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The Sticky Information Macro Model: Beyond Perfect Foresight

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Abstract

Sticky information monetary models have been used in the macroeconomic literature to explain some of the observed features regarding inflation dynamics. In this paper, we explore the consequences of relaxing the rational expectations assumption usually taken in this type of model; in particular, by considering expectations formed through adaptive learning, it is possible to arrive to results other than the trivial convergence to a fixed point long-term equilibrium. The results involve the possibility of endogenous cyclical motion (periodic and a-periodic), which emerges essentially in scenarios of hyperinflation. In low inflation settings, the introduction of learning implies a less severe impact of monetary shocks that, nevertheless, tend to last for additional time periods relative to the pure perfect foresight setup.

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1 Introduction

The sticky information paradigm was born as a reply to the New Keynesian price stickiness setup, widely used in monetary policy analysis (see Clarida, Gali and Gertler [3] and Woodford [23]). Mankiw and Reis [8] have stressed that their interpretation of monetary policy has the virtue of fitting better a series of macroeconomic facts. Unlike the New Keynesian model, the Mankiw-Reis sticky information framework is able to explain how and why: (i) inflation is persistent; (ii) monetary policy shocks have a delayed and gradual effect on inflation; (iii) desinflations are always contractionary. Above all, the sticky information setup may be viewed as an alternative to a paradoxical point of the New Keynesian approach: a model that has in its essence the prevalence of price rigidity is a model in which inflation changes fast; it cannot replicate the inflation inertia found in empirical evidence.

The information stickiness model imposes no constraint on the timing of pricing decisions. Firms can modify prices whenever they choose, however there is a constraint on information propagation: information concerning macroeconomic conditions diffuses slowly throughout the economy, making price changes to be often decided on the basis of outdated information. The slow information diffusion is assumed to be the result of the costs of acquiring and processing information, i.e., the costs of re-optimization. As the result of the sticky information hypothesis, the economy's current price level becomes a function of the weighted average of past expectations concerning the present state of the economy, with each firm having updated its information set at any past date, a decision that is independent of the same decision made by any other firm.

The initial version of the model was relatively straightforward: it focused essentially on the shape of the Phillips curve and the model was closed with a simple money demand equation. The same authors have then sophisticated and complemented the model in various directions. First, Mankiw and Reis [9] explicitly integrate into the model the dynamics of wages and productivity; the additional conclusion is that productivity slowdowns have a same type of impact as desinflations, i.e., they both lead to an employment trajectory that falls below the full information employment level. Second, Reis [15] [16] builds a theory of inattentiveness that provides the microfoundations for the sticky information approach; both producers and consumers face costs of acquiring, absorbing and processing information and these costs have to be taken into account when agents solve their optimal control problems (in practice, the standard profit maximization and utility maximization problems will involve an additional constraint relating information costs). The result is an optimal degree of inattentiveness, i.e., agents rationally choose to be inattentive to news, updating information on a sporadic basis. Rational inattention is also modelled resorting to alternative approaches, as in Sims [19] who alternatively to an intertemporal optimization approach supports the analysis of limited ability to process information on the modelling of a communication channel with finite Shannon capacity.

A third relevant extension of the information stickiness model consists in the attempt to combine the framework's central assumption with a general equilibrium setting. This is what Mankiw and Reis [10] [11] suggest. In this new setting, inattentiveness is pervasive: all the agents in the economy, i.e., consumers, firms and workers, choose to update the corresponding set of information only sporadically, meaning that past expectations on present events will be a common feature of the various central macroeconomic relations: the Phillips curve, the IS curve and a wage curve. Prices, consumption plans and wages are all set, to some extent, based on outdated information. The relevant fact concerning this version of the model is that it proves better at matching key empirical facts concerning economic fluctuations than the less sophisticated model in which only firms resort to outdated information to decide about their activity.

This paper intends to offer an additional contribution to the sticky information literature, by exploring the consequences of departing from the rational expectations assumption. To undertake the analysis, we restrict the model to a deterministic setting.[1] Under perfect foresight, the predictions of the sticky information model are rather simple: the inflation rate will converge to the rate of money supply, and it will remain on this value as long as the equilibrium remains undisturbed. Nevertheless, information stickiness continues to have a relevant role in such a purely deterministic setting: the degree of information stickiness will indicate the velocity of convergence towards the equilibrium point. If there is no stickiness, the adjustment is instantaneous; on the contrary, if the information stickiness is absolute, the inflation level remains fixed and no convergence occurs. Between these extreme circumstances, a lower degree of information stickiness is synonymous of faster adjustment. The main point is that independently of the degree of sluggishness in updating information, convergence to the steady state is guaranteed under perfect foresight (except in the unrealistic case in which information is never updated). Once we depart from rational expectations, different equilibrium outcomes are eventually obtained and fluctuations may be endogenously triggered and not just the result of extrinsic forces.

[1] Although, at the end of the paper (section 7), the possibility of stochastic shocks is introduced in order to address impulse-response dynamics in the discussed scenario.

To depart from perfect foresight, we will consider that agents have access to two sets of information: (i) information about the state of the economy, which is updated only sporadically and (ii) information about the past behavior of the assumed endogenous variable(s) (in the case, the inflation rate), which is known by all agents in every time period and that allows to learn the current value of the variable(s). Agents that have updated their information about the state of the economy in the past may have insufficient data in order to predict perfectly today's outcomes and they will resort essentially to past performance data on the variable in order to predict or learn the current value; on the contrary, agents who have updated information on the state of the economy recently will be able, to a large extent, to form rational expectations and therefore will not need to largely resort to learning. The learning component of expectations is modelled in a trivial way: we consider an adaptive learning algorithm under which the growth rate of the price level is estimated resorting to a simple least squares regression applied over a perceived law of motion for prices.

The characterization of the dynamics of the model, involving the above mentioned expectations assumptions, will attribute relevance to four parameters: the income elasticity of money demand, a parameter measuring real rigidities, the degree of information stickiness and the growth rate of money supply. For a large set of possible combinations of parameters, stability is found, i.e., there is convergence to the monetary steady state that prevails under perfect foresight. Two-period, quasi-periodic and chaotic cycles are found, following a flip bifurcation, for a significant level of information stickiness and a strong rate of growth of money supply. Endogenous cycles are, in this model, attached to scenarios of hyperinflation. This is a relevant result since some studies, e.g. Marcet and Sargent [13] and Marcet and Nicolini [12], point precisely to an association between fast growth of the price level and learning processes with outcomes that may deviate from the perfect foresight equilibrium.

Studies on the empirical validity of the sticky-information Phillips curve (e.g., Khan and Zhu [7], Dopke et al. [5] and Coibion [4], among others), point to an acceptable fit between evidence and models based on the sticky-information assumption; furthermore, as Reis [17] highlights, the observed misalignments between the data and the time series generated by the theoretical structure are not mainly caused by inattentiveness / information sluggishness, but rather by the assumption that in these models agents will form expectations on the grounds of perfect rationality, given the information sets they face. Such an evidence furnishes a strong motivation to continue exploring the sticky-information setup, analyzing it further beyond the rational expectations

/ perfect foresight benchmark framework. By exploring distinct assumptions concerning the formation of expectations, one will possibly find new avenues for achieving a better match with the data and also to attribute a stronger logical coherence to the modelling structure.

The remainder of the paper is organized as follows. Section 2 presents the sticky information model. In section 3 the dynamics of the model under rational expectations is addressed. Section 4 introduces adaptive learning and proposes a rule concerning the formation of aggregate expectations. Sections 5 and 6 proceed with the dynamic analysis of the model, both locally and in a global perspective. Section 7 studies how a monetary policy shock impacts on the long-run evolution of the inflation rate, allowing to compare the response to shocks between the scenarios of full rationality and learning. Finally, section 8 concludes. Three appendices are also added at the end of the paper: appendix A derives the central dynamic equation of the model; appendix B indicates how the model could be sophisticated through the introduction of a more complex monetary policy structure, namely by resorting to an IS relation; in appendix C we change the rule governing the formation of expectations, in order to further explore the implications of learning in the sticky information environment and to evaluate the robustness of the sticky-information / adaptive learning analytical structure.

2 The Sticky Information Model: General Features

The presentation of the sticky information model follows closely the setup by Mankiw and Reis [8]; thus, we just present the basic features of the model that are necessary for the discussion that follows in next sections. In the considered environment, prices are fully flexible in the sense that firms have the possibility of changing them in every time period. However, agents may be inattentive and, thus, base their pricing decisions on outdated information. As remarked in Reis [15] [16], firms' inattentiveness is not necessarily a synonymous of lack of rationality; an optimal degree of inattentiveness arises as the result of equating benefits of price changes and costs of acquiring and processing information. Therefore, relevant information will diffuse slowly throughout the economy, as firms choose to optimally update their information set only sporadically, remaining inattentive in the time length between adjustment dates.

Let $p_t \in \mathbb{R}$ be the logarithm of the price level and $y_t \in \mathbb{R}$ the logarithm of the output gap. The output gap is defined as the difference between effective and potential output. Consider as well a parameter $\alpha > 0$ defining a measure of sensitivity of the firm's target price to the output gap. Firms are monopolistically competitive, produce a single homogeneous good and act optimally to derive the following target price:

$$p_t^* = p_t + \alpha y_t \tag{1}$$

Equation (1) reveals that firms will desire to maintain the existing price only if there is a coincidence between effective and potential output at date t; otherwise, the target price will rise above the observable price level when firms are faced with a positive output gap, and the target price will fall below p_t in the opposite circumstance.

Parameter α has an important role in the analysis and therefore we should further clarify its meaning. This parameter is also known as reflecting 'real rigidities' and according to Romer [18] such real rigidities may involve the following items: thick-market externalities, economies of scale and agglomeration economies, financial market imperfections, imperfect information that makes existing consumers more responsive to increases in prices than potential consumers are to price decreases, just to cite some of the most relevant.

Under the assumption of sticky information, a firm that last updated its price plans j periods ago will set a price that corresponds to the expectations formed in period $t-j$ concerning the target price in t, i.e., $p_t^j = E_{t-j}p_t^*$ with p_t^j the price set by a firm in t when the firm updated its information in period $t-j$.

Aggregation of individual prices will follow a mechanism similar to the one introduced by Calvo [2] concerning nominal rigidities. It is assumed that, at each time moment, a share of firms $\lambda \in [0,1]$ collects information on the state of the economy and updates prices resorting to such information; the other firms also set prices in a given time moment but resorting to outdated information. If $\lambda = 1$, there is full information flexibility [rule (1) is applied in every moment by every firm]; if $\lambda = 0$, there are no firms updating information, and therefore prices remain constant. In practice, it is admissible to consider that some intermediate level of information updating exists and thus λ is some value between 0 and 1 with the degree of information sluggishness rising as λ becomes closer to the lower bound of the interval.

In Reis [15], it is demonstrated that in an economy with many agents the distribution of information updating converges to a Poisson distribution, implying that each firm has an equal probability of being one of the firms updating prices with the most recent information, independently of the date of the last information review. The aggregate price level will be the weighted average of the prices set by each one of the firms what, under the Poisson process, translates into $p_t = \lambda \sum_{j=0}^{\infty} (1-\lambda)^j p_t^j$. Replacing, in this expression, each firm's price by the expected target price j periods ago and having in consideration equation (1), aggregate prices in period t will be given by

$$p_t = \lambda \sum_{j=0}^{\infty} (1-\lambda)^j E_{t-j}(p_t + \alpha y_t) \tag{2}$$

In expression (2), the aggregate price level is basically the direct result of two assumptions: reaction of each firm to the output gap in triggering price changes and information stickiness.

Defining $\pi_t := p_t - p_{t-1}$ as the inflation rate and $g_t := y_t - y_{t-1}$ as the growth rate of the output gap, simple algebra allows to transform the price equation (2) into the well known sticky information Phillips curve,

$$\pi_t = \frac{\alpha\lambda}{1-\lambda} y_t + \lambda \sum_{j=0}^{\infty} (1-\lambda)^j E_{t-1-j}\left(\pi_t + \alpha g_t\right) \tag{3}$$

Equation (3) indicates the existence of a contemporaneous relation between the output gap and inflation; besides this, today's inflation rate is determined by past expectations concerning the current state of the economy.

To proceed with the analysis of the model, some consideration about monetary policy must be added. In appendix (appendix A at the end of the paper), we show how a simple equation describing the equilibrium in the money market, combined with the displayed Phillips relation, allows for presenting the dynamics of inflation as follows:

$$\begin{aligned} \pi_{t+1} \quad = \quad & \frac{\alpha\lambda}{\beta} \Delta m + \frac{\beta - \alpha\lambda}{\beta} \pi_t + \frac{\beta - \alpha}{\beta} \lambda \sum_{j=0}^{\infty} (1-\lambda)^j E_{t+1-j}\left(\pi_{t+1}\right) \\ & - \frac{\beta - \alpha}{\beta} \lambda \sum_{j=0}^{\infty} (1-\lambda)^j E_{t-j}\left(\pi_t\right) \end{aligned} \tag{4}$$

In equation (4), Δm is a constant rate of money growth and $\beta > 0$ is a parameter measuring the income elasticity of money demand. The second part of the appendix (appendix B) discusses how one could sophisticate the model by considering a more structured approach to monetary policy. The choice of a simple rule of adjustment of money supply to demand conditions is justified on the basis of proposing a framework that is analytically tractable in a simple way once we introduce learning.

It is the study of the dynamics underlying equation (4) that will concern us in the following sections. It can be interpreted as the reduced form of the sticky information model. If one defines the steady state as the scenario in which inflation remains constant, it is straightforward to obtain the monetary steady state $\pi^* = \Delta m$. The dynamics of (4) will depend on which assumption one takes about the formation of expectations.

3 Perfect Foresight Dynamics

The perfect foresight assumption implies that independently of the considered past moments of time, agents will be able to accurately predict today's level of inflation. Under this assumption, equation (4) will be equivalent to

$$\pi_{t+1} = \lambda \Delta m + (1 - \lambda)\pi_t \quad (5)$$

The only relevant parameter to characterize inflation dynamics in the simple sticky information monetary model is the parameter representing the degree of information stickiness. Because $\lambda \in (0, 1)$ one observes that the system is stable: convergence towards the monetary steady state is guaranteed independently of the initial state π_0. Parameter λ will translate the velocity of adjustment towards $\pi^* = \Delta m$; if $\lambda \to 0$, no adjustment will take place (the inflation rate suffers no change); if $\lambda = 1$, there is an instantaneous adjustment. The larger is the value of λ, i.e., the more flexible information is, the faster is the adjustment process.

Figure 1 draws the time trajectory of the inflation rate for $\pi_0 = 0.04$ and $\Delta m = 0.02$, considering three different degrees of information stickiness: $\lambda = 0.25$, $\lambda = 0.5$, $\lambda = 0.75$. Observe that the lower is λ, the slower will be the adjustment process. In deterministic terms this is the only information the model is capable of conveying, as long as one sticks with the perfect foresight / rational expectations assumption.

It is straightforward to understand that no dynamic convergence would occur if $\pi_0 = \Delta m$; in this case, the system is initially at the steady-state and, in the absence of any external shock, it will remain there independently of the degree of information stickiness. For any $\pi_0 \neq \Delta m$, the level of stickiness will determine how many periods are needed in order to asymptotically reach the steady-state. Note that the role of information updating gains relevance when a disturbance over the equilibrium in which the system rests occurs; a more infrequent information updating leads to a more inertial response to the shock, i.e., to a lengthier reestablishment of the initial steady-state.

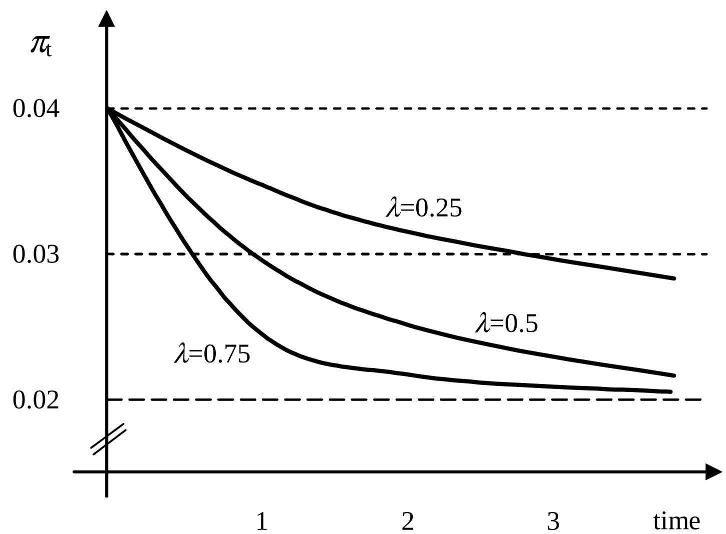

Figure 1: Inflation dynamics in the deterministic sticky information model (perfect foresight).

A crucial assumption in this paper is that the initial state is one in which the system deviates from rational expectations: by taking $\pi_0 \neq \Delta m$, one is considering that the economic system is 'born' in a state of disequilibrium and that the rational behavior of the economic agents will allow for a convergence towards the equilibrium (at a rate that depends on the available information). A possible explanation for this assumption can be found in the complete absence of knowledge about the evolution of the economy that agents might display at the initial state, i.e., initially firms will have to form prices without any information on how monetary policy is conducted; once economic agents perceive how money supply evolves, they will adjust their behavior accordingly. Therefore, $t = 0$ can be interpreted as the time period when all firms initiate their activity and, thus, have insufficient knowledge on how the economy behaves and how authorities undertake their policies.

4 Learning Past Results

In this section, we assume that the ability of economic agents in predicting perfectly current inflation values, independently of how far in the past they formulate expectations about the current state of the economy, is limited.

More specifically, we consider the assumption described in the introduction: agents possess two information sets, one concerning the state of the economy which is updated sporadically, and another one consisting on the observation of all past inflation levels prior to moment t. This last information set allows agents to predict the value of the inflation rate through a process of adaptive learning.

We will consider the following rule in what respects the formation of expectations:

$$E_t(\pi_t) = \pi_t; E_{t-j}(\pi_t) = \widehat{\pi}_t, \forall j = 1, 2, ... \tag{6}$$

According to rule (6), if expectations are formed in the current period, agents will know the true value of this rate; otherwise, agents will ignore any information on the state of the economy and they will just resort to the observation of the past performance of the inflation rate through time $t-1$. Variable $\widehat{\pi}_t$ represents the estimated level of inflation at t computed by resorting to all the information on π through $t-1$. Under this hypothesis, it is straightforward to regard that $\lambda \sum_{j=0}^{\infty} (1-\lambda)^j E_{t-j}(\pi_t) = \lambda\pi_t + (1-\lambda)\widehat{\pi}_t$.

The monetary model equation (4) takes, under the established assumption, the form

$$\pi_{t+1} = \frac{1}{(\beta/\alpha)(1-\lambda)+\lambda} \cdot [\lambda\Delta m + (\beta/\alpha)(1-\lambda)\pi_t + (\beta/\alpha - 1)(1-\lambda)(\widehat{\pi}_{t+1} - \widehat{\pi}_t)] \tag{7}$$

Note that if one eliminates the learning component (by taking $\widehat{\pi}_{t+1} = \pi_{t+1}$ and $\widehat{\pi}_t = \pi_t$), we return to the simple perfect foresight equation (5).

Having no anticipated knowledge on the value of $\widehat{\pi}$, agents estimate it by acting as econometricians, i.e., by running a least squares regression. Following Bullard [1], we consider that agents do not estimate directly a constant inflation rate but a constant rate of price change, that is, the adopted perceived law of motion is the following,

$$P_t = \exp(\widehat{\pi})P_{t-1} \tag{8}$$

In (8), P_t is the price level, i.e., $p_t = \ln P_t$. Estimating in t the inflation rate using data available through time $t-1$ yields,

$$\exp(\widehat{\pi}_t) = \frac{\sum_{s=1}^{t-1} P_{s-1}P_s}{\sum_{s=1}^{t-1} P_{s-1}^2} \tag{9}$$

The expectations feedback system composed by (8) and (9) can be transformed into a recursive dynamic system. This requires introducing a gain sequence variable, which is defined as $\sigma_t := P_{t-1}^2 / \sum_{s=1}^{t} P_{s-1}^2$. Variable σ_t is known as a decreasing gain sequence since as time passes and the number of observations increases, the gain or impact of each new observation becomes smaller. The derived system of difference equations is

$$\exp(\widehat{\pi}_{t+1}) = \exp(\widehat{\pi}_t) + \sigma_t \left[\frac{P_t}{P_{t-1}} - \exp(\widehat{\pi}_t) \right] \tag{10}$$

$$\sigma_{t+1} = \frac{\sigma_t}{\left(\frac{P_{t-1}}{P_t}\right)^2 + \sigma_t} \tag{11}$$

Under our definition of variables, $\frac{P_t}{P_{t-1}} = \exp(\pi_t)$.

Equation (7) together with the learning scheme [i.e., equations (10) and (11)] form a three first-order equations system with three endogenous variables: π_t, $\widehat{\pi}_t$ and σ_t; observe that equation (10) can be presented as $\widehat{\pi}_{t+1} = \ln\left[(1-\sigma_t)\exp(\widehat{\pi}_t) + \sigma_t \exp(\pi_t)\right]$. In the next sections, we address this model's dynamics. From a local dynamics point of view, a general analysis may be pursued; globally, one will need to calibrate the model in order to undertake a meaningful characterization of dynamic results. The benchmark calibration will be $\alpha = 0.1$, $\beta = 0.5$ and $\lambda = 0.25$ (these are the values of parameters considered by Mankiw and Reis [9] in their simulation of inflation dynamics under sticky information).

We will consider that parameter values reflect quarterly data, and therefore the value of λ can be interpreted in terms of concrete calendar dates. If $\lambda = 1$, then information about the state of the economy is updated every quarter of year; for instance if $\lambda = 0.25$, this will imply that economic agents update their information on average once a year. Generalizing, one can say that for some stickiness parameter $\lambda = \widetilde{\lambda}$, the average frequency of information updating will be $1/\widetilde{\lambda}$ quarters, or $3/\widetilde{\lambda}$ months. Although theoretically λ might assume any value between zero and one, it is admissible to impose a plausible positive lower bound; for instance, the assumption that $\lambda > 0.05$ indicates that information will be updated at least every 20 quarters or 60 months.

In the subsequent analysis, it does not make sense to consider a separate discussion on the impact of changes in the values of parameters α and β; their effect over the economic system arises always through the ratio β/α. The displayed quotient can assume only positive values. The benchmark case is such that the real rigidities parameter is lower than the income elasticity of money demand; however, one will observe that departures from stability will require a smaller than 1 value for the ratio β/α.

5 Dynamics under Learning

The steady-state value of the gain sequence variable is $\sigma^* = 1 - [\exp(\Delta m)]^{-2}$. This is a positive and lower than 1 value for any $\Delta m > 0$. The monetary steady-state can, then, be fully displayed: $(\pi^*, \widehat{\pi}^*, \sigma^*) = \{\Delta m, \Delta m, 1 - [\exp(\Delta m)]^{-2}\}$. Local stability results are addressable after computing a linearized system in the vicinity of the presented steady-state. The obtained system is

$$\begin{bmatrix} \widehat{\pi}_{t+1} - \widehat{\pi}^* \\ \pi_{t+1} - \pi^* \\ \sigma_{t+1} - \sigma^* \end{bmatrix} = \begin{bmatrix} 1 - \sigma^* & \sigma^* & 0 \\ -\frac{\theta - 1}{\theta}\sigma^* & \frac{\theta - \lambda + (\theta - 1)\sigma^*}{\theta} & 0 \\ 0 & \frac{2\sigma^*}{[\exp(\Delta m)]^3} & 1 - \sigma^* \end{bmatrix} \begin{bmatrix} \widehat{\pi}_t - \widehat{\pi}^* \\ \pi_t - \pi^* \\ \sigma_t - \sigma^* \end{bmatrix} \tag{12}$$

with $\theta := (\beta/\alpha)(1 - \lambda) + \lambda > 0$.

Resorting to the center manifold theorem, one can exclude the third dynamic equation from the analysis, i.e., we can concentrate on the sub-matrix of the Jacobian matrix in (12) respecting to its two first rows and two first columns. This matrix will be designated by J and it will furnish relevant information regarding local dynamics.[2]

Proposition 1 states a first result concerning stability.

Proposition 1 *If $\frac{\beta}{\alpha} \geq \frac{2-3\lambda}{4(1-\lambda)}$, then the system is locally stable $\forall \Delta m > 0$. Furthermore, for an information updating schedule such that $\lambda \geq 2/3$, stability will prevail $\forall \Delta m$, α, $\beta > 0$.*

[2] The center manifold theorem applies when the state space of the system can be split in more than one invariant subspace; invariant subspaces are found when the corresponding eigenvalues are obtained resorting solely to a sub-dimension of the system, i.e., without the need for considering the whole system. This is the case at hand: the matrix in (12) displays two invariant subspaces; the first is composed by sub-matrix J and the other one by the element in the last row and last column of the matrix (this corresponds to the eigenvalue associated to variable σ_t; this is an eigenvalue locating inside the unit circle, and therefore one identifies from the start a stable dimension). See Medio and Lines [14] for details on the center manifold theorem.

Proof. Trace and determinant of J are, respectively, $Tr(J) = \frac{2\theta-\lambda-\sigma^*}{\theta}$ and $Det(J) = \frac{\theta-\lambda-(1-\lambda)\sigma^*}{\theta}$. Stability conditions for two-dimensional discrete time systems are

$1 - Det(J) > 0$

$1 - Tr(J) + Det(J) > 0$

$1 + Tr(J) + Det(J) > 0$

The stability conditions $1 - Det(J) = \frac{\lambda+(1-\lambda)\sigma^*}{\theta} > 0$ and $1 - Tr(J) + Det(J) = \frac{\lambda\sigma^*}{\theta} > 0$ hold, for any possible parameter values. The only constraint on stability comes from the third condition, which requires $\sigma^* < \frac{4(\beta/\alpha)(1-\lambda)+2\lambda}{2-\lambda}$. The equilibrium value of the gain sequence variable is bounded below 1, and therefore if $\frac{4(\beta/\alpha)(1-\lambda)+2\lambda}{2-\lambda} \geq 1$, then stability is guaranteed. This last condition is equivalent to the one in the proposition.

The result in the second part of the proposition is straightforward to obtain once we evaluate inequality $\frac{\beta}{\alpha} \geq \frac{2-3\lambda}{4(1-\lambda)}$ numerically. Because the quotient β/α must be larger than zero, we replace the ratio by zero in the inequality and solve in order to λ. Straightforward computation allows to reach the boundary $2/3$ ∎

Constraint $\lambda \geq 2/3$ in proposition 1 indicates that introducing learning into the formation of expectations in the way we have done implies stability of the monetary steady state as long as information is updated at least every 1.5 quarters (1/0.667) or 4.5 months (3/0.667); more infrequent information updating can lead to a departure from a stability outcome, depending on the values assumed by the degree of real rigidities, the income elasticity of money demand and the growth rate of money supply.

One can approach graphically the model's local results by representing areas of local stability and instability in the space of parameters. In figure 2, we draw the regions where stability holds for any admissible Δm (we take in consideration the constraint in proposition 1). The other region, close to the origin (i.e., for low values of both β/α and λ), is the region in which instability is possible for at least some values of the growth rate of m.

Let us focus attention in the case of possible loss of stability. The following result applies:

Proposition 2 *Let* $\frac{\beta}{\alpha} < \frac{2-3\lambda}{4(1-\lambda)}$. *In this case, local stability will require an upper bound on the rate of money growth. In the point in which* $\Delta m = \ln\sqrt{\frac{2-\lambda}{2-3\lambda-4(\beta/\alpha)(1-\lambda)}}$, *a flip bifurcation occurs.*

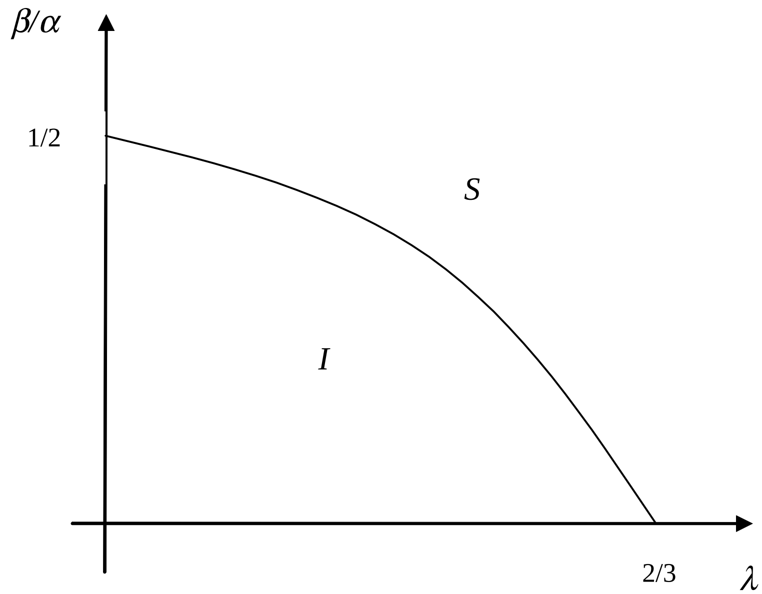

Figure 2: Stability properties in the space of parameters.

Proof. In the proof of proposition 1, we have remarked that the first two stability conditions hold independently of parameter values, while the third condition imposes a constraint on the steady state value of the gain sequence: $\sigma^* < \frac{4(\beta/\alpha)(1-\lambda)+2\lambda}{2-\lambda}$. Recalling that $\sigma^* = 1 - [\exp(\Delta m)]^{-2}$, and solving in order to Δm, we obtain the following inequality:

$$\Delta m < \ln\sqrt{\frac{2-\lambda}{2-3\lambda-4(\beta/\alpha)(1-\lambda)}}$$

Therefore, one realizes that stability requires the money growth rate to be below a given combination of parameters α, β and λ. The inequality in the proposition guarantees that the value inside the square root is positive.

Stability is eventually lost through a flip bifurcation, because it is condition $1+Tr(J)+Det(J) > 0$ that may fail to hold. In two dimensional systems, it is well known that a flip bifurcation occurs when the line $1+Tr(J)+Det(J) = 0$ is crossed with the other two stability conditions being satisfied (see Medio and Lines [14] for formal definitions of local bifurcations). No other kind of bifurcation can occur in the specified case ■

The relevant result that the two previous propositions imply is that when we introduce learning, we can only obtain a long term outcome that differs from the perfect foresight equilibrium if there is a large degree of information

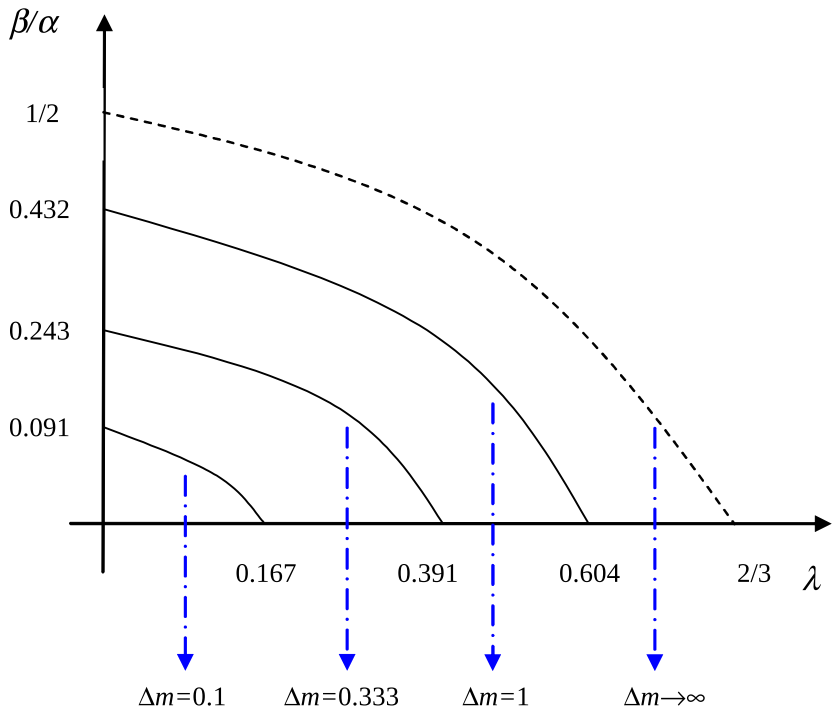

Figure 3: Bifurcation line for different values of Δm.

stickiness, the ratio β/α is relatively low and the rate of growth of money supply is relatively high (implying long run scenarios of fast price growth and even situations of hyperinflation).

To illustrate the information proposition 2 furnishes, consider figure 3. This presents the bifurcation line for different possible values of Δm in the space of parameters (i.e., taking into account, as in figure 2, the relation between λ and β/α). The figure is drawn by rewriting the equality in proposition 2 as $\frac{\beta}{\alpha} = \frac{(2-3\lambda)[\exp(\Delta m)]^2-(2-\lambda)}{4(1-\lambda)[\exp(\Delta m)]^2}$ and choosing an array of values for Δm (in particular, we consider $\Delta m = 0.1$, $\Delta m = 0.333$ and $\Delta m = 1$); to the right of each one of these lines stability holds and local instability will prevail otherwise.

The dashed line in figure 3 represents the boundary already displayed in figure 2; as remarked then, independently of how high the rate of money growth is, stability will hold to the right of such frontier. As one considers lower possible values for this growth rate, the requirements to depart from stability become more strict, i.e., the closer the rate of money growth is to zero, the stickier prices will have to be and the lower must be the elasticity of money demand - real rigidities ratio in order for stability to be lost.

We have remarked that only fast money growth can lead to the loss of stability and eventual nonlinear dynamics (which will be characterized in the following section). Nevertheless, non conventional dynamics can arise for not too high money growth (and, thus, not too high equilibrium inflation rates) in extreme cases of information stickiness, significant real rigidities and a small income elasticity of money demand. For instance, if $\lambda = 0.05$ (the lower bound on this parameter that we have assumed earlier and that implied that information must be updated at least every 5 years) and β/α is a very small value, e.g., 0.0005, then the bifurcation occurs at $\Delta m = 0.027$, i.e., for a rate of money growth such that the equilibrium inflation rate is 2.7% per quarter or 10.73% per year.

The main intuition one withdraws from the previous arguments is that we can exclude any departure from stability as long as:

a) The economy follows a responsible monetary policy;

b) Agents are not excessively inattentive;

c) The degree of real rigidities is not too strong when compared with the income change occurring as a reaction to a variation in money demand.

If one of the above conditions fails significantly, even if the others remain at relatively admissible levels, the steady-state inflation rate may become unattainable; the failure to converge to the inflation rate equilibrium level may take various forms and involve different consequences, as we will discuss in the next sections.

6 Endogenous Cycles

In the previous section, one has remarked that the introduction of learning will not change the perfect foresight outcome of convergence towards the monetary steady state, as long as economic conditions remain 'normal'. By a normal economic environment we understand a setting where the rate of money growth is not excessively high, the degree of inattentiveness is kept at a relatively low value, and the degree of real rigidities is relatively low when compared with the income elasticity of money demand.

In this section, we look at the consequences over global dynamics of considering such a learning environment for the formation of expectations. First, the stable outcome is addressed in terms of velocity of convergence (we compare the obtained results to the ones found in the perfect foresight case through figure 1); second, we approach the exception, i.e., what are the implications of extreme economic conditions that lead to a departure from the scenario of convergence towards the monetary steady-state. In this second case, we find that the flip bifurcation generates an area (the region locally characterized by the saddle-path outcome) where endogenous cycles emerge; these cycles are in some circumstances simple two-period cycles and, for other combinations of parameter values, quasi-periodic and chaotic fluctuations. The existence of endogenous cycles in situations where the rate of money growth is excessively high constitutes a candidate explanation for why business cycles become harder to control and predict in scenarios of hyperinflation (besides the stochastic component of such fluctuations, a deterministic fluctuation process is also highlighted).

Consider as a benchmark case of stability under learning the one referred in section 4 ($\alpha = 0.1$, $\beta = 0.5$, $\lambda = 0.25$). To draw a figure similar to figure 1, consider some initial inflation rate value and a long run rate of money growth: $\pi_0 = 0.04$ and $\Delta m = 0.02$. Besides $\lambda = 0.25$, take also the two alternative cases $\lambda = 0.5$ and $\lambda = 0.75$. Figure 4 presents the process of convergence of the inflation rate towards the equilibrium point in the learning model. By comparing this figure with figure 1, no relevant differences are found; the degree of inattentiveness continues to determine the velocity of convergence towards the steady-state: the stickier the information is (the lengthier is the process of expectations adjustment), the slower will be the convergence in the direction of Δm, i.e., in the direction of the inflation rate level that prevails in the long run. Thus, although other parameters become relevant once we introduce learning (namely the ratio β/α), the degree of information stickiness continues to mainly determine the pace of convergence towards the steady-state.

Recover now the exceptional cases in which the flip bifurcation is crossed. The numerical - graphical evaluation of the system allows to identify circumstances in which the bifurcation simply leads to period two cycles that are perpetuated over time and cases for which complete a-periodicity and chaos prevail. Table 1 indicates, for each considered combination of parameters, the maximum periodicity cycles that may be found in each case. Note that there are also cases in which the bifurcation directs the system immediately to an unstable result, existing in these cases no cyclical motion.

An important point to remark at this stage is that initial conditions (in particular, the initial level of inflation) are not relevant in what respects the long-term dynamic outcome. Every numerical experiment allows to find a basin of attraction that is coincident with the state space; this means that the long-run result is independent of π_0.

The presence of chaotic motion for particular parameter values can be confirmed by computing the corresponding Lyapunov characteristic exponents

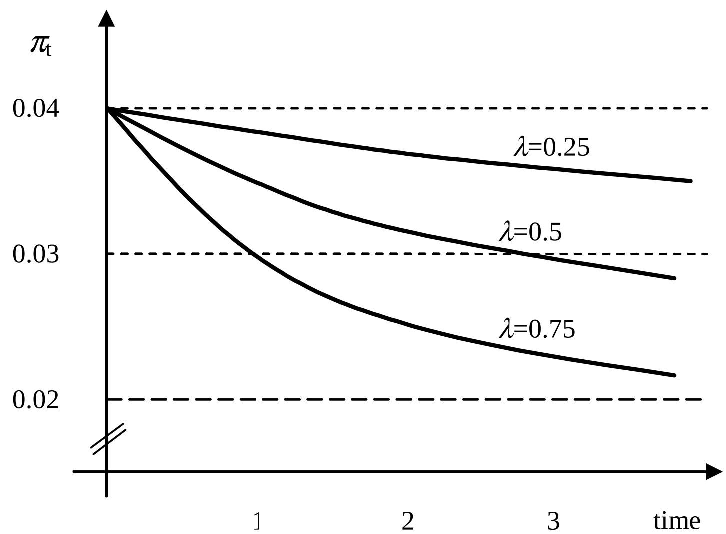

Figure 4: Inflation stability dynamics (with learning).

λ	β/α	Maximum periodicity cycles	λ	β/α	Maximum periodicity cycles
0.5	0.1	Period 2	0.25	0.3	Period 2
0.5	0.2	Period 2	0.25	0.4	Period 2
0.333	0.1	Chaos (e.g. $\Delta m = 1$)	0.125	0.1	No cycles - instability
0.333	0.2	Period 2	0.125	0.2	Chaos (e.g. $\Delta m = 0.37$)
0.333	0.3	Period 2	0.125	0.3	Period 2
0.25	0.1	Chaos (e.g. $\Delta m = 0.35$)	0.125	0.4	Period 2
0.25	0.2	Chaos (e.g. $\Delta m = 1$)			

Table 1: Maximum periodicity cycles found after the bifurcation is crossed (for selected values of parameters).

(LCEs); these are a measure of exponential divergence of nearby orbits or sensitive dependence on initial conditions (SDIC). SDIC implies that the same dynamic process with distinct initial values (even if very close) will generate completely different long term time trajectories, characterized by irregular cycles that do not converge to any steady-state or periodic orbit. Thus, SDIC is commonly associated with the notion of topological chaos.

In a three dimensional system as the one considered, three LCEs exist. In practice, only the value of the largest LCE is relevant: if this is negative, a result other than chaos characterizes long term dynamics; if it is positive, we confirm the presence of SDIC. Take, for instance, two of the cases involving chaos, presented in table 1: $\lambda = 1/3$, $\beta/\alpha = 0.1$, $\Delta m = 1$ and $\lambda = 0.25$, $\beta/\alpha = 0.1$, $\Delta m = 0.35$. For these, one effectively encounters positive largest Lyapunov exponents (respectively, $LCE = 0.077$ and $LCE = 0.031$). All the cases for which chaos is not found will involve solely negative Lyapunov exponents.

A graphical illustration of the nonlinear results allows to better understand how the learning assumption may decisively deviate the economy's outcome from the one suggested by a world where perfect foresight prevails. In figure 5, a bifurcation diagram is represented for a case where chaotic motion is identified; this is the case in which $\lambda = 1/3$, $\beta/\alpha = 0.1$. In this specific setting, the bifurcation occurs for $\Delta m = 0.41$; above this value of the rate of money growth, a period-two cycle is first generated and then it leads to a region in which quasi-periodicity and then chaos will prevail. Figure 6 displays a strange attractor (the long term relation between the inflation level and the learned rate of inflation for the selected parameter values and for $\Delta m = 1$), relatively to which the presence of chaotic motion is unequivocal.[3]

It is relevant the kind of nonlinearity one finds when evaluating the global dynamics of a system that involves a given learning algorithm, as it is the case. Our learning specification follows closely, as referred earlier, the work by Bullard [1] who studies the implications of introducing learning into a standard overlapping generations (OLG) model with a fixed point perfect foresight steady-state (this is the so called monetary steady-state). The referred work, further discussed in Schonhofer [20] and Tuinstra and Wagener [22], raises a relevant question: although learning schemes may provide a useful explanation for the existence of a rational expectations equilibrium, the nonlinearities introduced by the mechanism of learning (e.g., least squares) may imply the endogenous generation of complex long term dynamics. Forecasting errors will eventually never vanish and attractors different from the steady-state equilibrium might emerge. Introducing learning generates, in the referred studies as well as in the sticky information case we have considered above, long run endogenous cycles which have gained the designation of learning equilibria; their main feature is that they depend entirely on fundamental factors, i.e., it is not a third variable or an ad-hoc assumption that forces the presence of nonlinearities. In other words, learning alone produces multiple equilibria.

Using the same analytical structure and employing the same least squares learning scheme as Bullard [1], Schonhofer [20] focuses the analysis on chaotic equilibrium trajectories. The finding of chaos in this kind of model gives a

[3] Figures 5, 6, 11 and 12 are drawn using iDMC (interactive Dynamical Model Calculator). This is a free software program available at www.dss.uniud.it/nonlinear, and copyright of Marji Lines and Alfredo Medio.

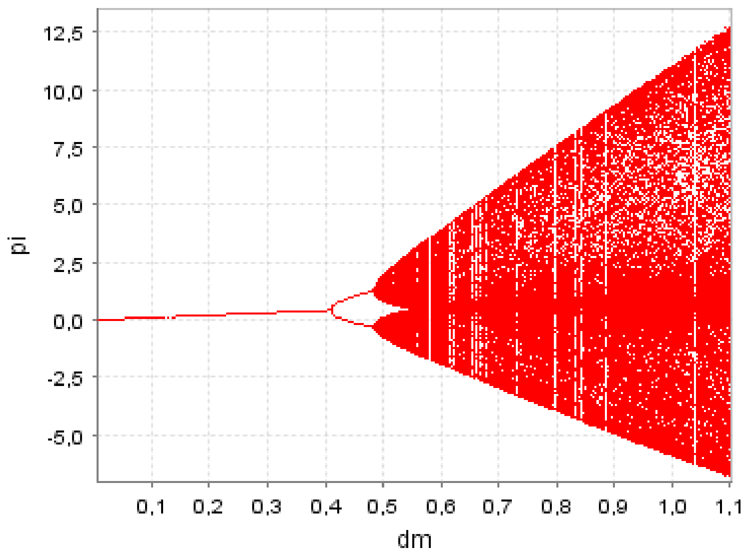

Figure 5: Bifurcation diagram ($\lambda = 1/3, \beta/\alpha = 0.1$).

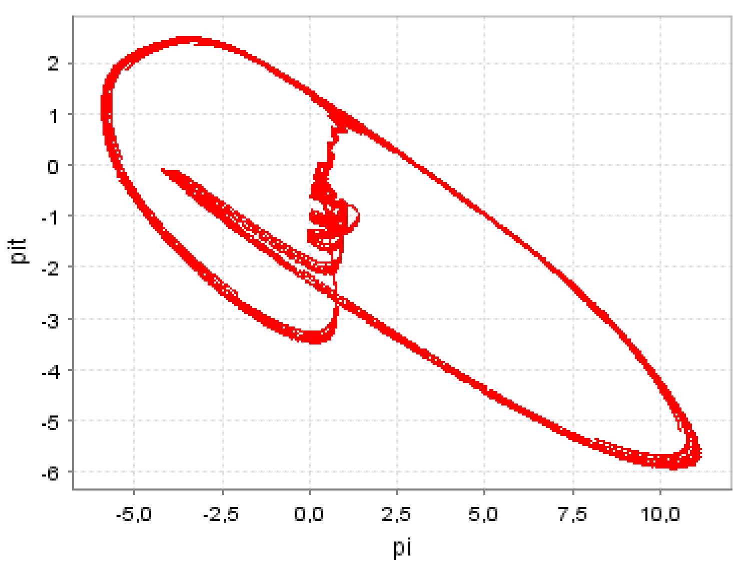

Figure 6: Chaotic attractor ($\lambda = 1/3, \beta/\alpha = 0.1, \Delta m = 1$).

better reply to the persistent use of a learning scheme that generates systematic errors. Under a simple periodic cycles result it would be straightforward to perceive that systematic errors are being made, what would lead the representative agent to change her forecasting rule. In the presence of deterministic chaos, the persistent use of the same inefficient rule may be perpetuated as agents are unable to distinguish between irregular deterministic behavior and random deviations from the fundamental solution. This justifies the relevance we put in finding, for some combinations of parameter values, chaotic solutions. Schonhofer [21] raises precisely the question of whether agents have the ability to learn their way out of chaos and Grandmont [6] discusses the concept of 'self-fulfilling mistake' which is directly attached to the idea that agents are unable to distinguish between chaotic deterministic behavior and random noise, and tend to simplify reality by considering that every fluctuation has a stochastic nature.

7 Response to Shocks

The main goal with which the sticky-information paradigm was born was to measure the impact of external shocks over the equilibrium inflation rate. In Mankiw and Reis [8], four different types of shocks and the corresponding effects over the long-run inflation trajectory are evaluated. In each one of the specified scenarios, information stickiness will imply an inertial response to disturbances on aggregate demand (or on money supply), i.e., the maximum impact of the shock is felt only after a few time periods (8 to 10 quarters); once the maximum impact is reached, the inflation rate will gradually converge once again to the steady-state value, that in this type of exercise is assumed to be $\pi^* = 0$, for simplifying purposes.

To illustrate how learning changes the response to monetary shocks, we consider the fourth experience described in Mankiw and Reis [8]. The variation in money supply is modelled as a first-order autoregressive process, i.e., $\Delta m_t = \rho\Delta m_{t-1} + \varepsilon_t$, with $|\rho| < 1$ and ε_t a disturbance term representing eventual perturbations over the money supply growth rate. We consider $\rho = 0.5$ and the shock is modelled as $\varepsilon_{t=0} = -0.007$ ($\varepsilon_{t\neq 0} = 0$), i.e., a negative shock over the evolution of Δm_t occurs at $t = 0$. Under perfect foresight, the inertia effect is evident: the impact of the monetary policy shock reveals its maximum strength only some time after the disturbance takes place; this negative effect vanishes after around 20 periods. The policy experiment is conducted for $\alpha = 0.1$, $\beta = 1$ and $\lambda = 0.25$.

We apply precisely the same policy experiment to our learning setup (using the same parameter values); no significant qualitative differences are found: the shock disturbs the inflation rate path negatively, the effect involves some inertia and, again, after a few periods the impact of the shock begins to vanish. The main differences are that with learning the impact of the shock is less accentuated (the maximum impact over inflation corresponds to less than a half of the impact in the rational expectations case) and with learning the process of recovery towards the initial state is slower. These results are displayed in figure 7, where we compare the impact of the disturbance for the rational expectations case and for two learning possibilities. Learning model 1 is the case characterized in the previous sections; learning model 2 respects to an alternative specification that is described in appendix C and that involves a more sophisticated learning rule, where rational expectations gradually lose weight and learning gradually gains weight as older time periods are considered.

The evident feature in figure 7 is that under learning the impact of the shock is less pronounced (it will be more pronounced for the learning specification in the text than for the case in appendix C, because the first attributes more weight to learning). We also observe that the larger is the importance of learning (first learning model) the lengthier will be the convergence towards the steady-state; this is apparently an intuitive result because with learning we introduce a second element of inertia: not only information is sticky (i.e., agents are inattentive) but agents will also need time to learn the way economic aggregates evolve over time.

One can also evaluate how a change in the growth of money supply changes inflation paths when nonlinearities are present. Focus the attention on the first learning model and on the example that allowed to build the bifurcation diagram in figure 5. In this case, $\lambda = 1/3$ and $\beta/\alpha = 0.1$. A first bifurcation occurs at $\Delta m = 0.41$ and a second bifurcation, in the transition between a period two-cycle and quasi-periodicity is observed at around $\Delta m = 0.48$. Thus, a one time change in the rate of money growth can have dramatic implications over the observed type of dynamics. This is illustrated in figure 8. In this figure, we begin by considering that $\Delta m = 0.4$, situation in which the system converges towards a unique equilibrium inflation rate; a shock on the money supply growth rate may imply a change on the topological nature of the system, generating a different evolution pattern of inflation over time. In this case, by assuming that at a moment $t = 50$, the growth rate of m_t becomes $\Delta m = 0.5$, one effectively observes a relevant change in the long-run behavior of the inflation rate - now, it evolves a-periodically. As a result, a permanent change in the equilibrium rate of money growth not only changes the average

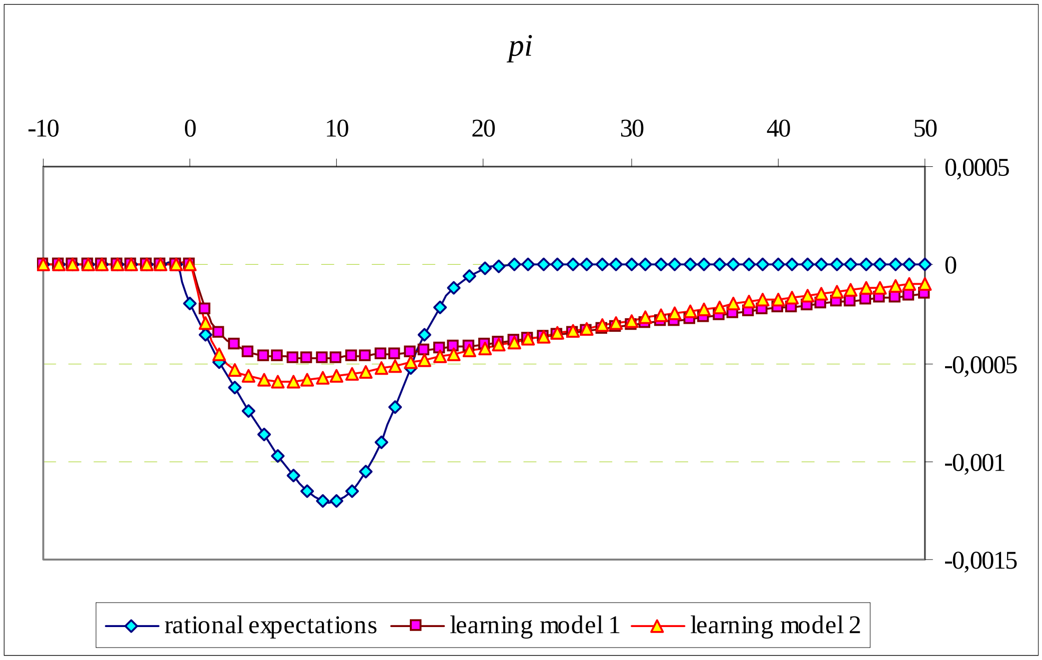

Figure 7: Response to a monetary shock.

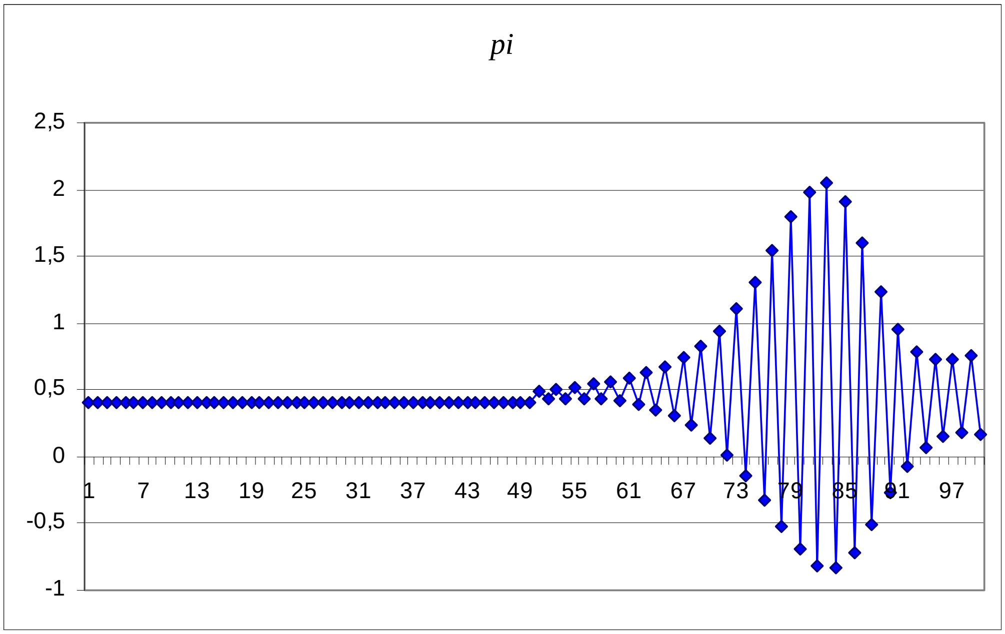

Figure 8: Inflation dynamics with a monetary shock occurring at a bifurcation point.

inflation rate, it also modifies the qualitative behavior of the time path of inflation: from a fixed-point stability result $\pi^* = 0.4$, the system has jumped to an irregular movement around the new steady-state $\pi^* = 0.5$. Therefore, by allowing for the possibility of stronger rates of money growth, the central bank may be opening the door for an ineffective monetary policy: although the rate of money growth remains constant over time, the inflation rate will no longer evolve following a pattern of stability.

8 Conclusion

The question asked in this paper is whether the departure from the perfect foresight assumption in a deterministic version of the sticky information macro model produces relevant changes in terms of the obtained stability results. In the assumed setting, if expectations about the value of the inflation rate today are generated today, then perfect foresight prevails; if the update of information is made in any moment of the past, such information about the state of the economy will be simply ignored in favour of resorting to the known past performance of the inflation rate, which allows to predict the current value.[4]

The adopted least squares learning algorithm through which inflation is (partially) forecasted discloses a relevant role of the degree of sticky information in determining stability. While under perfect foresight the degree of information updating only influences the velocity of convergence towards the steady-state, the inclusion of learning allows for the loss of stability when the information updating process is strongly sluggish. Other parameters also determine the stability outcome: if the extent of real rigidities (e.g., externalities or financial market imperfections) is large (relatively to the income elasticity of money demand), convergence towards the monetary steady-state may as well be compromised. One has also remarked that divergence from the equilibrium only occurs for relatively large rates of growth of money supply, indicating an association, which appears elsewhere in the literature, between nonlinearities and cases of hyperinflation.

[4] A second, more elaborated, setting is considered in appendix: in this, expectations about the current inflation level will always include a component of perfect foresight relating to previous periods' information updating about the state of the economy; however, the weight of this component will diminish as we consider further away in the past time moments. As perfect foresight loses weight, agents will use progressively more a learning rule that allows to forecast current inflation resorting solely to past information regarding this variable - the two specifications allow for similar results from a qualitative point of view, although in the first setting the loss of stability is possible for a wider range of parameter combinations, because we attribute a stronger role to learning.

The studied dynamic system allows for a loss of stability only through a flip bifurcation. This leads in most of the circumstances to cycles of periodicity 2, which can degenerate in some of the cases into quasi-periodicity and chaos. Chaotic results are particularly relevant because they possibly imply the presence of systematic errors that the agents cannot separate from random noise. Imagining that cycles are identifiable with stochastic processes, agents will have no incentive to switch to another, better performing, learning algorithm (which in the present case would mean to estimate directly the inflation rate and not the dynamic behavior of prices).

The analysis of the reaction of inflation to monetary policy shocks indicates that learning produces a less severe effect of disturbances over price changes, although such effect is felt over a wider range of time periods. Furthermore, monetary policy shocks gain a deeper relevance when nonlinearities are observed, since such shocks may change the qualitative nature of the results allowing a fixed-point outcome to be transformed into an irregular path or vice-versa. This last feature has important implications concerning public policy; if the monetary authorities are able to perceive that the main source of fluctuations is endogenous, then they may be able to identify in what extent the rate of money growth needs to be changed in order to stabilize the evolution of prices.

Appendix

Appendix A - Derivation of Equation (4)

Following Mankiw and Reis [9], we take the simplest possible assumption concerning monetary policy. Namely, we consider a money market equilibrium condition of the form

$$m_t - p_t = \beta y_t \tag{a1}$$

Variable m_t is the money supply, i.e., it is the policy instrument to which public authorities can resort to. Parameter $\beta > 0$ is the income elasticity of money demand.

Monetary policy will, then, be modelled by simply considering a constant rate of money growth, i.e., $m_{t+1} - m_t = \Delta m > 0$, with m_0 given.

Replacing $(a1)$ into (3) in order to remove the output gap and its growth rate from the expression of the sticky-information Phillips curve, the following relation is obtained:

$$\pi_t = \frac{\alpha\lambda}{\beta}(m_{t-1} - p_{t-1}) + \frac{\beta - \alpha}{\alpha}\lambda\sum_{j=0}^{\infty}(1-\lambda)^j E_{t-j}(\pi_t) + \frac{\alpha}{\beta}\Delta m \qquad (a2)$$

To present $(a2)$, one has kept in mind that $\sum_{j=0}^{\infty}(1-\lambda)^j = 1/\lambda$. Equation $(a2)$ establishes a relation between prices and money supply; this relation becomes easier to analyze if one applies first-differences. In this way, the relation will simply involve the constant rate of money growth and the value of the inflation rate in two consecutive time moments. Straightforward algebra will conduct to equation (4) as presented in the text.

Appendix B - Details on Monetary Policy

The paper has considered the most simple approach to monetary policy. This was done for the sake of tractability, however the model can be easily extended in order to include a more realistic characterization of monetary policy. In this appendix, we briefly explain how monetary policy decisions could be endogenously modelled, by assuming that the nominal interest rate is an endogenous variable that the central bank may control in order to influence the macroeconomic outcome, both in real and nominal terms.

Consider a static and deterministic IS relation[5]:

$$y_t = \varphi_0 - \varphi_1(i_t - \pi_t), \quad \varphi_0, \varphi_1 > 0 \qquad (a3)$$

where i_t represents the nominal interest rate. The IS equation presents the conventional relation of opposite sign between the real interest rate and the output gap. The interest rate can be determined optimally, given some objective function of the monetary authority or, alternatively, a Taylor rule can be applied. Such rule can take the form:

$$i_t = i^* + \gamma_\pi \pi_t + \gamma_y y_t, \quad \gamma_\pi, \gamma_y > 0 \qquad (a4)$$

[5] If one intended to further sophisticate the presentation, the IS could be transformed into a dynamic relation, by assuming an expected inflation rate rather than observed inflation at moment t. Since one just intends to illustrate how monetary policy could be given a more relevant role in the analysis, we simply assume a static conventional relation between the output gap and the real interest rate.

Parameter i^* represents the equilibrium nominal interest rate. The Taylor rule describes how the central bank will react to changes in the state variables, i.e., to changes in the values of the inflation rate and of the output gap.[6] By replacing the interest rate in $(a4)$ into $(a3)$, we find the following relation between the output gap and the inflation rate:

$$y_t = \frac{\varphi_0 - \varphi_1 i^* + \varphi_1(1-\gamma_\pi)\pi_t}{1+\varphi_1\gamma_y} \qquad (a5)$$

If one replaces the value of the output gap as presented above into the sticky-information Phillips curve, (3), we will obtain a dynamic equation with a single endogenous variable - the inflation rate. That equation could be analyzed under learning in the same way one has proceeded with equation (4). The main difference is that in such case one would not have the change in money supply as the policy instrument; the parameters subject to the eventual manipulation of the authorities would be the ones we have just specified: $\varphi_0, \varphi_1, i^*, \gamma_\pi$ and γ_y. Thus, the option for such a non elaborated specification of monetary policy has served the purpose of analyzing inflation dynamics having in mind a single bifurcation parameter that comprises the whole of the monetary policy actions that can be followed by the central bank.

Appendix C - An Alternative Specification

In this appendix, we take a different assumption in terms of formation of expectations relatively to the one in section 4; the goal is to check the robustness of the model's results by analyzing a more sophisticated rule of inflation expectations; results will be similar in qualitative terms, although they will be closer to the ones in the original model, given that a less relevant role is attributed to the learning component in the formation of expectations. Instead of considering learning for every time period previous to t, we will assume that the capacity to perfectly predict today's inflation rate fades away as we consider more distant in the past time moments. The assumption is analytically translated in the following expectations formation rule:

$$E_{t-j}(\pi_t) = \frac{1}{1+j}\pi_t + \frac{j}{1+j}\widehat{\pi}_t \qquad (a6)$$

[6] As with the IS equation, the Taylor rule can take slightly different forms. For instance, the interest rate may react not to the observed inflation rate but to the expected inflation rate (or to the difference between the expected inflation and an inflation rate target defined by the central bank).

As before, $\widehat{\pi}_t$ respects to the estimated level of inflation at t computed by resorting to all the information on π through $t-1$. The intuition is as follows: when $j=0$, expectations about inflation are formed in the present moment, and therefore agents know with absolute certainty that the inflation rate is π_t; in the opposite circumstance, when $j\to\infty$, information about the state of the economy was updated so far in the past that agents will display no ability to generate accurate perfect foresight expectations on π_t; thus, they will fully resort to the learning procedure in order to predict inflation, i.e., $\lim_{N\to\infty} E_{t-N}\left(\pi_t\right)=\widehat{\pi}_t$. Between the extreme time moments t and $t-N$, expectations will be formed by weighting both the information on the state of the economy and the time series of the inflation rate. The weight of the perfect foresight component falls the larger is the distance of the generation of information period relatively to the present moment. Note that the sum of the weights in $(a6)$ is 1, meaning that the relative increase of weight of one of the expectation rules is made at the expenses of the other: as the ability to produce perfect foresight expectations falls, the larger will be the relevance agents attribute to learning. This departs from the first specification in the sense that then perfect foresight was possible only in the present moment and it would be absent from the formation of expectations in any past moment, while now such ability will fade away in a progressive way as we go back in time.

Recover the sticky-information term of the inflation dynamic equation in (4). Replacing in such term the inflation expectations by the expression in $(a6)$ yields the following equality:

$$\lambda\sum_{j=0}^{\infty}(1-\lambda)^j E_{t-j}\left(\pi_t\right)=\lambda\pi_t\sum_{j=0}^{\infty}\frac{(1-\lambda)^j}{1+j}+\lambda\widehat{\pi}_t\sum_{j=0}^{\infty}\frac{j(1-\lambda)^j}{1+j}$$

The following (exact) series results hold,

$$\sum_{j=0}^{\infty}\frac{(1-\lambda)^j}{1+j}=-\frac{\ln\lambda}{1-\lambda}$$

$$\sum_{j=0}^{\infty}\frac{j(1-\lambda)^j}{1+j}=\frac{1}{\lambda}+\frac{\ln\lambda}{1-\lambda}$$

Thus, we further simplify the past expectations term,

$$\lambda\sum_{j=0}^{\infty}(1-\lambda)^j E_{t-j}\left(\pi_t\right)=\widehat{\pi}_t-\frac{\lambda\ln\lambda}{1-\lambda}(\pi_t-\widehat{\pi}_t) \qquad (a7)$$

To better understand the relation between the degree of information stickiness and the series of past expectations, we rearrange $(a7)$ to write the equality as

$$\frac{\lambda \sum_{j=0}^{\infty}(1-\lambda)^j E_{t-j}(\pi_t) - \widehat{\pi}_t}{\pi_t - \widehat{\pi}_t} = -\frac{\lambda \ln \lambda}{1-\lambda} \qquad (a8)$$

Expression $(a8)$ allows to characterize how different degrees of stickiness lead to a departure from the perfect foresight outcome. Consider precisely the perfect foresight benchmark case. In this situation, the l.h.s. of $(a8)$ is equal to 1; such result arises only in the instantaneous information updating case, i.e., for $\lambda = 1$. The value of the expression in $(a8)$ will fall below 1 for $\lambda < 1$, with such value being as much lower as the lower is the value of the information

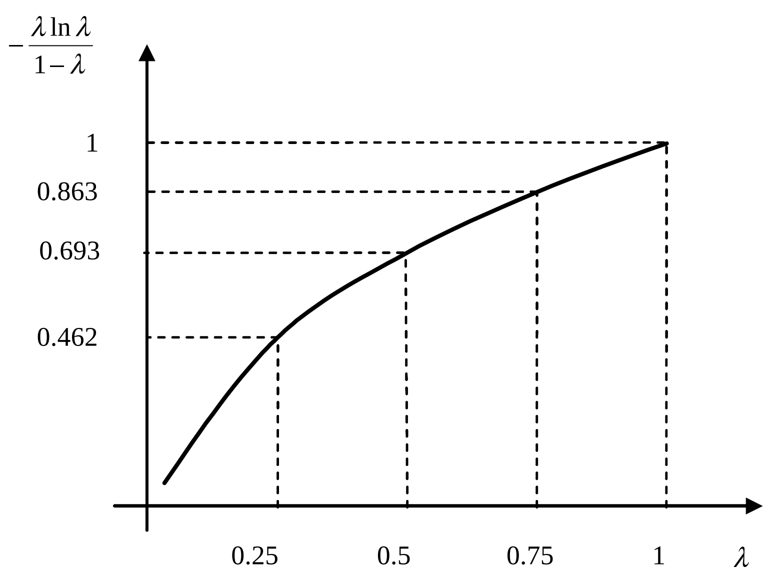

Figure 9: Information stickiness and expectations under learning (2^{nd} model).

stickiness parameter. Figure 9 represents function $f(\lambda) = -\frac{\lambda \ln \lambda}{1-\lambda}$; this is an increasing and concave function where we effectively observe that $f(1) = 1$. Therefore, one concludes that the larger is the degree of information stickiness, the more the ratio in the l.h.s. of $(a8)$ will depart from 1, and this essentially means that the larger is the degree of information stickiness, the more the series $\lambda \sum_{j=0}^{\infty}(1-\lambda)^j E_{t-j}(\pi_t)$ will depart from the perfect foresight outcome π_t.

Replacing expectations, as presented in $(a7)$, into the inflation dynamics equation (4), one obtains:

$$\begin{aligned}\pi_{t+1} \quad = \quad & \frac{1}{(\beta/\alpha)(1-\lambda+\lambda\ln\lambda)-\lambda\ln\lambda}\cdot \\ & \{\lambda(1-\lambda)\Delta m+[(\beta/\alpha)(1-\lambda+\lambda\ln\lambda)-\lambda(1-\lambda+\ln\lambda)]\,\pi_t \\ & +(\beta/\alpha-1)(1-\lambda+\lambda\ln\lambda)(\widehat{\pi}_{t+1}-\widehat{\pi}_t)\} \qquad (a9)\end{aligned}$$

As in the first explored case, taking $\widehat{\pi}_{t+1}=\pi_{t+1}$ and $\widehat{\pi}_t=\pi_t$ would imply returning to the benchmark perfect foresight setting, i.e., the learning component would be eliminated.

Equation $(a9)$ together with (10) and (11), will constitute the new dynamic model relatively to which one intends to address stability properties. We follow the same procedure as in the first model, i.e., we begin by linearizing the system in the steady-state vicinity [the steady-state $(\pi^*,\widehat{\pi}^*,\sigma^*)$ is exactly the same as in the first version of the model],

$$\begin{bmatrix}\widehat{\pi}_{t+1}-\widehat{\pi}\\ \pi_{t+1}-\pi^*\\ \sigma_{t+1}-\sigma^*\end{bmatrix}=\begin{bmatrix}1-\sigma^* & \sigma^* & 0\\ \frac{(1-\lambda)\sigma^*}{\widetilde{\theta}}-\sigma^* & 1+\sigma^*-\frac{(1-\lambda)(\lambda+\sigma^*)}{\widetilde{\theta}} & 0\\ 0 & \frac{2\sigma^*}{[\exp(\Delta m)]^3} & 1-\sigma^*\end{bmatrix}\begin{bmatrix}\widehat{\pi}_t-\widehat{\pi}^*\\ \pi_t-\pi^*\\ \sigma_t-\sigma^*\end{bmatrix} \qquad (a10)$$

with $\widetilde{\theta}:=(\beta/\alpha)(1-\lambda)+(\beta/\alpha-1)\lambda\ln\lambda$.

Again, resorting to the center manifold theorem, which allows to restrict the local analysis to the sub-matrix $\widetilde{J}$ of the Jacobian matrix in expression $(a10)$ composed by the first two rows and the first two columns, one withdraws the main stability properties of the system.

Proposition 3 *If* $\frac{\beta}{\alpha}\geq\frac{(1-\lambda)(2-\lambda)+2\lambda(1-\lambda+2\ln\lambda)}{4(1-\lambda+\lambda\ln\lambda)}$*, then the system is locally stable* $\forall\Delta m>0$*. For a degree of information stickiness such that* $\lambda\geq 0.383$*, stability will hold independently of the values of* Δm*,* α *and* β*.*

Proof. Trace and determinant of $\widetilde{J}$ are, respectively, $Tr(\widetilde{J})=2-\frac{(1-\lambda)\lambda}{\widetilde{\theta}}-\frac{1-\lambda}{\widetilde{\theta}}\sigma^*$ and $Det(\widetilde{J})=1-\frac{(1-\lambda)\lambda}{\widetilde{\theta}}-\frac{(1-\lambda)^2}{\widetilde{\theta}}\sigma^*$.

The stability conditions $1-Det(\widetilde{J})=\frac{1-\lambda}{\widetilde{\theta}}[\lambda+(1-\lambda)\sigma^*]>0$ and $1-Tr(\widetilde{J})+Det(\widetilde{J})=\frac{(1-\lambda)\lambda}{\widetilde{\theta}}\sigma^*>0$ hold, for any admissible combination of parameter values. The only constraint on stability comes from condition

$1+Tr(\widetilde{J})+Det(\widetilde{J})>0$, which requires $\sigma^*<\frac{4\widetilde{\theta}-2(1-\lambda)\lambda}{(1-\lambda)(2-\lambda)}$. The equilibrium value of the gain sequence variable is bounded below 1, and therefore if $\frac{4\widetilde{\theta}-2(1-\lambda)\lambda}{(1-\lambda)(2-\lambda)}\geq 1$, then stability is guaranteed. This last inequality is equivalent to the one presented in the proposition.

Recalling that β/α must be larger than zero, if the r.h.s. of the inequality is non negative one guarantees the presence of stability for any positive rate of money supply; this value is above (or equal to) zero as long as $\lambda \geq 0.383$ ■

The condition $\lambda \geq 0.383$ indicates that introducing learning into the formation of expectations in the way we have just done implies stability of the monetary steady state as long as information is updated at least every 2.611 quarters (1/0.383) or 7.8329 months (3/0.383); more infrequent information updating can lead to a departure from the stability outcome, depending on the values assumed by the degree of real rigidities, the income elasticity of money demand and the growth rate of money supply.

The limit value of λ above which stability is guaranteed is lower in the present case than in the first model's specification. This is intuitively correct, because in the current case we attribute a less significant role to learning (relatively to perfect foresight); therefore, the loss of stability can only occur for relatively larger values of information stickiness (lower values of parameter λ).

If the possibility of loss of stability is present, we can state the following result.

Proposition 4 *Let* $\frac{\beta}{\alpha}<\frac{(1-\lambda)(2-\lambda)+2\lambda(1-\lambda+2\ln\lambda)}{4(1-\lambda+\lambda\ln\lambda)}$. *In this case, local stability will require an upper bound on the rate of money growth. In the point in which* $\Delta m=\ln\sqrt{\frac{(1-\lambda)(2-\lambda)}{(1-\lambda)(2-\lambda)+2\lambda(1-\lambda+2\ln\lambda)-4(\beta/\alpha)(1-\lambda+\lambda\ln\lambda)}}$, *a flip bifurcation occurs.*

Proof. The proof of the proposition is straightforward: consider the inequality found in the previous proposition's proof, $\sigma^*<\frac{4\widetilde{\theta}-2(1-\lambda)\lambda}{(1-\lambda)(2-\lambda)}$; by recalling that $\sigma^*=1-[\exp(\Delta m)]^{-2}$ and by rearranging this inequality, one arrives to:

$$\Delta m<\ln\sqrt{\frac{(1-\lambda)(2-\lambda)}{(1-\lambda)(2-\lambda)+2\lambda(1-\lambda+2\ln\lambda)-4(\beta/\alpha)(1-\lambda+\lambda\ln\lambda)}}$$

Therefore, one realizes that stability requires the money growth rate to be below a given combination of parameters α, β and λ. The inequality in the proposition guarantees that the value inside the square root is positive.

As in the first discussed case, it is condition $1+Tr(\widetilde{J})+Det(\widetilde{J})>0$ that may not hold. In the point at which $1+Tr(\widetilde{J})+Det(\widetilde{J})=0$, a flip bifurcation takes place ■

A contour plot similar to the one in figure 3 can be presented for the case now in appreciation. Figure 10 takes, once again, the values $\Delta m = 0.1$, $\Delta m = 0.333$ and $\Delta m = 1$ to draw the corresponding bifurcation lines. These have a similar pattern relatively to the one characterized in figure 3, but now the conditions for the loss of stability are more strict: a stronger level of information stickiness is required, and the value of the ratio β/α needs to be lower for each value of Δm.

To arrive to the lines in figure 10, one has to rewrite the flip bifurcation condition in proposition 4 in the following way:

$$\frac{\beta}{\alpha} = \frac{\left[(1-\lambda)(2-\lambda) + 2\lambda(1-\lambda+2\ln\lambda)\right]\left[\exp(\Delta m)\right]^2 - (1-\lambda)(2-\lambda)}{4(1-\lambda+\lambda\ln\lambda)\left[\exp(\Delta m)\right]^2}$$

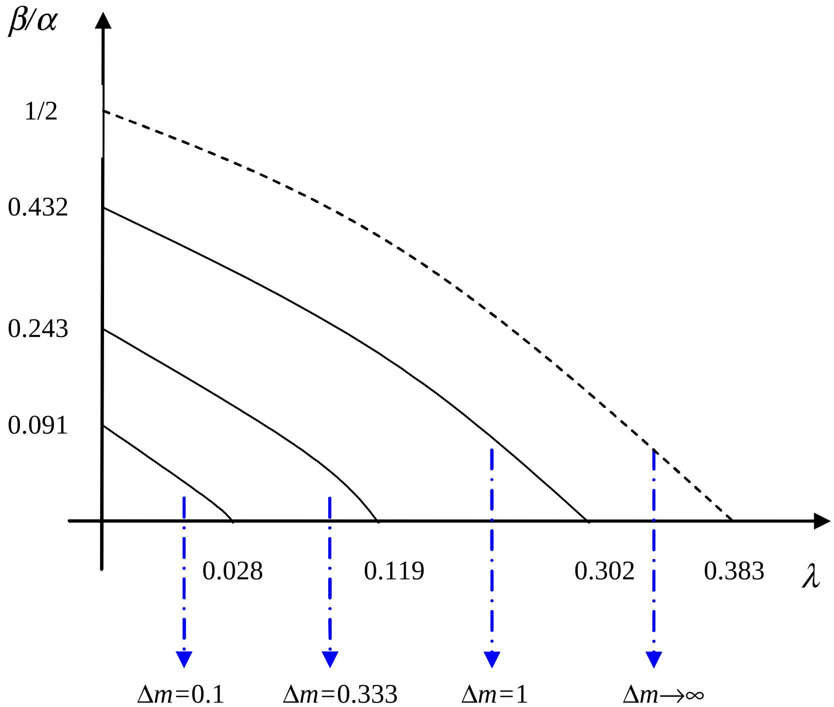

Figure 10: Bifurcation line for different values of Δm (2^{nd} model).

The values of β/α in the limit case $\lambda = 0$ is the same as in the first proposed scenario, but the values of λ for each Δm in the extreme point $\beta/\alpha = 0$ are lower; as a result, larger degrees of inattentiveness are now needed to depart from stability (for the same rate of money growth). As in figure 3, the area below each bifurcation line represents local instability, while stability prevails for combinations of parameter values locating above the represented lines.

Given the qualitative similarities with the case addressed in the body of text, we omit a thorough graphical analysis concerning both local and global dynamics. From a global point of view, we find a similar type of outcomes: under stability, the frequency of information updating is determinant in what respects the velocity of convergence (once more, the smaller is λ, the slower will be the convergence process). The breakdown of stability generates as well a region of cyclical motion. Table 2 indicates, for selected parameter values, the type of dynamics that is installed. As in the benchmark case, period 2 cycles dominate but quasi-periodicity and chaotic motion are possible in some cases, with the relevant consequences we have already mentioned.

Figure 11 presents a bifurcation diagram for a setting where quasi-periodicity is identified, but chaotic motion is absent (irregular cycles are displayed, however they do not represent a situation of complete divergence of nearby orbits); this is the case in which $\lambda = 0.1, \beta/\alpha = 0.15$. In this specific setting, the bifurcation occurs at $\Delta m = 0.552$; above this value of the rate of money growth, a period two cycle is first generated and then it leads to a region in which quasi-periodic cycles will hold. Figure 12 displays an attractor, i.e., the long term relation between the inflation level and the learned rate of inflation for a set of parameter values ($\lambda = 0.1$, $\beta/\alpha = 0.1$, $\Delta m = 2$) relatively to which the presence of chaotic motion is clear.

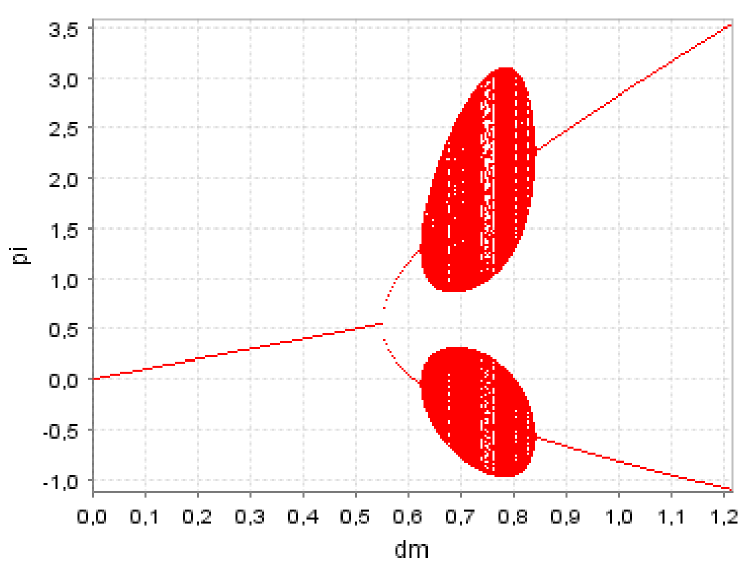

Figure 11: Bifurcation diagram ($\lambda = 0.1$, $\beta/\alpha = 0.15$) – 2^{nd} model.

λ	β/α	Maximum periodicity cycles	λ	β/α	Maximum periodicity cycles
0.35	0.05	Period 2	0.15	0.25	Period 2
0.3	0.05	Period 2	0.1	0.05	No cycles -instability
0.3	0.1	Period 2	0.1	0.1	Chaos (e.g. $\Delta m = 2$)
0.25	0.05	Period 2	0.1	0.15	Quasi-periodicity (e.g. $\Delta m = 0.7$)
0.25	0.1	Period 2	0.1	0.2	Period 2
0.25	0.15	Period 2	0.1	0.25	Period 2
0.2	0.05	Period 2	0.1	0.3	Period 2
0.2	0.1	Period 2	0.05	0.05	No cycles -instability
0.2	0.15	Period 2	0.05	0.1	No cycles -instability
0.2	0.2	Period 2	0.05	0.15	No cycles -instability
0.15	0.05	Chaos (e.g. $\Delta m = 1.8$)	0.05	0.2	Chaos (e.g. $\Delta m = 1.5$)
0.15	0.1	Period 2	0.05	0.25	Period 2
0.15	0.15	Period 2	0.05	0.3	Period 2
0.15	0.2	Period 2	0.05	0.35	Period 2

Table 2: Maximum periodicity cycles found after the bifurcation is crossed (2^{nd} model).

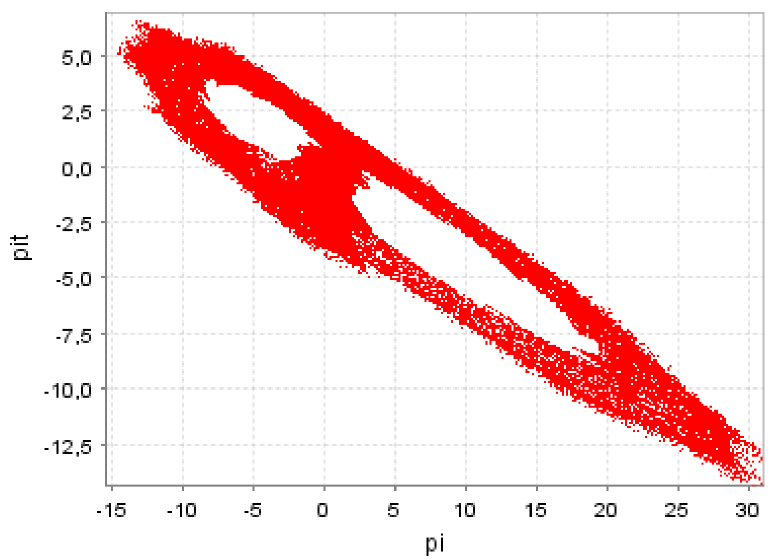

Figure 12: Chaotic attractor ($\lambda = 0.1$, $\beta/\alpha = 0.1$, $\Delta m = 2$) – 2^{nd} model.

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