of good deals for every coherent and expectation bounded (scalar or vector) measure of risk, and the incorporation of transaction costs will not guarantee the solution of this caveat.

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1 Introduction

Since Artzner *et al.* (1999) introduced the "Coherent Measures of Risk" there has been a growing interest in risk measures beyond the variance, and many authors have extended the discussion. So, among many other interesting contributions, Föllmer and Schied (2002) defined the Convex Risk Measures, Goovaerts *et al.* (2004) introduced the Consistent Risk Measures, Rockafellar *et al.* (2006) defined the Expectation Bounded Risk Measures, Zhiping and Wang, (2008) presented the Two-Sided Coherent Risk Measures, Brown and Sim (2009) introduced the Satisfying Measures, and Aumann and Serrano (2008) and Foster and Hart (2009) defined Indexes of Riskiness. All of these measures are more and more used by researchers, practitioners, regulators and supervisors.

Actuarial and financial applications of risk measures have been more and more developed in the literature. Interesting examples are Portfolio Theory and Equilibrium (Rockafellar *et al.*, 2007, Miller and Ruszczynski, 2008, etc.), Pricing Issues (Hamada and Sherris, 2003, Staum, 2004, Goovaerts and Laeven, 2008, etc.), Optimal Reinsurance (Cai *et al.,* 2008, Balbás *et al.,* 2009, etc.), etc.

The notion of "Good Deal" was introduced in the paper by Cochrane and Saa-Requejo (2000). Mainly, a good deal is an investment strategy providing traders with a "very high return/risk ratio", in comparison with the value of this ratio for the Market Portfolio. Risk is measured with the standard deviation, and the absence of good deals is imposed in an arbitrage free model so as to price in incomplete markets. This line of research has been extended for more general risk functions.[1]

Besides, some recent papers deal with risk measures and impose conditions that are strictly stronger than the absence of arbitrage. For instance, Stoica and Lib (2010) fix a risk measure and its subgradient must contain "Equivalent Risk Neutral Probabilities".[2] However, the fulfillment of these assumptions, stronger than the arbitrage absence, is not so obvious in very important Pricing Models of Financial Economics. Balbás *et al.* (2010) have shown the existence of "pathological results" when combining some risk measures (VaR, $CVaR$, Dual Power Transform or DPT, etc.) and very popular pricing models (Black and Scholes,

[1] See Staum (2004) or Arai (2011), amongst many other interesting contributions.

[2] Thus, the existence of "Equivalent Risk Neutral Probabilities" is not sufficient. Some of them must belong to the risk measure subgradient.

Heston, etc.). Indeed, for the examples above, the Stochastic Discount Factor (SDF) of the pricing model and the risk measure subgradient do not satisfy some conditions, which implies the existence of sequences of portfolios whose expected returns tend to plus infinite and whose risk levels tend to minus infinite or remain bounded ($risk = -\infty$ *and* $return = +\infty$, *or bounded risk and* $return = +\infty$). In this paper we will use the expression "good deal" to represent these sequences making the managers as rich as desired and obviously outperforming the Market Portfolio.

It is needless to say that the existence of these good deals is a meaningless finding from a financial point of view, and it is not supported by the empirical evidence either. A possible solution could be the incorporation of frictions, that may make the traders lose many potential earnings. The objective of this paper is to analyze the existence of good deals in presence of transaction costs, even when trading the riskless asset.

The article's outline is as follows. Section 2 will summarize the basic properties of the risk measures and the imperfect pricing rules we are going to deal with. We will draw on a slight extension of the representation theorem of expectation bounded risk measures of Rockafellar *et al.* (2006), and the most important results of this section are a new representation theorem of the pricing rule (Corollary 2) and a mean value theorem (Lemma 3).

Section 3 will be devoted to introducing a general portfolio choice problem that minimizes the portfolio risk for every desired expected return. Both a primal and a dual approach will be given, and the most important result is Theorem 5, which guarantees the absence of duality gap between both problems. Corollary 2 and Lemma 3 above play a critical role in the proof of Theorem 5.

Section 4 will deal with the main problem of this paper, which is the absence or existence of good deals under frictions. The main results are Theorem 7 and its remark. They give necessary and necessary and sufficient conditions to prevent the existence of good deals. In particular, some SDF of the pricing rule must be a convex combination of the risk measure subgradient and the riskless asset, and the weight of the riskless asset must be strictly positive. It is also remarkable that most of the necessary conditions do not affect the risk measure, and only the pricing rule is involved. We will prove that pricing rules without essentially bounded SDF will provide traders with good deals for every risk measure that may be extended to the whole space L^1 ($CVaR$, for instance) and pricing rules without SDF bounded from below by a strictly positive value will provide traders with good deals regardless of the coherent and expectation bounded risk measures they use. These properties still hold for vector risk measures (*i.e.*, vector valued functions whose components are risk measures). Thus, the existence of $0 < b \leq B$ and a SDF z such that $b \leq z \leq B$ must hold. It is worth remarking that bounded SDF are not usual at all in financial literature. In particular, pricing rules having a unique SDF (*i.e.*, perfect markets) with the Log-Normal (Black and Scholes) or heavier tailed (stochastic volatility pricing models) distributions will generate good deals for every coherent and expectation bounded risk measure.

Though the transaction costs are represented in Section 2 in a very general setting, more complex frictions might be considered. The focus of Section 5 is on short selling restrictions and further market imperfections. It will be shown that the presence of short selling restrictions or cone constraints might very partially solve the caveat we found in Theorem 7, but it is not easy to accept the existence of these restrictions just to solve a theoretical problem. Actually, these restrictions will not be justified by high risk levels in the composed portfolios. On the contrary, the global risk will decrease if further short sales are allowed.

Section 6 presents the most important conclusions of the paper.

2 Preliminaries and notations

2.1 The risk measure

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ composed of the set of "states of the world" Ω that may occur within the time interval $[0, T]$, the $\sigma-$algebra $\mathcal{F}$ and the probability measure $\mathbb{P}$. If $p \in [1, \infty)$, L^p will denote the space of $\mathbb{R}-$valued random variables y on Ω such that $\mathbb{E}\left(|y|^p\right) < \infty$, $\mathbb{E}\left(\right)$ representing the mathematical expectation. If $q \in (1, \infty]$ is its conjugate value (*i.e.*, $1/p + 1/q = 1$), then the Riesz Representation Theorem (Rudin, 1973) guarantees that L^q is the dual space of L^p, where L^∞ is composed of the essentially bounded random variables. A special important case arises for $p = q = 2$.

Let $p \in [1,2]$ and $q \in [2,\infty]$.

$$\rho : L^p \longrightarrow \mathbb{R}$$

will be the general risk function that a trader uses in order to control the risk level of his wealth at T. Denote by

$$\Delta_\rho = \{z \in L^q; -\mathbb{E}(yz) \leq \rho(y), \ \forall y \in L^p\} \tag{1}$$

the subgradient of ρ. The set Δ_ρ is obviously convex. We will assume that Δ_ρ is also $\sigma(L^q, L^p)$ −compact,[3] and

$$\rho(y) = Max \ \{-\mathbb{E}(yz) : z \in \Delta_\rho\} \tag{2}$$

holds for every $y \in L^p$. Furthermore, we will also impose that $z = 1$ $a.s.$ is in Δ_ρ,

$$\Delta_\rho \subset \{z \in L^q; \mathbb{E}(z) = 1\}, \tag{3}$$

and

$$\Delta_\rho \subset L^q_+ = \{z \in L^q; \mathbb{P}(z \geq 0) = 1\}. \tag{4}$$

Summarizing, we have:

Assumption 1. The set Δ_ρ given by (1) is convex and $\sigma(L^q, L^p)$ −compact, (2) holds for every $y \in L^p$, $z = 1$ $a.s.$ is in Δ_ρ and (3) and (4) hold. □

The assumption above is closely related to the Representation Theorem of Risk Measures stated in Rockafellar *et al.* (2006). Following their ideas, it is easy to prove that the fulfillment of Assumption 1 holds if and only if ρ is continuous and:

$a)$ Translation invariant,

$$\rho(y+k) = \rho(y) - k$$

for every $y \in L^p$ and $k \in \mathbb{R}$.

$b)$ Homogeneous,

$$\rho(\alpha y) = \alpha \rho(y)$$

for every $y \in L^p$ and $\alpha > 0$.

[3]See Rudin (1973) for further details about $\sigma(L^q, L^p)$ −compact sets.

$c)$ Sub-additive,

$$\rho\left(y_1+y_2\right)\leq\rho\left(y_1\right)+\rho\left(y_2\right)$$

for every $y_1, y_2 \in L^p$.

$d)$ Mean dominating,

$$\rho\left(y\right)\geq-\mathbb{E}\left(y\right)$$

for every $y \in L^p$.

$e)$ Decreasing,

$$\rho\left(y_1\right)\leq\rho\left(y_2\right)$$

whenever $y_1, y_2 \in L^p$ and $y_1 \geq y_2$ $a.s.$

Particular interesting examples are the Conditional Value at Risk ($CVaR$) and the Weighted Conditional Value at Risk ($WCVaR$, Rockafellar *et al.*, 2006, or Cherny, 2006), the Dual Power Transform (DPT, Wang, 2000) and the Wang Measure (Wang, 2000), among many others.

Remark 1 *With the Hahn-Banach Separation Theorem (Rudin, 1973) and Expressions (1) and (2) it is easy to prove that there is a one to one bijection*

$$\begin{array}{ccc}\mathcal{M} & \leftrightharpoons & \mathcal{S}\\ \rho & \leftrightharpoons & \Delta_\rho\end{array}$$

between the set $\mathcal{M}$ of risk measures satisfying Assumption 1 and the set $\mathcal{S}$ of convex and $\sigma\left(L^q, L^p\right)-$compact subsets of L^q containing the constant random variable whose value is 1 and fulfilling (3) and (4). This bijection is increasing, i.e., higher risk measures are associated with higher sets of $\mathcal{S}$. Consequently, given a finite family of risk measures satisfying Assumption 1

$$\left\{\rho_1, \rho_2, ..., \rho_k\right\}\subset\mathcal{M},$$

one can consider the family of subgradients

$$\left\{\Delta_{\rho_1}, \Delta_{\rho_2}, ..., \Delta_{\rho_k}\right\}\subset\mathcal{S}.$$

Then, taking the convex hull

$$\Delta_\rho = Co\left(\bigcup_{i=1}^{k}\Delta_{\rho_i}\right),$$

which is obviously $\sigma(L^q, L^p)$ −compact, we easily prove that there exists ρ satisfying Assumption 1 *and such that $\rho_k \leq \rho$, $i = 1, 2, ..., k$. Furthermore, ρ is the minimum element in $\mathcal{M}$ fulfilling both properties.* □

2.2 The pricing rule

There are several ways to introduce pricing rules in a market with transaction costs (see, among many other interesting contributions, Jouini and Kallal, 1995 and 2001, or Schachermayer, 2004). Nevertheless, all of them are closely related and lead to quite similar assumptions. In the line of previous literature, we will consider the function

$$\Phi : L^2 \longrightarrow \mathbb{R}$$

that provides us with the initial (at $t = 0$) price $\Phi(y)$ of final (at T) pay-off $y \in L^2$.[4] We will adopt usual conventions for imperfect markets, so

$$\Phi(y_1 + y_2) \leq \Phi(y_1) + \Phi(y_2) \tag{5}$$

for every $y_1, y_2 \in L^2$ and

$$\Phi(\alpha y) = \alpha \Phi(y) \tag{6}$$

for every $y \in L^2$ and every $\alpha > 0$.

$\Phi(y)$ is usually interpreted as the ask price of $y \in L^2$, whereas $-\Phi(-y)$ is the bid price. Since (6) leads to $\Phi(0) = 0$,[5] inequality

$$-\Phi(-y) \leq \Phi(y) \tag{7}$$

trivially follows from (5). We will also assume that the lending rate is non negative and not higher than the borrowing one, *i.e.*,

$$0 < -\Phi(-1) \leq \Phi(1) \leq 1 \tag{8}$$

[4] We will price securities in L^2 becasue we are assuming returns with finite variance. This property holds for the most important models of Financial Economics (*CAPM*, *APT*, Black and Scholes, Stochastic Volatility Models, etc.) and is also supported by the empirical evidence (see, among other interesting contributions, Grabchak and Smorodnitski, 2010).

[5] Otherwise, $\Phi(0) = \Phi(2 \times 0) = 2\Phi(0)$ would lead to the contradiction $1 = 2$.

must hold. Summarizing, we have:

Assumption 2. The pricing rule $\Phi : L^2 \longrightarrow \mathbb{R}$ is continuous, it satisfies (5) and (6), and (8) holds. $\square$

The following version of the Hahn-Banach Theorem is adopted from Rudin (1973), and the proof is omitted since it is provided in this reference.

Theorem 1 *Consider the linear manifold $L \subset L^2$ and the linear function $\varphi : L \longrightarrow \mathbb{R}$ such that $\varphi(y) \leq \Phi(y)$ for every $y \in L$. Then, there exists $\phi : L^2 \longrightarrow \mathbb{R}$ linear and such that*

$$\phi(y) = \varphi(y) \tag{9}$$

for every $y \in L$ and

$$\phi(y) \leq \Phi(y) \tag{10}$$

for every $y \in L^2$. $\square$

Corollary 2 *The subgradient of Φ given by*

$$\Delta_\Phi = \left\{ z \in L^2; \mathbb{E}(yz) \leq \Phi(y), \ \forall y \in L^2 \right\} \tag{11}$$

is convex and $\sigma(L^2, L^2)-$compact, and Expression

$$\Phi(y) = Max \ \left\{ \mathbb{E}(yz) : z \in \Delta_\Phi \right\} \tag{12}$$

holds for every $y \in L^2$.

Proof. The convexity of Δ_Φ is obvious, so let us prove its weak-compactness. Since Δ_Φ is obviously weakly-closed we only have to show that it is norm-bounded (Alaoglu's Theorem, see Rudin, 1973). The continuity of Φ implies the existence of $\delta > 0$ such that

$$\|y\| \leq \delta \Longrightarrow |\Phi(y)| \leq 1$$

holds. Then,

$$\|y\| \leq \delta \Longrightarrow |\mathbb{E}(yz)| \leq 1, \ \forall z \in \Delta_\Phi$$

holds, *i.e.*,

$$\|y\| \leq 1 \Longrightarrow |\mathbb{E}(yz)| \leq 1/\delta, \ \forall z \in \Delta_\Phi \tag{13}$$

holds. Expression (13) obviously implies that $\|z\| \leq 1/\delta$ for every $z \in \Delta_\Phi$.

Next, let us see the fulfillment of (11). Obviously, it is sufficient to show the inequality $\Phi(y) \leq Max\ \{\mathbb{E}(yz) : z \in \Delta_\Phi\}$.[6] Fix $y_0 \in L^2$ and the linear manifold generated by y_0, given by

$$L = \{\lambda y_0;\ \lambda \in \mathbb{R}\}.$$

Consider the linear function $\varphi : L \longrightarrow \mathbb{R}$ given by

$$\varphi(\lambda y_0) = \lambda \Phi(y_0)$$

for every $\lambda \in \mathbb{R}$. The inequality $\varphi(\lambda y_0) \leq \Phi(\lambda y_0)$ is obvious from Assumption 2 if $\lambda \geq 0$, and for $\lambda < 0$ Expressions (7) and (6) imply that

$$\Phi(\lambda y_0) \geq -\Phi(-\lambda y_0) = \lambda \Phi(y_0) = \varphi(\lambda y_0).$$

Consider now the extension ϕ of φ of Theorem 1. The continuity of Φ (Assumption 2) implies the continuity of ϕ. Indeed, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|y\| \leq \delta \Longrightarrow |\Phi(y)| \leq \varepsilon$$

holds. Thus, (10) implies that

$$\begin{array}{l} \|y\| \leq \delta \Longrightarrow \\ \left\{ \begin{array}{ll} \|y\| \leq \delta & \Longrightarrow \phi(y) \leq \Phi(y) \leq |\Phi(y)| \leq \varepsilon \\ \|-y\| \leq \delta & \Longrightarrow -\phi(y) = \phi(-y) \leq \Phi(-y) \leq |\Phi(-y)| \leq \varepsilon \end{array} \right\} \\ \Longrightarrow |\phi(y)| \leq \varepsilon \end{array}$$

According to the Riesz Representation Theorem, take $z \in L^2$ with $\phi(y) = \mathbb{E}(yz)$ for every $y \in L^2$. Then, (10) shows that $z \in \Delta_\Phi$, and (9) shows that $\mathbb{E}(y_0 z) = \varphi(y_0) = \Phi(y_0)$. $\square$

Remark 2 *For frictionless markets the set Δ_Φ contains a unique element usually called Stochastic Discount Factor (SDF). Further details may be found, for instance, in Duffie (1988). In our more general framework we will say that every element of Δ_Φ is a SDF of Φ.* $\square$

[6] Notice that the compactness of Δ_Φ implies that the maximum is attained.

Next, we will end this section by providing without proof a Mean Value Theorem. The first statement applies for the risk measure ρ, and it is adopted from Balbás *et al.* (2009), where a complete proof may be found. The main arguments are implied by Assumption 1. The second statement applies for the pricing rule Φ, and its proof is similar if one bears in mind Corollary 2.

Henceforth, $\mathcal{C}(\Delta_\rho)$ and $\mathcal{C}(\Delta_\Phi)$ will denote the Banach spaces composed of the real valued $\sigma(L^q, L^p)$ $-$continuous and $\sigma(L^2, L^2)$ $-$continuous functions. $\mathcal{B}_\rho$ and $\mathcal{B}_\Phi$ will denote the Borel $\sigma-$algebras of Δ_ρ and Δ_Φ endowed with topologies $\sigma(L^q, L^p)$ and $\sigma(L^2, L^2)$. $\mathcal{M}(\Delta_\rho)$ and $\mathcal{M}(\Delta_\Phi)$ will denote the Banach spaces of inner regular $\sigma-$additive measures on $\mathcal{B}_\rho$ and $\mathcal{B}_\Phi$. $\mathcal{P}(\Delta_\rho)$ and $\mathcal{P}(\Delta_\Phi)$ will denote the subsets of $\mathcal{M}(\Delta_\rho)$ and $\mathcal{M}(\Delta_\Phi)$ composed of those measures that are probabilities (non-negative and total mass equal to 1). Recall that the Riesz Representation Theorem (Rudin, 1973) guarantees that $\mathcal{M}(\Delta_\rho)$ and $\mathcal{M}(\Delta_\Phi)$ are the dual spaces of $\mathcal{C}(\Delta_\rho)$ and $\mathcal{C}(\Delta_\Phi)$.

Lemma 3 *(Mean Value Theorem). a) For every probability measure $m \in \mathcal{P}(\Delta_\rho)$ there exists a unique $z_m \in \Delta_\rho$ such that*

$$\mathbb{E}(yz_m) = \int_{\Delta_\rho} \mathbb{E}(yz)\, dm(z)$$

holds for every $y \in L^p$.

b) For every probability measure $m \in \mathcal{P}(\Delta_\Phi)$ there exists a unique $z_m \in \Delta_\Phi$ such that

$$\mathbb{E}(yz_m) = \int_{\Delta_\Phi} \mathbb{E}(yz)\, dm(z)$$

holds for every $y \in L^2$. □

3 Primal and dual portfolio choice problems

Balbás *et al.* (2010) have proposed a general portfolio choice problem involving coherent and expectation bounded risk measures and perfect pricing models. The natural extension for a market with frictions is

$$\begin{cases} Min\ \rho(y) \\ \Phi(y) \le 1,\ \mathbb{E}(y) \ge R \end{cases}. \tag{14}$$

$y \in L^2$ being the decision variable and $R > 0$ representing the minimum required expected return. Problem (14) minimizes the risk of a portfolio whose global ask price is not higher than one dollar and whose expected value is at least R. Thus, it is a standard risk/return approach with ρ as the risk measure.

Next, let us give conditions so as to guarantee that (14) is feasible (*i.e.*, the feasible set is non void).

Assumption 3. There exists $y_1 \in L^2$ such that $0 < \Phi(y_1) < -\Phi(-\mathbb{E}(y_1))$. □

Assumption 3 is not restrictive at all, since it only imposes the existence of a portfolio y_1 whose expected return is higher than the borrowing rate. Indeed, suppose that some investor accepts a debt with value $\mathbb{E}(y_1)$ to be paid at T. Then, he receives the bid price $-\Phi(-\mathbb{E}(y_1))$ at $t = 0$, which is higher than the price $\Phi(y_1)$ of y_1. Thus, he can buy y_1 and conserve the strictly positive quantity $-\Phi(-\mathbb{E}(y_1)) - \Phi(y_1)$, but his expected final wealth vanishes.

Proposition 4 *Problem (14) is feasible for every $R > 0$.*[7]

Proof. Obviously, Portfolio $y_1 - \mathbb{E}(y_1)$ has a negative price, since Assumption 3 implies that

$$\Phi(y_1 - \mathbb{E}(y_1)) \leq \Phi(y_1) + \Phi(-\mathbb{E}(y_1)) < 0. \tag{15}$$

Consider $k > 0$, $R > 0$ and Portfolio

$$x_{k,R} = k(y_1 - \mathbb{E}(y_1)) + R \in L^2.$$

One has that,

$$\Phi(x_{k,R}) \leq k\Phi(y_1 - \mathbb{E}(y_1)) + R\Phi(1).$$

Bearing in mind (15), we have that

$$k \geq \frac{1 - R\Phi(1)}{\Phi(y_1 - \mathbb{E}(y_1))} \Longrightarrow \Phi(x_{k,R}) \leq 1. \tag{16}$$

[7] The proof of this proposition will show that Assumption 3 may be sligtly relaxed. It is sufficient to impose the inequality

$$\Phi(y_1 - \mathbb{E}(y_1)) < 0,$$

though we think that the given condition is more intuitive from a financial point of view.

Besides,

$$\mathbb{E}\left(x_{k,R}\right)=k\left(\mathbb{E}\left(y_1\right)-\mathbb{E}\left(y_1\right)\right)+R=R.$$

Therefore, $x_{k,R}$ is (14)$-$feasible as long as one chooses $k>0$ so as to satisfy the left hand side condition in (16). □

Remark 3 *Since (14) is feasible, hereafter*

$$\infty>\rho_R^*\geq-\infty$$

will represent its optimal value. □

According to the financial intuition, Problem (14) should be bounded, and its infimum value (the optimal risk level ρ_R^*) should increase if so does the expected return R. We will deal with duality theory so as to analyze whether the intuitive properties above do hold.

First of all, Assumption 1 and Corollary 2 allow us to substitute (14) by an equivalent problem. Indeed, consider Problem

$$\begin{cases} Min\ \theta \\ \theta+\mathbb{E}\left(yz\right)\geq 0, & \forall z\in\Delta_\rho \\ \mathbb{E}\left(yz\right)\leq 1, & \forall z\in\Delta_\Phi \\ \mathbb{E}\left(y\right)\geq R \end{cases}\tag{17}$$

$(\theta,y)\in\mathbb{R}\times L^2$ being the decision variable. It is easy to see that $y\in L^2$ solves (14) if and only if there exists $\theta\in\mathbb{R}$ such that (θ,y) solves (17), in which case $\theta=\rho\left(y\right)$ holds.

Assumptions 1 and 2 imply that (14) and (17) are convex problems, so we can deal with the general duality theory for convex optimization problems of Luenberger (1969). Therefore, let us consider the Lagrangian function of (17)

$$\mathbb{R}\times L^2\times\mathcal{M}\left(\Delta_\rho\right)\times\mathcal{M}\left(\Delta_\Phi\right)\times\mathbb{R}\ni\left(\theta,y,m_1,m_2,\lambda\right)\to\mathcal{L}\left(\theta,y,m_1,m_2,\lambda\right)\in\mathbb{R}$$

given by

$$\mathcal{L}\left(\theta,y,m_1,m_2,\lambda\right)=$$
$$\theta\left(1-\int_{\Delta_\rho}dm_1\right)-\int_{\Delta_\rho}\mathbb{E}\left(yz_1\right)dm_1\left(z_1\right)+\int_{\Delta_\Phi}\left(\mathbb{E}\left(yz_2\right)-1\right)dm_2\left(z_2\right)+\lambda\left(R-\mathbb{E}\left(y\right)\right).$$

Then, $(m_1, m_2, \lambda) \in \mathcal{M}(\Delta_\rho) \times \mathcal{M}(\Delta_\Phi) \times \mathbb{R}$ is dual-feasible if and only if $m_1 \geq 0$, $m_1 \geq 0$, $\lambda \geq 0$, and $\mathcal{L}(\theta, y, m_1, m_2, \lambda)$ is bounded from below for $(\theta, y) \in \mathbb{R} \times L^2$, which obviously implies that $m_1(\Delta_\rho) = 1$, *i.e.*, $m_1 \in \mathcal{P}(\Delta_\rho)$. Thus, the dual problem of (17) becomes

$$\begin{cases} Max \ \left(Inf_{y \in L^2} \left(-\int_{\Delta_\rho} \mathbb{E}(yz_1)\, dm_1(z_1) + \int_{\Delta_\Phi} (\mathbb{E}(yz_2) - 1)\, dm_2(z_2) + \lambda (R - \mathbb{E}(y)) \right) \right) \\ (m_1, m_2, \lambda) \in \mathcal{P}(\Delta_\rho) \times \mathcal{M}(\Delta_\Phi) \times \mathbb{R} \\ m_2, \lambda \geq 0 \end{cases} \tag{18}$$

However, bearing in mind Lemma 3, and denoting $\mu = m_2(\Delta_\Phi)$, it is obvious that the problem above is equivalent to

$$\begin{cases} Max \ \left(Inf_{y \in L^2} \left(-\mathbb{E}(yz_1) + \mu \mathbb{E}(yz_2) - \mu + \lambda (R - \mathbb{E}(y)) \right) \right) \\ (z_1, z_2, \mu, \lambda) \in \Delta_\rho \times \Delta_\Phi \times \mathbb{R} \times \mathbb{R} \\ \mu, \lambda \geq 0 \end{cases}$$

Since

$$\begin{aligned} &-\mathbb{E}(yz_1) + \mu \mathbb{E}(yz_2) - \mu + \lambda (R - \mathbb{E}(y)) \\ &= \mathbb{E}(y(-z_1 + \mu z_2 - \lambda)) - \mu + \lambda R, \end{aligned}$$

the infimum becomes higher than $-\infty$ if and only if $-z_1 + \mu z_2 - \lambda$ vanishes, so the dual problem becomes

$$\begin{cases} Max \ -\mu + \lambda R \\ z_1 = \mu z_2 - \lambda \\ (z_1, z_2, \mu, \lambda) \in \Delta_\rho \times \Delta_\Phi \times \mathbb{R} \times \mathbb{R}, \ \mu, \lambda \geq 0 \end{cases} \tag{19}$$

$(z_1, z_2, \mu, \lambda) \in \Delta_\rho \times \Delta_\Phi \times \mathbb{R} \times \mathbb{R}$ being the decision variable.

Next, let us prove that there is no duality gap between (14) and (19).[8]

Theorem 5 *Consider $R > 0$. There is strong duality between (14) and (19), i.e., (14) is bounded if and only if (19) is feasible. In such a case (19) is also bounded and solvable, and both optimal values coincide with $\rho_R^* > -\infty$.*

Proof. Since (14) is equivalent to (17) and (19) is equivalent to (18), it is sufficient to prove that there is no duality gap between (17) and (18). According to Luenberger (1969),

[8]Notice that there are several problems in Mathematical Finance leading to the existence of duality gaps. See, for instance, Jin *et al.* (2008).

it is sufficient to prove that (17) satisfies the Slater Qualification, *i.e.*, there exists (θ_0, y_0) satisfying all the constraints of (17) in terms of strict inequality. Fix, $R > 0$ and Proposition 4 implies the existence of y_1 such that $\Phi(y_1) \leq 1$, $\mathbb{E}(y_1) \geq 4R$. Then, $y_0 = \frac{y_1}{2}$ satisfies $\Phi(y_0) \leq 1/2 < 1$ and $\mathbb{E}(y_0) \geq 2R > R$. Therefore, (12) implies that $\mathbb{E}(y_0 z) \leq 1/2 < 1$ for every $z \in \Delta_\Phi$. Finally, choose

$$\theta_0 > \rho(y_0) = Max \ \{-\mathbb{E}(y_0 z) : z \in \Delta_\rho\}$$

and the first constraint of (17) will be strictly satisfied. $\square$

4 The no good deal condition

As already said, the financial intuition indicates that Problem (14) should be bounded, and its infimum value (the optimal risk level) should increase as so does the expected return R. Let us show that these properties do not hold in general, unless there exists an appropriate SDF of Φ.

Lemma 6 *a)* $-\Phi(-1) \leq \mathbb{E}(z) \leq \Phi(1) \leq 1$ *holds for every* $z \in \Delta_\Phi$.

b) If (z_1, z_2, μ, λ) *is (19)-feasible then*

$$\frac{1+\lambda}{-\Phi(-1)} \geq \mu \geq \frac{1+\lambda}{\Phi(1)}. \tag{20}$$

Proof. *a)* Expressions (8) and (12) imply that

$$\mathbb{E}(z) \leq \Phi(1) \leq 1$$

and

$$-\mathbb{E}(z) \leq \Phi(-1).$$

b) Bearing in mind (3), and taking expectations in the first constraint of (19), we have that

$$1 = \mu \mathbb{E}(z_2) - \lambda.$$

Thus, bearing in mind that $\mu \geq 0$, Statement *a)* leads to

$$-\mu\Phi(-1) - \lambda \leq 1 \leq \mu\Phi(1) - \lambda,$$

and (20) trivially follows. $\square$

Remark 4 *Notice that the (19)-feasible set does not depend on the required return $R > 0$. If it is void then Theorem 5 shows that the optimal value of (14) becomes $\rho_R^* = -\infty$ for every $R > 0$.* $\square$

Remark 5 *If the (19)-feasible set is not empty then Theorem 5 and (20) show that*

$$b_R \leq \rho_R^* \leq B_R, \tag{21}$$

b_R and B_R being the optimal values of Problems

$$\begin{cases} Max\ \dfrac{1+\lambda}{\Phi(-1)} + \lambda R = \dfrac{1}{\Phi(-1)} + \left(R + \dfrac{1}{\Phi(-1)}\right)\lambda \\ z_1 = \mu z_2 - \lambda \\ (z_1, z_2, \mu, \lambda) \in \Delta_\rho \times \Delta_\Phi \times \mathbb{R} \times \mathbb{R},\ \mu, \lambda \geq 0 \end{cases} \tag{22}$$

and

$$\begin{cases} Max\ -(1+\lambda) + \lambda R = -1 + (R-1)\lambda \\ z_1 = \mu z_2 - \lambda \\ (z_1, z_2, \mu, \lambda) \in \Delta_\rho \times \Delta_\Phi \times \mathbb{R} \times \mathbb{R},\ \mu, \lambda \geq 0 \end{cases} \tag{23}$$

Bearing in mind (8), for

$$R > \frac{1}{-\Phi(-1)}$$

both problems have the same solution $(z_1^*, z_2^*, \mu^*, \lambda^*)$, which also solves

$$\begin{cases} Max\ \lambda \\ z_1 = \mu z_2 - \lambda \\ (z_1, z_2, \mu, \lambda) \in \Delta_\rho \times \Delta_\Phi \times \mathbb{R} \times \mathbb{R},\ \mu, \lambda \geq 0 \end{cases}. \tag{24}$$

Notice that, according to Theorem 5, Problem (19) is bounded, and therefore (21) shows that (22) is bounded. Then, (23) and (24) are bounded too, and (21) shows that the optimal value $\lambda^* \geq 0$ of (24) will satisfy

$$\frac{1}{\Phi(-1)} + \left(R + \frac{1}{\Phi(-1)}\right)\lambda^* \leq \rho_R^* \leq -1 + (R-1)\lambda^* \tag{25}$$

for every $R > \dfrac{1}{-\Phi(-1)}$. $\square$

Let us extend the notion of compatibility of Balbás *et al.* (2010) for models with transaction costs.

Definition 1 *The couple* (ρ, Φ) *is said to be compatible if* $\rho_R^* > -\infty$ *or, equivalently, the (24)-feasible set is non void.*[9] *The couple* (ρ, Φ) *is said to be strongly compatible if there exists a (24)-feasible element* (z_1, z_2, μ, λ) *such that* $\lambda > 0$. □

Remark 6 *If* (ρ, Φ) *is not compatible then we are facing a meaningless situation from a financial point of view. For every* $R > 0$ *the optimal risk level becomes* $-\infty$*, so traders may compose portfolios whose return is as large as desired and whose risk is as small as desired too. We will say that the value* $(risk,\ return) = (-\infty,\ \infty)$ *is available to investors.*

If (ρ, Φ) *is compatible but it is not strongly compatible then (25) shows that* $\rho_R^* \leq -1$ *for every* $R > \dfrac{1}{-\Phi(-1)}$.[10] *Once again traders may compose portfolios with risk level non higher than* -1 *and with the desired expected return. We will say that the value* $(risk,\ return) = (-1,\ \infty)$ *is available to investors.*

In both cases we will say that there are good deals, which is unacceptable from a financial perspective. □

Let us prove the main results of this paper, *i.e.*, let us give conditions that the SDF of Φ must satisfy so as to prevent the pathological existence of good deals.

Theorem 7 *a) If* Φ *satisfies Assumptions* 2 *and* 3 *and for every* SDF $z \in \Delta_\Phi$ *and every* $\delta > 0$ *the inequality*

$$\mathbb{P}\left(z < \delta\right) > 0 \tag{26}$$

holds, then Φ *is not strongly compatible with every* ρ *satisfying Assumption* 1.

b) If Φ *satisfies Assumptions* 2 *and* 3 *and for every* SDF $z \in \Delta_\Phi$ *and every* $\delta > 0$ *the inequality*

$$\mathbb{P}\left(z \geq \delta\right) > 0 \tag{27}$$

holds, then Φ *is not compatible with every* ρ *satisfying Assumption* 1 *and such that* ρ *may be extended to the space* L^1.

[9] Notice that $\rho_R^* > -\infty$ holds for every $R > 0$ if and only if it holds for some $R_0 > 0$, since the (19)-feasible set does not depend on R and Theorem 5 applies.

[10] And therefore $\rho_R^* \leq -1$ for every $R > 0$, since the (14)-feasible set obviously increases as $R > 0$ decreases, and thus ρ_R^* decreases too.

c) If Φ satisfies Assumptions 2 and 3 then there exists $\rho : L^1 \longrightarrow \mathbb{R}$ satisfying Assumption 1 and strongly compatible with Φ if and only if there exist $0 < b \leq B$ and a SDF z of Φ such that

$$b \leq z \leq B$$

out of a null set.[11]

d) If $-\Phi(-1) = \Phi(1)$,[12] *Φ satisfies Assumptions 2 and 3, and ρ satisfies Assumption 1, then Φ and ρ are compatible if and only if there exists $(z_1, z_2) \in \Delta_\rho \times \Delta_\Phi$ such that $z_2/\Phi(1)$ is a linear convex combination of z_1 and the riskless asset $y = 1$.*

e) If $-\Phi(-1) = \Phi(1)$, Φ satisfies Assumptions 2 and 3, and ρ satisfies Assumption 1, then Φ and ρ are strongly compatible if and only if there exists $(z_1, z_2) \in \Delta_\rho \times \Delta_\Phi$ such that $z_2/\Phi(1)$ is a linear convex combination of z_1 and the riskless asset $y = 1$ such that the weight of $y = 1$ is strictly positive.

Proof. *a*) If Φ were strongly compatible with some ρ satisfying Assumption 1, then there would exist a (19)-feasible element (z_1, z_2, μ, λ) with $\lambda > 0$. Then, $z_1 = \mu z_2 - \lambda$ would imply that $\mu > 0$, since otherwise $z_1 = -\lambda < 0$ would contradict (3). Consequently, bearing in mind (4),

$$z_2 = \frac{z_1 + \lambda}{\mu} \geq \frac{\lambda}{\mu}$$

and (26) does not hold for z_2 and $0 < \delta < \lambda/\mu$.

b) If Φ were compatible with some ρ, then there should exist a (19)-feasible element (z_1, z_2, μ, λ). As above, $\mu > 0$, so

$$z_2 = \frac{z_1 + \lambda}{\mu}.$$

Since $q = \infty$, we have that z_1 is essentially bounded, and therefore so is z_2. Thus, (27) does not hold for z_2 if δ is large enough.

c) The necessity of the given condition trivially follows from *a*) and *b*). It is also sufficient because one can choose the risk measure ρ such that Δ_ρ is the "segment" $[z_1, 1]$, where

$$z_1 = \frac{(1 + \lambda)}{\mathbb{E}(z)} z - \lambda \tag{28}$$

[11] Theorem's proof will show that Statements *b*) and *c*) may be easily adapted so as to involve every $p \in [0, 1)$, rather than $p = 1$.

[12] *i.e.*, if there are no frictions when trading the riskless asset, or, equivalently, if lending rates equal borrowing rates.

(see Remark 1, (8) and Lemma 6a) and $\lambda > 0$ is chosen so as to satisfy

$$\frac{(1+\lambda)}{\mathbb{E}(z)} \geq \frac{\lambda}{b}.$$

Obviously, (3) and (4) hold, and the rest of conditions in Assumption 1 become trivial. Expression (28) proves that

$$\left(z_1, z, \frac{(1+\lambda)}{\mathbb{E}(z)}, \lambda\right)$$

is (19)-feasible, and $\lambda > 0$ implies that (ρ, Φ) is strongly compatible.

d) If (z_1, z_2, μ, λ) is (19)-feasible then Lemma 6b) implies that $\mu = (1+\lambda)/\Phi(1)$, so the constraint of (19) leads to

$$\frac{z_2}{\Phi(1)} = \frac{z_1}{1+\lambda} + \frac{\lambda}{1+\lambda}.$$

Conversely, suppose that

$$\frac{z_2}{\Phi(1)} = (1-t)\, z_1 + t$$

with $0 \leq t \leq 1$. If $t = 1$, then $(z_1 = 1, z_2 = \Phi(1), \mu = 2/\Phi(1), \lambda = 1)$ is (19)-feasible. If $t \neq 1$, then

$$z_1 = \frac{1}{(1-t)\,\Phi(1)} z_2 - \frac{t}{(1-t)},$$

and

$$(z_1, z_2, \mu, \lambda) = \left(z_1, z_2, \frac{1}{(1-t)\,\Phi(1)}, \frac{t}{(1-t)}\right)$$

becomes (19)-feasible.

e) Analogous to d). □

Remark 7 *Theorem 7 implies the necessity of a SDF with a strictly positive lower bound, since otherwise the pathologies* $(risk,\ return) = (-\infty,\ \infty)$ *or* $(risk,\ return) = (-1,\ \infty)$ *will arise for every coherent and expectation bounded risk measure* ρ*. Moreover, these pathologies still hold if* ρ *is replaced by a vector* $(\rho_1, \rho_2, ..., \rho_k)$ *and (14) becomes a vector optimization problem, i.e., for vector problems one will get the solution*

$$(risk_1, risk_2, ..., risk_k, return) = (-1, -1, ..., -1, \infty).$$

Indeed, considering the risk measure ρ *of Remark* 1*, the inequality* $\rho \geq \rho_i$*,* $i = 1, 2, ..., k$*, holds, and the solution of (14) is* $(risk,\ return) = (-\infty,\ \infty)$ *or* $(risk,\ return) = (-1,\ \infty)$ *if we use the risk measure* ρ*.*

Besides, the existence of SDF of Φ with a finite upper bound is convenient too, since otherwise the pathology $(risk,\ return) = (-\infty, \infty)$ will hold for many important risk measures such as the $CVaR$ or many versions the $WCVaR$, amongst others.[13] *Furthermore, the inequality $VaR \leq CVaR$ for every level of confidence shows that he pathology $(risk,\ return) = (-\infty, \infty)$ will hold for VaR too.*

It is worthwhile to point out that for perfect markets there is only one SDF z, which must satisfy the existence of $0 < b \leq B$ such that

$$\mathbb{P}\left(b \leq z \leq B\right) = 1$$

so as to prevent the existence of good deals. An obvious implication is that a SDF with a Log-Normal distribution (for example, the Black and Scholes model, see Wang, 2000, or Hamada and Sherris, 2003) or distributions with heavier tails (most of the stochastic volatility pricing models) will never be strongly compatible with any coherent and expectation bounded risk measure, and they will not be compatible with VaR or with measures that can be extended to L^1. Thus, the already described pathologies will arise, i.e., good deals will be available to traders. $\square$

5 Short selling restrictions and further imperfections

Though Assumption 2 implies a very general setting, let us incorporate further imperfections. First, suppose the existence of lower bounds for the selected portfolio of (14), *i.e.*, the existence of $M \in \mathbb{R}$ such that $y \geq M$ must hold. If $M = 0$ we will be facing short selling restrictions. Problem (14) becomes

$$\begin{cases} Min\ \rho\left(y\right) \\ \Phi\left(y\right) \leq 1,\ y \geq M,\ \mathbb{E}\left(y\right) \geq R \end{cases} \tag{29}$$

As in Section 3, the dual problem becomes

$$\begin{cases} Max\ f\left(M\right) - \mu + \lambda R \\ z_1 \leq \mu z_2 - \lambda \\ \left(z_1, z_2, \mu, \lambda\right) \in \Delta_\rho \times \Delta_\Phi \times \mathbb{R} \times \mathbb{R},\ \mu, \lambda \geq 0 \end{cases}, \tag{30}$$

[13] Among many other interesting properties of the $CVaR$, this coherent and expectation bounded risk measure is compatible with the second order stochastic dominance. This property is not satisfied by the variance in presence of asymmetries (Ogryczak and Ruszczynski, 1999 and 2002).

$f(M) \in \mathbb{R}$ depending on M. As in Theorem 5, if (29) is feasible then there is no duality gap and (30) is solvable. As in the proof of Theorem 7a), if (z_1, z_2, μ, λ) is (30)-feasible and $\lambda > 0$ then $\mu > 0$ and

$$z_2 \geq \mu/\lambda > 0, \tag{31}$$

so one still needs SDF with a strictly positive lower bound. Otherwise, the solution of (30) will be some $(z_1, z_2, \mu, \lambda = 0)$ or $(z_1, z_2, \mu, \lambda = -\infty)$, and the optimal risk level will be $f(M) - \mu \leq f(M)$ or $-\infty \leq f(M)$, which does not depend on the required return R and will be bounded from above. Thus, the caveat $(risk, return) = (f(M), \infty)$ applies again. In particular, for perfect markets involving the Log-normal or heavier tailed distributions, (31) will not hold and short selling restrictions will not solve the caveat.

However, the existence of upper bounds for some SDF is not necessary now, since the arguments in the proof of theorem 7b) do not apply. The reason is that the first constraint in (30) is an inequality, instead of the equality of (19).

One could relax the constraint $\mathbb{E}(y) \geq R$, since $y \geq M$ imposes some kind of "minimum achievement" in the investment. Then, the portfolio selection problem becomes

$$\left\{ \begin{array}{l} Min\ \rho(y) \\ \Phi(y) \leq 1,\ y \geq M \end{array} \right. . \tag{32}$$

Suppose that (32) is feasible. As in Section 3, the dual problem is

$$\left\{ \begin{array}{l} Max\ f(M) - \mu \\ z_1 \leq \mu z_2 \\ (z_1, z_2, \mu) \in \Delta_\rho \times \Delta_\Phi \times \mathbb{R},\ \mu \geq 0 \end{array} \right. . \tag{33}$$

Once again there is no duality gap and (33) is solvable. This couple of optimization problems is not very interesting because, obviously, (32) should not be feasible for M large enough. Nevertheless, the lack of the $\lambda - variable$ in (33) and the lack of equalities in the constraints of this problem implies that the proofs of Theorem 7a) and 7b) do not apply any more. Actually, the lack of bounded SDF of Φ would not necessarily lead to any pathological property of (32). For instance, if $\rho = CVaR_\alpha$, $0 < \alpha < 1$ being the level of confidence, then (33) is often feasible, and therefore (32) is bounded and the optimal risk level will not be $-\infty$. Indeed, bearing in mind that

$$\Delta_{CVaR_\alpha} = \left\{ z \in L^\infty; 0 \leq z \leq \frac{1}{1-\alpha},\ \mathbb{E}(z) = 1 \right\}$$

(see Rockafellar *et al.*, 2006), (8) and Lemma 6a) will easily show that if there exists $z_2 > 0$ belonging to Δ_Φ then

$$(z_1, z_2, \mu) \in \Delta_\rho \times \Delta_\Phi \times \mathbb{R}$$

will be (33)-feasible for $\mu > 0$ large enough.

6 Conclusions

In a recent paper Balbás *et al.* (2010) have proved that the usual frictionless complete pricing models (Black and Scholes, Heston, etc.) imply the existence of good deals (*i.e.*, investors may compose portfolios with ($risk$, $return$) values as close as desired to $(-\infty, \infty)$ or $(-1, \infty)$) for every coherent and expectation bounded measure of risk. It is natural to analyze whether the existence of frictions may modify this finding, which is obviously meaningless from a financial viewpoint.

This paper have addressed the caveat above by considering a general pricing rule generating transaction costs, even when trading the riskless asset. Under general conditions about this pricing rule we have provided necessary and necessary and sufficient conditions which must hold so as to prevent the pathology above. These conditions do not depend on the concrete risk measure we are dealing with, and they mainly affect the pricing rule. The existence of bounded SDF must hold, and the lower bound must be strictly positive. If there are no bounded SDF, or the lower bound is not bigger than zero, then the caveat above will arise even for vector risk measures, and the existence of transaction costs will not solve the problem. It is worth remarking that bounded SDF are not usual at all in financial literature. In particular, for perfect markets, Log-Normal of heavier tailed distributions for the SDF will imply the existence of good deals for every coherent and expectation bounded scalar or vector measure of risk. □

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