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ORIGINAL RESEARCH PAPER

Fractal Laplace transform: analyzing fractal curves

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Abstract
The concept of Laplace transform has been extended to fractal curves, enabling the solution of fractal differential equations with constant coefficients. This extension, known as the fractal Laplace transform, is particularly useful for handling inhomogeneous differential equations that involve delta Dirac functions and step functions within the realm of fractal functions. A comprehensive table of essential formulas for the fractal Laplace transform has been compiled to facilitate its application in various scenarios. By utilizing this transformative approach, researchers can now delve into the study of fractal functions and address complex problems involving non-traditional geometries. To illustrate the practicality of the fractal Laplace transform, several examples are provided, showcasing its effectiveness in solving fractal differential equations. This advancement represents a significant augmentation of the classical Laplace transform, tailored to suit the distinctive characteristics of fractal systems and functions.

Keywords Fractal calculus · Fractal Laplace transform · Fractal Dirac function · Fractal curves

Mathematics Subject Classification 28A80 · 44A10

1 Introduction

Fractals, which are recurring patterns found in various natural phenomena such as clouds, mountains, coastlines, blood vessels, and more, have been studied extensively [1]. Fractal geometry is employed to characterize these structures and

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understand their unique properties, such as their measures. Fractals often possess fractional dimensions, exhibit self-similarity, and their fractal dimension surpasses their topological dimension [2]. Researchers have explored the analysis of fractals using diverse approaches, including measure theory, harmonic analysis, fractional space, fractional calculus, and stochastic processes [3–12]. The door to understanding the recently developed area of analysis on fractals is opened by the exploration of Laplacians on the Sierpinski gasket and related fractals [8, 11].

In the context of rough spaces represented by fractals, ordinary calculus may not suffice. Therefore, researchers have explored the extension of differentiation and integration to fractal supports with regular or multifractal properties. Differentiation of functions with respect to finite atomless measures on fractals and the related harmonic calculus, similar to that of the classical Lebesgue case, have been introduced [3, 13].

The classical concepts have been extended, and definitions of derivatives and integrals for fractal cases have been proposed. The fractal derivability of variational functions has been explored to obtain a Newton-Leibniz type formula under certain properties and natural conditions [14].

Riemann-type integrals, including the s-Riemann integral, the s-HK integral, and the s-first-return integral, have been suggested. The Fundamental Theorem of Calculus in the context of fractal calculus has been derived. The relationship between these integrals and the Lebesgue integral with respect to the Hausdorff measure has also been explored, providing a characterization of the primitives of the s-HK integral [15–17].

To accommodate functions with fractal support, a generalized ordinary calculus has been established, which is algorithmic, geometric, and physically meaningful. This generalization enables the study of functions with fractal shapes like Koch curves and Cantor sets [18–21]. The concept of fractal local derivatives has been utilized to study sub-diffusion and super-diffusion, maintaining locality and the central limit theorem [22, 23]. Moreover, non-local fractal calculus has been proposed to model incompressible viscous fluids in fractal media and processes with memory, analogous to how Riemann-Liouville and Caputo generalizations extended ordinary local calculus [24–26].

Fractal calculus has found applications in physics, yielding new models in the context of fractal space and time. These models often exhibit power law and self-similar solutions [25, 27–33]. Additionally, the calculus has been used to derive derivatives and integrals of functions like the Weierstrass function [34]. Fractal Laplace, Sumudu, and Fourier transforms on Cantor sets have been defined and applied to model system dynamics with fractal times [35–38].

By utilizing nonstandard analysis encompassing hyperreal and hyperinteger numbers, the framework of left and right limits as well as derivatives was established on fractal curves. Furthermore, the concept of fractal integral and differential forms was defined through the application of nonstandard analysis [39].

To expand the scope of fractal calculus, generalizations have been made to include unbounded functions through the use of gauge functions [40]. The stability of fractal differential equations has been explored, providing conditions for unique and stable solutions [41]. Furthermore, Tsallis entropy on fractal sets has been defined, with its parameter related to the fractal dimension of the Hadron system [42]. In the pursuit of research in this direction, this paper aims to introduce a novel concept known as the “fractal Laplace transform,” which focuses on applying Laplace transforms to functions that have support on fractal curves.

The outline of the paper includes sections on fractal calculus on curves, the definition of the Laplace transform on fractal curves, and the inverse fractal Laplace transform and fractal convolution. The conclusion summarizes the findings and implications of the study.

2 Preliminaries

In this section, we provide a concise summary of the fractal calculus applied to fractal curves, drawing insights from various works such as [18, 19, 21].

2.1 Fractal calculus on fractal curve

In this section, we introduce fractal calculus on fractal curves, including the concepts of fractal dimension, mass function, staircase function, and fractal continuity. We also define fractal derivatives and integrals for functions on fractal curves [18, 19, 21].

Definition 1 A fractal curve $\mathbf{F} \subset \mathbb{R}^n$ is said to be “continuously parameterizable” if there exists a function $\mathbf{u}$ such that

$$\mathbf{u} : [a_1, b_1] \to \mathbf{F}, \tag{1}$$

where $[a_1, b_1] \subset \mathbb{R}$, and $\mathbf{u}$ is continuous, one-to-one, and onto $\mathbf{F}$.

Definition 2 We define a “subdivision” $\mathbf{Q}[a, b]$ of $[a, b] \subset [a_1, b_1]$ as a set of points:

$$\mathbf{Q}[a, b] = \{a = t_0, t_1, \ldots, t_n = b\}, \tag{2}$$

where each interval $[t_i, t_{i+1}]$ is a component of the subdivision $\mathbf{Q}_{[a,b]}$. If $\mathbf{Q} \subset \mathbf{P}$, then we say $\mathbf{P}$ is a refinement of $\mathbf{Q}$.

Definition 3 The quantity $\mathfrak{M}^\alpha[\mathbf{F}, \mathbf{Q}]$ for a fractal curve $\mathbf{F}$ and a subdivision $\mathbf{Q}[a, b]$ can be defined as follows:

$$\mathfrak{M}^\alpha[\mathbf{F}, \mathbf{Q}] = \sum_{j=0}^{m} \frac{|\mathbf{u}(t_j) - \mathbf{u}(t_{j-1})|^\alpha}{\Gamma(\alpha + 1)}, \tag{3}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. This expression quantifies a measure related to the subdivision and the curve $\mathbf{F}$ in a specific way using the parameter α.

Definition 4 The coarse-grained mass $\mathcal{M}_\delta^\alpha(\mathbf{F}, a, b)$, with a given δ, is defined as follows:

$$\mathcal{M}_\delta^\alpha(\mathbf{F}, a, b) = \inf_{\{\mathbf{Q}_{[a,b]} : |\mathbf{Q}| \leq \delta\}} \mathfrak{M}^\alpha[\mathbf{F}, \mathbf{Q}], \tag{4}$$

where $|\mathbf{Q}| = \max_{0 \leq j \leq m}(t_j - t_{j-1})$ for the subdivision $\mathbf{Q}_{[a,b]}$. This definition provides a way to compute the coarse-grained mass of the fractal curve **F** within the interval $[a,b]$ by considering various subdivisions and finding the infimum value based on the parameter α.

Definition 5 The mass function $\mathcal{M}^{\alpha}(\mathbf{F},a,b)$ is defined as the limit of the coarse-grained mass $\mathcal{M}^{\alpha}_{\delta}(\mathbf{F},a,b)$ as δ approaches zero:

$$\mathcal{M}^{\alpha}(\mathbf{F},a,b) = \lim_{\delta\to 0}\mathcal{M}^{\alpha}_{\delta}(\mathbf{F},a,b). \tag{5}$$

Since $\mathcal{M}^{\alpha}_{\delta}(\mathbf{F},a,b)$ is a monotonic function of δ, the limit exists. This mass function provides a measure of the "size" or "content" of the fractal curve **F** within the interval $[a,b]$, as quantified by the parameter α.

Definition 6 The fractal dimension of the curve **F**, denoted by $dim_{\gamma}(\mathbf{F})$, is defined as follows [19, 21]:

$$\begin{aligned} dim_{\gamma}(\mathbf{F}) &= \inf\{\alpha : \mathcal{M}^{\alpha}(\mathbf{F},a,b) = 0\} \\ &= \sup\{\alpha : \mathcal{M}^{\alpha}(\mathbf{F},a,b) = \infty\}. \end{aligned} \tag{6}$$

This definition establishes the fractal dimension of **F** as the infimum value of α for which the mass function $\mathcal{M}^{\alpha}(\mathbf{F},a,b)$ becomes zero, and the supremum value of α for which it becomes infinite. The fractal dimension is a measure of how the curve **F** fills the space, indicating its self-similarity and complexity.

Definition 7 The staircase function $S^{\alpha}_{\mathbf{F}} : [a_1,b_1] \to \mathbb{R}$ associated with the fractal curve **F** is defined as follows:

$$S^{\alpha}_{\mathbf{F}}(v) = \begin{cases} \mathcal{M}^{\alpha}(\mathbf{F},p_0,v), & v \geq p_0 \\ -\mathcal{M}^{\alpha}(\mathbf{F},v,p_0), & v < p_0, \end{cases} \tag{7}$$

Here, v belongs to the interval $[a_1,b_1]$, which is a subset of the interval $[a,b]$. The function $\mathbf{u}(v)$ maps v to a point ϑ on the fractal curve **F**, and $S^{\alpha}_{\mathbf{F}}(v) = J(\vartheta)$ corresponds to the value of $J(\vartheta)$, where $J(\vartheta)$ represents the fractal dimension of the portion of **F** from the starting point p_0 to ϑ.

The function $S^{\alpha}_{F}(v)$ is strictly increasing, making it invertible. Therefore, we can express $J(\vartheta)$ in terms of v through the inverse function $S^{\alpha}_{\mathbf{F}}(\mathbf{u}^{-1}(\vartheta))$. This relationship provides a one-to-one correspondence between the values of v and the corresponding fractal dimensions $J(\vartheta)$ along the fractal curve **F**.

Remark 1 We observe that there exist constants c_1 and c_2 such that:

$$c_1[\mathbf{L}(\mathbf{u}(v))]^{\alpha} < S^{\alpha}_{\mathbf{F}}(v) < c_2[\mathbf{L}(\mathbf{u}(v))]^{\alpha} \tag{8}$$

where $\mathbf{L}(\vartheta) = \mathbf{L}(\mathbf{u}(v)) = |\mathbf{u}(v)|$. Consequently, we can deduce that:

$$S^{\alpha}_{\mathbf{F}}(v) \sim [\mathbf{L}(\mathbf{u}(v))]^{\alpha} \quad or \quad S^{\alpha}_{\mathbf{F}} \sim \mathbf{L}^{\alpha} \tag{9}$$

This implies that the behavior of the staircase function $S^{\alpha}_{\mathbf{F}}(v)$ is proportional to the fractal dimension $\mathbf{L}(\mathbf{u}(v))$ raised to the power of α. In other words, the fractal dimension of the portion of **F** from the starting point p_0 to ϑ has a direct influence on the values of the staircase function.

Definition 8 We define the limit of a function $f : \mathbf{F} \rightarrow \mathbb{R}$ as follows: The limit of f through points of **F** is denoted by l, and it is attained when for any given $\epsilon > 0$, there exists a $\delta > 0$ such that for all $\vartheta' \in \mathbf{F}$, if $|\vartheta' - \vartheta| < \delta$, then $|f(\vartheta') - l| < \varepsilon$. In mathematical notation, this is expressed as:

$$l = \mathbf{F}\_\underset{\vartheta' \rightarrow \vartheta}{lim} f(\vartheta'). \tag{10}$$

This means that as the points ϑ' on the fractal curve **F** approach the point ϑ, the function $f(\vartheta')$ approaches the value l within a certain tolerance level ϵ. In simpler terms, it characterizes the behavior of the function f at a particular point ϑ on the fractal curve.

Definition 9 The fractal right-sided limit of a function $f(\vartheta) : \mathbf{F} \rightarrow \mathbb{R}$ is denoted by R as ϑ approaches the point a from the positive side (a^+). This limit is achieved when, for any given ϵ, there exists a positive value $\delta > 0$ such that for all ϑ in the fractal curve **F**, if $0 < \vartheta - a < \delta$, then $|f(\vartheta) - R| < \epsilon$. In mathematical notation, it is represented as:

$$F\_\underset{\vartheta \rightarrow a^+}{lim} f(\vartheta) = R \tag{11}$$

This definition essentially captures the behavior of the function f as it approaches the point a from the right side on the fractal curve **F**. If the function approaches a specific value R in this scenario, it is said to have a right-sided fractal limit at a.

Definition 10 The fractal left-sided limit of a function $f(\vartheta) : \mathbf{F} \rightarrow \mathbb{R}$ is denoted by L as ϑ approaches the point a from the negative side (a^-). This limit is attained when, for any given ϵ, there exists a positive value $\delta > 0$ such that for all ϑ in the fractal curve **F**, if $0 < a - \vartheta < \delta$, then $|f(\vartheta) - L| < \epsilon$. In mathematical notation, it is represented as:

$$\mathbf{F}\_\underset{\vartheta \rightarrow a^-}{lim} f(\vartheta) = L \tag{12}$$

This definition captures the behavior of the function f as it approaches the point a from the left side on the fractal curve **F**. If the function approaches a specific value L in this context, it is said to have a left-sided fractal limit at a.

Definition 11 A function $f(\vartheta) : \mathbf{F} \rightarrow \mathbb{R}$ defined on the fractal curve **F** is considered left-sided F-continuous if, for any given ϵ, there exists a positive value $\delta > 0$ such that for all ϑ in **F**, if $0 < a - \vartheta < \delta$, then $|f(\vartheta) - f(a)| < \epsilon$. In other words, as ϑ approaches the point a from the left side on the fractal curve, the function's values approach $f(a)$. This concept is denoted by the equation:

$$\mathbf{F}\_\underset{\vartheta \rightarrow a^-}{lim} f(\vartheta) = f(a). \tag{13}$$

In summary, a left-sided F-continuous function ensures that the function's values become arbitrarily close to the value of the function at a as the fractal parameter ϑ gets sufficiently close to a from the left side.

Definition 12 A function $f(\vartheta):\mathbf{F}\to\mathbb{R}$ defined on the fractal curve **F** is considered right-sided **F**-continuous if, for any given ϵ, there exists a positive value $\delta>0$ such that for all ϑ in **F**, if $0<\vartheta-a<\delta$, then $|f(\vartheta)-f(a)|<\epsilon$. In other words, as ϑ approaches the point a from the right side on the fractal curve, the function's values approach $f(a)$.

This concept is denoted by the equation:

$$\mathbf{F}\_\underset{\vartheta\to a^+}{lim} f(\vartheta)=f(a). \tag{14}$$

In summary, a right-sided **F**-continuous function ensures that the function's values become arbitrarily close to the value of the function at a as the fractal parameter ϑ gets sufficiently close to a from the right side.

Definition 13 Let $f(\vartheta):\mathbf{F}\to\mathbb{R}$ be a function defined on the fractal curve **F**. We say that $f(\vartheta)$ is **F**-continuous at a specific point $a\in\mathbf{F}$ if any of the following conditions holds:

1. For any given $\varepsilon>0$, there exists a positive value $\delta>0$ such that for all $\vartheta\in\mathbf{F}$, if $|\vartheta-a|<\delta$, then $|f(\vartheta)-f(a)|<\varepsilon$.
2. Alternatively, $f(a)$ is equal to the right and left-sided fractal limit of $f(\vartheta)$. This can be expressed as:

$$f(a)=\mathbf{F}\_\lim_{\vartheta\to a^+} f(\vartheta)=\mathbf{F}\_\lim_{\vartheta\to a^-} f(\vartheta)=\mathbf{F}\_\lim_{\vartheta\to a} f(\vartheta). \tag{15}$$

Definition 14 A function $f(\vartheta)$ is **F**-continuous over $[\vartheta_1,\vartheta_2]$ if it is **F**-continuous at every point in $(\vartheta_1,\vartheta_2)$, and it is **F**-continuous from the right at ϑ_1 and from the left at ϑ_2.

Definition 15 A function $f:F\to\mathbb{R}$ is called piecewise **F**-continuous if there exist $a=\vartheta_0<\vartheta_1<\cdots<\vartheta_n=b$ so that:

1. f is F-continuous on $(\vartheta_k,\vartheta_{k+1})$ all $k=0,\ldots,n-1$
2. The fractal limits $\mathbf{F}\_\lim_{\vartheta\to\vartheta_{k+1}^-} f(\vartheta)$ and $\mathbf{F}\_\lim_{\vartheta\to\vartheta_k^+} f(\vartheta)$ exist and are finite for all $k=0,\ldots,n-1$.

Definition 16 The right $\mathbf{F}^\alpha$-derivative of a function $f:\mathbf{F}\to\mathbb{R}$ at the point $\vartheta_0\in\mathbf{F}$ is defined as follows:

$$D_F^\alpha f(\vartheta')|_{\vartheta_0^+}=\mathbf{F}\_\underset{\vartheta\to\vartheta_0^+}{lim}\frac{f(\vartheta')-f(\vartheta_0)}{J(\vartheta')-J(\vartheta_0)} \tag{16}$$

Here, ϑ' represents a generic point on the fractal curve **F**, and $J(\vartheta)$ is the staircase function defined earlier. The right $\mathbf{F}^\alpha$-derivative measures the rate of change of the function f as ϑ approaches ϑ_0 from the right side of the fractal curve. If the limit of the expression exists, then the right $\mathbf{F}^\alpha$-derivative is well-defined at ϑ_0.

Definition 17 The left $\mathbf{F}^\alpha$-derivative of a function $f : \mathbf{F} \to \mathbb{R}$ at the point $\vartheta_0 \in \mathbf{F}$ is defined as follows:

$$D^\alpha_F f(\vartheta')|_{\vartheta_0^-} = \mathbf{F}\_\underset{\vartheta' \to \vartheta_0^-}{lim} \frac{f(\vartheta') - f(\vartheta_0)}{J(\vartheta') - J(\vartheta_0)} \tag{17}$$

Here, ϑ' represents a generic point on the fractal curve $\mathbf{F}$, and $J(\vartheta)$ is the staircase function defined earlier. The left $\mathbf{F}^\alpha$-derivative measures the rate of change of the function f as ϑ approaches ϑ_0 from the left side of the fractal curve. If the limit of the expression exists, then the left $\mathbf{F}^\alpha$-derivative is well-defined at ϑ_0.

Definition 18 The $\mathbf{F}^\alpha$-derivative of a function $f : \mathbf{F} \to \mathbb{R}$ at the point $\vartheta_0 \in \mathbf{F}$ is defined as follows:

$$D^\alpha_F f(\vartheta_0) = \mathbf{F}\_\underset{\vartheta' \to \vartheta_0}{lim} \frac{f(\vartheta') - f(\vartheta_0)}{J(\vartheta') - J(\vartheta_0)} \tag{18}$$

Here, ϑ' represents a generic point on the fractal curve $\mathbf{F}$, and $J(\vartheta)$ is the staircase function defined earlier. The $\mathbf{F}^\alpha$-derivative measures the rate of change of the function f at the point ϑ_0 with respect to the fractal arc length. If the limit of the expression exists, then the $\mathbf{F}^\alpha$-derivative is well-defined at ϑ_0.

Definition 19 Consider the subdivision $Q_{[a,b]}$ as defined in Definition2. We introduce the concept of the $\mathbf{F}^\alpha$-integral of a bounded function $f : \mathbf{F} \to \mathbb{R}$, computed over a segment $\mathfrak{C}(a, b)$. Here, a and b are points on the fractal curve $\mathbf{F}$.

The $\mathbf{F}^\alpha$-integral of f is expressed by the following formula:

$$\int_{\mathfrak{C}(a,b)} f(\vartheta) d^\alpha_F \vartheta = \mathbf{F}\_\underset{\Delta_n \to 0}{lim} \sum_{i=0}^{n} f(\vartheta_{i'})[S^\alpha_F(u^{-1}(\vartheta_i)) - S^\alpha_F(u^{-1}(\vartheta_{i-1}))] \tag{19}$$

Here, ϑ represents a point on the fractal curve $\mathbf{F}$, and $\vartheta_{i'}$ denotes a point within the interval $[\vartheta_{i-1}, \vartheta_i)$ of the subdivision $Q_{[a,b]}$. The notation Δ_n corresponds to the maximum width of the subintervals in the subdivision, and we evaluate the integral by taking the limit as Δ_n approaches zero. The staircase function $S^\alpha_F(u^{-1}(\vartheta))$ (defined earlier) plays a critical role in computing this $\mathbf{F}^\alpha$-integral.

In essence, the $\mathbf{F}^\alpha$-integral extends the concept of the traditional integral to fractal curves, accounting for their self-similar properties and the fractal arc length of the curve.

Definition 20 We introduce the concept of the fractal improper integral of a function $f : \mathbf{F} \to \mathbb{R}$ over an unbounded interval $\mathfrak{C}(a, \infty)$, where a is a point on the fractal curve $\mathbf{F}$. The fractal improper integral is defined as follows:

$$\int_{\mathfrak{C}(a,\infty)} f(\vartheta) d_F^{\alpha}\vartheta = \mathbf{F}\_\underset{\vartheta' \to \infty}{lim} \int_{\mathfrak{C}(a,\vartheta')} f(\vartheta) d_F^{\alpha}\vartheta \tag{20}$$

when to be convergent.In this definition, the interval of integration extends to infinity, but since fractal curves do not have a traditional notion of infinite length, we evaluate the integral by taking the limit as ϑ' approaches infinity. This process accounts for the self-similar nature of the fractal curve **F** and captures the behavior of the function f over an unbounded range. The fractal improper integral allows us to integrate functions on fractal curves in a manner that aligns with their unique geometric properties.

Example 1 Let's examine an example to illustrate the fractal improper integral of the function $f(\vartheta) = \exp(cJ(\vartheta))$, where $J(\vartheta) \geq 0$ and c is a constant. We want to evaluate the following integral:

$$\begin{aligned} \int_{\mathfrak{C}(0,\infty)} \exp(cJ(\vartheta)) d_F^{\alpha}\vartheta &= \mathbf{F}\_\underset{\vartheta' \to \infty}{lim} \int_{\mathfrak{C}(0,\vartheta')} \exp(cJ(\vartheta)) d_F^{\alpha}\vartheta \\ &= \mathbf{F}\_\underset{\vartheta' \to \infty}{lim} \left. \frac{\exp(cJ(\vartheta))}{c} \right|_0^{\vartheta'} \\ &= \mathbf{F}\_\underset{\vartheta' \to \infty}{lim} \frac{1}{c}(\exp(cJ(\vartheta') - 1)). \end{aligned} \tag{21}$$

Depending on the value of the constant c, the fractal integral given by Eq.(21) can behave differently. Specifically, if $c < 0$, the fractal integral is convergent.

Example 2 Consider an example with the function $f(\vartheta) = J(\vartheta)^{-p}$, where $J(\vartheta) \geq 1$, and p is a constant. We want to evaluate the following fractal improper integral:

$$\begin{aligned} \int_{\mathfrak{C}(1,\infty)} J(\vartheta)^{-p} d_F^{\alpha}\vartheta &= \mathbf{F}\_\underset{\vartheta' \to \infty}{lim} \int_{\mathfrak{C}(1,\vartheta')} J(\vartheta)^{-p} d_F^{\alpha}\vartheta \\ &= \mathbf{F}\_\underset{\vartheta' \to \infty}{lim} \frac{1}{1-p}\left(J(\vartheta')^{1-p} - J(1)^{1-p}\right) \\ &= \mathbf{F}\_\underset{\vartheta' \to \infty}{lim} \frac{1}{1-p}\left(J(\vartheta')^{1-p} - \frac{1}{2}\Gamma(\alpha+1)^{1-p}\right) \end{aligned} \tag{22}$$

If the constant $p \neq 1$ is greater than 1, then $J(\vartheta')^{1-p}$ approaches zero as ϑ' tends to infinity, resulting in a convergent fractal integral.

3 Laplace transform on fractal curves

In this section, we explore the Laplace transform on fractal curves, a mathematical technique utilized for the analysis and solution of dynamic systems governed by fractal differential equations. By extending the traditional Laplace transform to fractal curves, we can convert fractal differential equations with constant coefficients into algebraic equations, making them more amenable to solving. Leveraging the inverse fractal Laplace transform, we can then deduce the solutions to these intricate differential equations. This approach proves particularly valuable when dealing with systems that exhibit fractal dynamics, where their behavior depends not only on the present time but also on their past history. Such systems are commonly encountered in various domains, including neutron transport, viscoelasticity, population dynamics, and other fields characterized by complex, self-similar patterns. To effectively conduct the Laplace transform on fractal curves/the fractal Laplace transform, we define and work with various mathematical concepts unique to fractals, such as fractal convolution integrals and the notion of fractal dimensions. These tools empower us to efficiently analyze and model the behavior of intricate and irregular systems, providing valuable insights into their dynamic properties. Ultimately, the fractal Laplace transform serves as a powerful tool for understanding and solving differential equations within the realm of fractal geometry.

Definition 21 This definition introduces the analogue of the fractal Laplace transform for a non-negative function $f(\vartheta) : \mathbf{F} \to \mathbb{R}$, where $\mathbf{F}$ is a fractal curve. The fractal Laplace transform of $f(\vartheta)$ is denoted as $\mathcal{L}[f(\vartheta)]$, and it is defined as follows:

$$\begin{aligned} \mathcal{L}[f(\vartheta)] = F(s) &= \mathbf{F}\_\lim_{T\to\infty} \int_{\mathfrak{C}(0,T)} f(\vartheta)\exp(-J(s)J(\vartheta))d_F^{\alpha}\vartheta \\ &= \int_{\mathfrak{C}(0,\infty)} f(\vartheta)\exp(-J(s)J(\vartheta))d_F^{\alpha}\vartheta, \end{aligned} \tag{23}$$

Here, $s \in \mathbb{C}$ represents a complex number. The integral used in Definition21 is a fractal improper integral. If this improper integral converges, then f possesses a fractal Laplace transform. The fractal Laplace transform operator allows us to transform a function defined in the ϑ-domain, associated with the fractal curve $\mathbf{F}$, into a function defined in the s-domain, where s is the complex variable. This transformation is a powerful tool for analyzing and solving differential equations involving fractal curves.

Theorem 1 *The theorem states that a function* $f(\vartheta) : \mathbf{F} \to \mathbb{R}$ *has a fractal Laplace transform* $F(s)$ *if it satisfies the following two conditions*:

1. The function f is piecewise $\mathbf{F}$-continuous on the interval $0 \le \vartheta \le E$ for all $E > 0$. In other words, f is continuous on this interval except for a set of isolated points.
2. There exist positive constants M, a, and c such that the absolute value of $f(\vartheta)$ is bounded by $Me^{aJ(\vartheta)}$ for values of $J(\vartheta)$ greater than c.

If these conditions are met, then $f(\vartheta)$ is termed fractal exponentially-restricted, and it possesses a well-defined fractal Laplace transform $F(s)$. The fractal Laplace transform of $f(\vartheta)$ allows us to explore the behavior and properties of $f(\vartheta)$ in the complex s-domain, making it easier to analyze and solve differential equations involving fractal curves.

Proof We begin by using Definition 21 of the fractal Laplace transform to express $\mathcal{L}[f(\vartheta)]$ as an improper integral over the entire fractal curve $\mathfrak{C}(0,\infty)$:

$$\mathcal{L}[f(\vartheta)] = \int_{\mathfrak{C}(0,\infty)} f(\vartheta)\exp(-J(s)J(\vartheta))d_F^\alpha\vartheta \\ \leq \int_{\mathfrak{C}(0,E)} f(\vartheta)\exp(-J(s)J(\vartheta))d_F^\alpha\vartheta + M\int_{\mathfrak{C}(E,\infty)} \exp(-(J(s)-a)J(\vartheta))d_F^\alpha\vartheta. \tag{24}$$

Next, we split the integral into two parts: one over the interval $\mathfrak{C}(0,E)$ and the other over the interval $\mathfrak{C}(E,\infty)$, where E is a positive constant.

For the first integral over $\mathfrak{C}(0,E)$, we note that $f(\vartheta)$ is piecewise **F**-continuous on this interval. Let $A = \max|f(\vartheta)| : \vartheta \in \mathfrak{C}(0,E)$ be the maximum absolute value of $f(\vartheta)$ in this interval. Using this information, we obtain the inequality:

$$\int_{\mathfrak{C}(0,E)} f(\vartheta)\exp(-J(s)J(\vartheta))d_F^\alpha\vartheta \leq A\int_{\mathfrak{C}(0,E)} \exp(-J(s)J(\vartheta))d_F^\alpha\vartheta \\ = A\left(\frac{1}{J(s)} - \frac{\exp(-J(s)c)}{J(s)}\right) < \infty \tag{25}$$

The integral on the right-hand side is finite and does not depend on E, as it involves only the behavior of the fractal curve within the interval $\mathfrak{C}(0,E)$.

For the second integral of Eq.(24) over $\mathfrak{C}(E,\infty)$, we used the given condition that $|f(\vartheta)| \leq M\exp(aJ(\vartheta))$ for $J(\vartheta) > c$, where a, and c are constants. The integral is convergent for $J(s) > a$ and approaches zero as ϑ approaches infinity.

Considering both parts of the integral separately, we have shown that $\mathcal{L}[f(\vartheta)]$ is well-defined and converges for $J(s) > a$, which completes the proof. □

Example 3 Consider the function $f(\vartheta) = 1$ for $\vartheta \geq 0$. We want to find its fractal Laplace transform using Definition 21. Substituting $f(\vartheta) = 1$ into the fractal Laplace transform integral and utilizing the conjugacy of fractal calculus with standard calculus, we obtain:

$$\mathcal{L}[1] = \int_{\mathfrak{C}(0,\infty)} \exp(-J(s)J(\vartheta))d_F^\alpha\vartheta = \frac{1}{J(s)}, \quad J(s) > 0. \tag{26}$$

Example 4 Consider the function $f(\vartheta) = \exp(aJ(\vartheta))$ for $\vartheta \geq 0$. We want to find its Laplace transform using Definition 21.

By substituting $f(\vartheta) = \exp(aJ(\vartheta))$ into the fractal Laplace transform integral, we obtain:

$$\mathcal{L}[\exp(aJ(\vartheta))] = \int_{\mathfrak{C}(0,\infty)} \exp(-J(s)J(\vartheta))\exp(aJ(\vartheta))d_F^\alpha\vartheta. \tag{27}$$

To evaluate this integral, we notice that the integrand contains the term $\exp(-J(s)J(\vartheta))\exp(aJ(\vartheta))$. We observe that the exponential terms can be combined as follows:

$$\exp(-J(s)J(\vartheta))\exp(aJ(\vartheta)) = \exp((a - J(s))J(\vartheta)) \tag{28}$$

Now, the integral effectively captures the analogue length of the fractal curve $\mathfrak{C}(0, \infty)$. Thus, we have:

$$\int_{\mathfrak{C}(0,\infty)} \exp((a - J(s))J(\vartheta))d_F^\alpha \vartheta = \frac{1}{J(s) - a} \tag{29}$$

Thus, the fractal Laplace transform of $f(\vartheta) = \exp(aJ(\vartheta))$ is $\frac{1}{J(s)-a}$ for $J(s) > a$.

Example 5 Consider the function $f(\vartheta) = \sin(aJ(\vartheta))$ for $\vartheta \geq 0$. We want to find its fractal Laplace transform using Definition 21. By substituting $f(\vartheta) = \sin(aJ(\vartheta))$ into the fractal Laplace transform integral, we have:

$$\mathcal{L}[\sin(aJ(\vartheta))] = F(s) = \int_{\mathfrak{C}(0,\infty)} \exp(-J(s)J(\vartheta))\sin(aJ(\vartheta))d_F^\alpha \vartheta, \quad J(s) > 0, \tag{30}$$

To evaluate this integral, we perform fractal integration by parts. The integration by parts formula for fractals yields:

$$\begin{aligned} F(s) &= \underset{E\to\infty}{\mathbf{F}\_lim}\left[-\frac{\exp(-J(s)J(\vartheta))\cos aJ(\vartheta)}{a}\Big|_0^E \right. \\ &\left. -\frac{J(s)}{a}\int_{\mathfrak{C}(0,E)} \exp(-J(s)J(\vartheta))\cos(aJ(\vartheta))d_F^\alpha \vartheta \right] \\ &= \frac{1}{a} - \frac{J(s)}{a}\int_{\mathfrak{C}(0,\infty)} \exp(-J(s)J(\vartheta))\cos(aJ(\vartheta))d_F^\alpha \vartheta. \end{aligned} \tag{31}$$

Since $\cos(aJ(\vartheta))$ is bounded for all $\vartheta \geq 0$, the first term vanishes as $\vartheta \to \infty$. Therefore, we are left with:

$$\begin{aligned} F(s) &= \frac{1}{a} - \frac{J^2(s)}{a^2}\int_{\mathfrak{C}(0,\infty)} \exp(-J(s)J(\vartheta))\sin(aJ(\vartheta))d_F^\alpha \vartheta \\ &= \frac{1}{a} - \frac{J^2(s)}{a^2}F(s). \end{aligned} \tag{32}$$

Thus, solving Eq.(32) for $F(s)$, we arrive at the final answer:

$$F(s) = \frac{a}{J^2(s) + a^2}, \quad J(s) > 0. \tag{33}$$

Lemma 1 *For functions $f_1(\vartheta)$ and $f_2(\vartheta)$ with fractal Laplace transforms, and constants c_1 and c_2, we have:*

$$\mathcal{L}[c_1 f_1(\vartheta) + c_2 f_2(\vartheta)] = c_1\mathcal{L}[f_1(\vartheta)] + c_2\mathcal{L}[f_2(\vartheta)]. \tag{34}$$

In other words, the fractal Laplace transform of a linear combination of functions is equal to the linear combination of their individual fractal Laplace transforms, where the coefficients c_1 and c_2 are constants. This property allows us to simplify the fractal Laplace transform of complex expressions involving multiple functions.

Proof The proof follows directly from Definition 21. We have:

$$\begin{aligned}\mathcal{L}[c_1f_1(\vartheta) + c_2f_2(\vartheta)] &= \int_{\mathfrak{C}(0,\infty)} \exp(-J(s)J(\vartheta)(c_1f_1 + c_2f_2]d_F^{\alpha}\vartheta \\ &= c_1 \int_{\mathfrak{C}(0,\infty)} \exp(-J(s)J(\vartheta)f_1 d_F^{\alpha}\vartheta + c_2 \int_{\mathfrak{C}(0,\infty)} \exp(-J(s)J(\vartheta)f_2 d_F^{\alpha}\vartheta \\ &= c_1\mathcal{L}[f_1] + c_2\mathcal{L}[f_2],\end{aligned} \tag{35}$$

This establishes the linearity property of the fractal Laplace transform, showing that the fractal Laplace transform of a linear combination of functions is equal to the linear combination of their individual fractal Laplace transforms, with the appropriate constants. □

Under the conditions specified, we have the following theorem:

Theorem 2 *Suppose $f(\vartheta)$ is $\mathbf{F}$-continuous, and its fractal derivative $D_F^{\alpha}f(\vartheta)$ is piecewise $\mathbf{F}$-continuous on any interval $\mathfrak{C}(0,E)$. Furthermore, let there exist constants M, K, and a such that*

$$|f(\vartheta)| \leq K\exp(aJ(\vartheta)), \quad \vartheta \geq M. \tag{36}$$

Then, the fractal Laplace transform of the fractal derivative of $f(\vartheta)$ is given by

$$\mathcal{L}[D_F^{\alpha}f(\vartheta)] = J(s)\mathcal{L}[f(\vartheta)] - f(0), \tag{37}$$

provided $J(s) > a$.

Proof To prove the theorem, we consider the fractal integral:

$$\int_{\mathfrak{C}(0,E)} \exp(-J(s)J(\vartheta))D_F^{\alpha}f(\vartheta)d_F^{\alpha}\vartheta. \tag{38}$$

If $D_F^{\alpha}f(\vartheta)$ has n points of discontinuity $\vartheta_1, \vartheta_2, \ldots, \vartheta_n$ in the interval $\mathfrak{C}(0,E)$, we can split the fractal integral (38) into n integrals over the subintervals $\mathfrak{C}(0,\vartheta_1)$, $\mathfrak{C}(\vartheta_1,\vartheta_2)$,..., $\mathfrak{C}(\vartheta_n,E)$.

By applying fractal integration by parts to each of these integrals, we obtain a sum of terms involving the function $f(\vartheta)$ evaluated at the endpoints of the subintervals, as well as fractal integrals of the form $\int_{\mathfrak{C}(a,b)} \exp(-J(s)J(\vartheta))f(\vartheta)d_F^{\alpha}\vartheta$.

Since $f(\vartheta)$ is $\mathbf{F}$-continuous, the contributions of the terms involving $f(\vartheta)$ evaluated at the endpoints of the subintervals vanish. Thus, the fractal integral reduces to:

$$\int_{\mathfrak{C}(0,E)} \exp(-J(s)J(\vartheta))D_F^\alpha f(\vartheta)d_F^\alpha\vartheta = -f(0) + J(s)\int_{\mathfrak{C}(0,E)} \exp(-J(s)J(\vartheta))f(\vartheta)d_F^\alpha\vartheta. \tag{39}$$

As E approaches infinity, the term $-f(0)$ becomes negligible, and the fractal integral becomes:

$$\int_{\mathfrak{C}(0,\infty)} \exp(-J(s)J(\vartheta))D_F^\alpha f(\vartheta)d_F^\alpha\vartheta = J(s)\int_{\mathfrak{C}(0,\infty)} \exp(-J(s)J(\vartheta))f(\vartheta)d_F^\alpha\vartheta. \tag{40}$$

Therefore, we have:

$$\mathcal{L}[D_F^\alpha f(\vartheta)] = J(s)\mathcal{L}[f(\vartheta)] - f(0), \tag{41}$$

as long as $J(s) > a$, where a is a constant associated with the bound on $f(\vartheta)$ as given in hypothesis of the theorem. This completes the proof. □

Remark 2 This corollary provides a general result for the fractal Laplace transform of higher-order fractal derivatives.

Corollary 1 *Suppose that the functions f, $D_F^\alpha f(\vartheta)$,..., $(D_F^\alpha)^{n-1}f(\vartheta)$ are* **F***-continuous, and $(D_F^\alpha)^n f(\vartheta)$ is piecewise* **F***-continuous on any interval $\mathfrak{C}(0,E)$. We assume that there exist constants K, a, and M satisfying*:

$$\begin{aligned} &|f(\vartheta)| \le K\exp(aJ(\vartheta)), |D_F^\alpha f(\vartheta)| \le K\exp(aJ(\vartheta)) \\ &,\ldots, |(D_F^\alpha)^{n-1}f(\vartheta)| \le K\exp(aJ(\vartheta)),\ J(\vartheta) \ge M. \end{aligned} \tag{42}$$

Therefore, the fractal Laplace transform of the n-th fractal derivative $(D_F^\alpha)^n f(\vartheta)$ exists for $J(s) > a$ and is given by:

$$\begin{aligned} \mathcal{L}[(D_F^\alpha)^n f(\vartheta)] &= J^n(s)\mathcal{L}[f(\vartheta)] - J^{n-1}(s)f(0) \\ &- \cdots - J(s)(D_F^\alpha)^{n-2}f(\vartheta)|_{\vartheta=0} - (D_F^\alpha)^{n-1}f(\vartheta)|_{\vartheta=0}. \end{aligned} \tag{43}$$

This result allows us to compute the fractal Laplace transform of higher-order fractal derivatives based on the fractal Laplace transform of the original function $f(\vartheta)$ and its lower-order fractal derivatives. By providing a recursive formula, the corollary facilitates the analysis and solution of differential equations involving fractal operators through the fractal Laplace transform method.

Definition 22 This definition introduces the concept of the inverse fractal Laplace transform of a function $F(s)$. The inverse transform, denoted as $f(\vartheta)$, is a piecewise **F**-continuous and fractal exponentially-restricted function, which allows us to retrieve the original function from its fractal Laplace transform. The inverse fractal Laplace transform is given by the following expression, often referred to as the fractal Mellin's inverse formula:

$$f(\vartheta) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\mathbf{F}\_\lim_{b\to\infty}\int_{\mathfrak{C}(a-ib,a+ib)} \exp(J(\vartheta)J(s))F(s)d_F^{\alpha}s \tag{44}$$

where $Re(s) = a$ represents the real part of the complex number s, and $i = \sqrt{-1}$. This formula allows us to recover the original function $f(\vartheta)$ by integrating its fractal Laplace transform $F(s)$ over a specific contour in the complex plane. The inverse fractal Laplace transform is a valuable tool in the analysis and solution of fractal differential equations, enabling us to find the time-domain behavior of systems with fractal dynamics.

Table 1 showcases essential formulas for the fractal Laplace transforms.

Table 1 presents various functions $f(\vartheta)$ along with their corresponding fractal Laplace transforms $F(s)$. These formulas are essential in transforming functions between the fractal time domain (ϑ-domain) and the fractal frequency domain (s-domain), enabling the analysis and solution of fractal differential equations.

Example 6 Consider the fractal differential equation given by

$$(D_F^{\alpha})^2 f(\vartheta) - D_F^{\alpha} f(\vartheta) - f(\vartheta) = 0, \tag{45}$$

with the initial conditions

$$f(0) = 1, \quad (D_F^{\alpha}) f(\vartheta)|_{\vartheta=0} = 0. \tag{46}$$

To solve Eq.(45), we apply the fractal Laplace transform, resulting in

$$\mathcal{L}[(D_F^{\alpha})^2 f(\vartheta)] - \mathcal{L}[D_F^{\alpha} f(\vartheta)] - 2\mathcal{L}[f(\vartheta)] = 0. \tag{47}$$

Using Corollary 1, we express $\mathcal{L}[(D_F^{\alpha})^2 f(\vartheta)]$ and $\mathcal{L}[(D_F^{\alpha}) f(\vartheta)]$ in terms of $\mathcal{L}[f(\vartheta)]$ as follows:

Table 1 Important Formulas of Fractal Laplace Transforms

$f(\vartheta) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(\vartheta)]$
1	$\frac{1}{J(s)}, \; J(s) > 0$
$\exp(a\vartheta)$	$\frac{1}{J(s)-a}, \; J(s) > a$
ϑ^n n is positive number	$\frac{n!}{J^{n+1}(s)}, \; J(s) > 0$
$\vartheta^p, p > -1$	$\frac{\Gamma(p+1)}{J^{p+1}(s)}, \; J(s) > 0$
$\sin a\vartheta$	$\frac{a}{J^2(s)+a^2}, \; J(s) > 0$
$\cos a\vartheta$	$\frac{J(s)}{J^2(s)+a^2}, \; J(s) > 0$
$(D_F^{\alpha})^n f(\vartheta)$	$J^n(s)F(s) - J^{n-1}(s)f(0) - \cdots - (D_F^{\alpha})^{n-1} f(\vartheta)\|_{\vartheta=0}$

$$J^2(s)\mathcal{L}[f(\vartheta)] - J(s)f(0) - (D_F^\alpha)f(\vartheta)|_{\vartheta=0} - [J(s)\mathcal{L}[f(\vartheta)] - f(0)] - 2\mathcal{L}[f(\vartheta)] = 0. \tag{48}$$

This simplifies to

$$(J^2(s) - J(s) - 2)F(s) + (1 - J(s))f(0) - DF^\alpha f(\vartheta)|_{\vartheta=0} = 0, \tag{49}$$

where $F(s) = \mathcal{L}[f(\vartheta)]$. After substituting the initial conditions from Eq.(46) and solving for $F(s)$, we find

$$F(s) = \frac{J(s) - 1}{J^2(s) - J(s) - 2} = \frac{J(s) - 1}{(J(s) - 2)(J(s) + 1)}. \tag{50}$$

By applying the inverse fractal Laplace transform, we obtain

$$f(\vartheta) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{1/3}{J(s) - 2} + \frac{2/3}{J(s) + 1}\right] = \frac{1}{3}\exp(2J(\vartheta)) + \frac{2}{3}\exp(-J(\vartheta)), \tag{51}$$

which represents the solution of Eq. (45). Figure 1 depicts the graph of Eq. (51).

Definition 23 The unit step function, also known as the Heaviside function, is denoted by $u_c(\vartheta)$, where $\vartheta \in \mathbf{F}$. It is defined as follows:

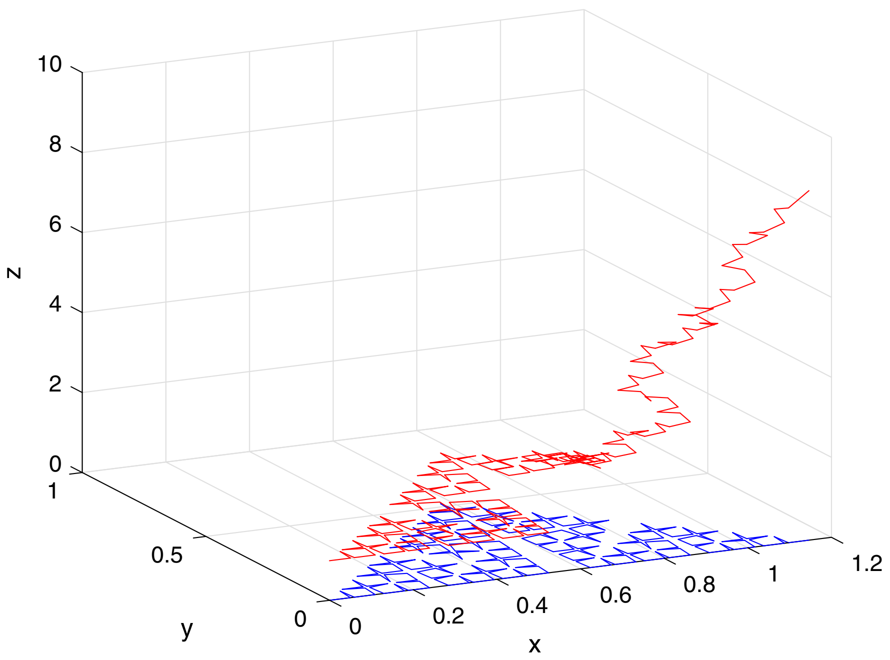

Fig. 1 Graph of Eq. (51)

$$u_c(\vartheta) = \begin{cases} 0, & J(\vartheta) < c; \\ 1, & J(\vartheta) \geq c. \end{cases} \tag{52}$$

Here, c is a non-negative constant. The unit step function serves as a mathematical representation of a sudden change or transition in a system, where the value of the function jumps from 0 to 1 at the critical point c within the fractal domain $\mathbf{F}$.

Example 7 The fractal Laplace transform of the unit step function $u_c(\vartheta)$ is :

$$\mathcal{L}[u_c(\vartheta)] = \frac{\exp(-cJ(s))}{J(s)}, \quad J(s) > 0. \tag{53}$$

This result provides a convenient way to transform the unit step function into the s-domain, where it is represented by a simple algebraic expression involving exponential and fractional terms. Using Definition 21, we can express the fractal Laplace transform of the unit step function as

$$\mathcal{L}[u_c(\vartheta)] = \int_{\mathfrak{C}(0,\infty)} \exp(-J(\vartheta)J(s))d_F^\alpha\vartheta = \int_{\mathfrak{C}(c,\infty)} \exp(-J(\vartheta)J(s))d_F^\alpha\vartheta. \tag{54}$$

Since $u_c(\vartheta) = 0$ for $J(\vartheta) < c$ and $u_c(\vartheta) = 1$ for $J(\vartheta) \geq c$, the integral is equivalent to integrating over the region $\mathfrak{C}(c,\infty)$, as the integrand is nonzero only in this interval. Solving the integral gives us the desired result:

$$\mathcal{L}[u_c(\vartheta)] = \frac{\exp(-cJ(s))}{J(s)}, \quad J(s) > 0. \tag{55}$$

This demonstrates the fractal Laplace transform of the unit step function, providing an expression that facilitates the analysis and solution of dynamic systems involving such functions.

Definition 24 A shift of the function $f(\vartheta)$ in the positive direction is defined as follows:

$$g(\vartheta) = u_c(\vartheta)f(\vartheta - c), \tag{56}$$

where c is a constant. In this definition, $u_c(\vartheta)$ is the unit step function, which acts as a switch that turns on for $\vartheta \geq c$ and turns off for $\vartheta < c$. The function $f(\vartheta - c)$ represents the original function f shifted to the right by a distance of c in the ϑ-domain.

Remark 3 In simpler terms, the function $g(\vartheta)$ is constructed by taking the function $f(\vartheta)$ and shifting it to the right by c units, but only for values of ϑ that are greater than or equal to c. For values of ϑ less than c, the function $g(\vartheta)$ remains zero. This concept of shifting functions is commonly used in mathematics to manipulate and analyze functions. It allows us to explore how changes in the input variable affect the behavior of a function, and it is a useful tool in various mathematical and engineering applications.

Proposition 3 *Let $f(\vartheta) : F \to \mathbb{R}$. If $J(s) > a \geq 0$ and $J(\vartheta) \geq c > 0$, therefore the fractal Laplace transform of the shifted function $u_c(\vartheta)f(\vartheta - c)$ is given by:*

$$\mathcal{L}[u_c(\vartheta)f(\vartheta - c)] = \exp(-cJ(s))\mathcal{L}[f(\vartheta)]. \tag{57}$$

Conversely, the shifted function $u_c(\vartheta)f(\vartheta - c)$ can be obtained by taking the inverse fractal Laplace transform of $\exp(-cJ(s))F(s)$:

$$u_c(\vartheta)f(\vartheta - c) = \mathcal{L}^{-1}[\exp(-cJ(s))F(s)]. \tag{58}$$

This proposition shows that when we shift a function in the ϑ-domain by an amount c, it corresponds to multiplying its fractal Laplace transform in the s-domain by the exponential term $\exp(-cJ(s))$. Similarly, we can retrieve the shifted function from its fractal Laplace transform by taking the inverse fractal Laplace transform after multiplying by the exponential term. This relationship is valuable in solving and analyzing differential equations involving shifted functions.

Proof To prove this theorem, we begin by using the fractal Laplace transform as follows:

$$\begin{aligned}\mathcal{L}[u_c(\vartheta)f(\vartheta - c)] &= \int_{\mathfrak{C}(0,\infty)} \exp(-J(s)J(\vartheta))u_c(\vartheta)f(J(\vartheta) - c)d_F^{\alpha}\vartheta \\ &= \int_{\mathfrak{C}(c,\infty)} \exp(-J(s)J(\vartheta))f(J(\vartheta) - c)d_F^{\alpha}\vartheta.\end{aligned} \tag{59}$$

By changing the variable $\xi = \vartheta - c$, we have:

$$\begin{aligned}\mathcal{L}[u_c(\vartheta)f(\vartheta - c)] &= \int_{\mathfrak{C}(0,\infty)} \exp(-(J(\xi) + c)J(s))f(\xi)d_F^{\alpha}\xi \\ &= \exp(-cJ(s)) \int_{C(0,\infty)} \exp(-J(s)J(\xi))f(\xi)d_F^{\alpha}\xi \\ &= \exp(-cJ(s))F(s),\end{aligned} \tag{60}$$

which establishes Eq. (57). Next, Eq. (58) follows by taking the inverse transform of the fractal Laplace transform on both sides of Eq. (57), which completes the proof. □

Remark 4 The proposition states the relationship between the fractal Laplace transform of a shifted function and the fractal Laplace transform of the original function.

Example 8 Consider the fractal function defined as:

$$f(\vartheta) = \sin(J(\vartheta)) + u_{\pi/4}(J(\vartheta))\cos(J(\vartheta) - \pi/4), \quad \vartheta \in \mathbf{F}, \tag{61}$$

To find its fractal Laplace transform, we express it as the sum of two terms:

$$\begin{aligned}\mathcal{L}[f(\vartheta)] &= \mathcal{L}[\sin(\vartheta)] + \mathcal{L}[u_{\pi/4}(\vartheta)\cos(\vartheta - \pi/4)] \\ &= \mathcal{L}[\sin(\vartheta)] + \exp(-\pi J(s)/4)\mathcal{L}[\cos(\vartheta)].\end{aligned} \tag{62}$$

Next, we use the fractal Laplace transforms of $\sin(\vartheta)$ and $\cos(\vartheta)$ to obtain:

$$\mathcal{L}[f(\vartheta)] = \frac{1 + J(s)\exp(-\pi J(s)/4)}{J^2(s) + 1}. \tag{63}$$

Thus, the fractal Laplace transform of the given function is given by the expression above.

Example 9 Consider the given fractal Laplace transform:

$$F(s) = \frac{1 - \exp(-2J(s))}{J^2(s)}. \tag{64}$$

To find the inverse fractal Laplace transform of Eq.(64), we express it as the difference of two terms:

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{1}{J^2(s)}\right] - \mathcal{L}^{-1}\left[\frac{\exp(-2J(s))}{J^2(s)}\right]. \tag{65}$$

Now, we apply the inverse fractal Laplace transform to each term separately. The inverse fractal Laplace transform of $\frac{1}{J^2(s)}$ is given by $J(\vartheta)$, and the inverse fractal Laplace transform of $\frac{\exp(-2J(s))}{J^2(s)}$ is given by $u_2(J(\vartheta))(J(\vartheta) - 2)$. Therefore, the inverse fractal Laplace transform of Eq.(64) is:

$$\mathcal{L}^{-1}[F(s)] = J(\vartheta) - u_2(J(\vartheta))(J(\vartheta) - 2). \tag{66}$$

This completes the computation of the inverse fractal Laplace transform.

The following theorem establishes a relationship between the fractal Laplace transform of a function and the fractal Laplace transform of its exponential with respect to a constant parameter.

Proposition 4 *Let $f(\vartheta) : F \to \mathbb{R}$ and $a \geq 0$. If $J(s) > a + c$, therefore the fractal Laplace transform of its exponential with respect to a positive constant c is*

$$\mathcal{L}[\exp(cJ(\vartheta)f(\vartheta))] = F(J(s) - c). \tag{67}$$

Conversely, we have:

$$\exp(cJ(\vartheta)f(\vartheta)) = \mathcal{L}^{-1}[F(J(s) - c)]. \tag{68}$$

In Eq. (67), the theorem states that the fractal Laplace transform of $\exp(cJ(\vartheta)f(\vartheta))$ with respect to ϑ is equal to $F(J(s) - c)$, where $F(s)$ is the fractal Laplace transform of the function $f(\vartheta)$. Conversely, it states that the function $\exp(cJ(\vartheta)f(\vartheta))$ can be obtained by taking the inverse fractal Laplace transform of $F(J(s) - c)$. However, it is essential to note that these results hold under the condition that $J(s) > a + c$.

Proof To proof this proposition, we require merely the evaluation of

$$\begin{aligned}\mathcal{L}[\exp(cJ(\vartheta)f(\vartheta)] &= \int_{\mathfrak{C}(0,\infty)} \exp(-J(\vartheta)J(s))\exp(cJ(\vartheta))f(\vartheta)d_F^{\alpha}\vartheta \\ &= \int_{\mathfrak{C}(0,\infty)} \exp(-(J(s)-c)J(\vartheta)f(\vartheta)d_F^{\alpha}\vartheta\end{aligned} \tag{69}$$

$$= F(J(s)-c) \tag{70}$$

We finish the proof by taking the inverse fractal Laplace transform of Eq.(67). □

Example 10 Consider the following fractal differential equation with initial conditions:

$$(D_F^{\alpha})^2 f(\vartheta) + 4f(\vartheta) = \frac{u_5(J(\vartheta))(J(\vartheta)-5) - u_{10}(J(\vartheta))(J(\vartheta)-10)}{5}, \tag{71}$$

with $f(0) = 0$ and $D_F^{\alpha} f(\vartheta)|_{\vartheta=0} = 0$. To find the solution of Eq.(71), we take the fractal Laplace transform of both sides of the equation and apply the initial conditions. This leads to the following equation in terms of then fractal Laplace transform $F(s) = \mathcal{L}[f(\vartheta)]$:

$$(J^2(s)+4)F(s) = \frac{\exp(-5J(s)) - \exp(-10J(s))}{5J^2(s)}. \tag{72}$$

By taking the inverse fractal Laplace transform, we find the solution as follows:

$$f(\vartheta) = \frac{[u5(J(\vartheta))h(J(\vartheta)-5) - u_{10}(J(\vartheta))h(J(\vartheta)-10)]}{5}, \tag{73}$$

where $h(\vartheta) = \frac{1}{4}J(\vartheta) - \frac{1}{8}\sin(2J(\vartheta))$.

The following definition introduces the concept of a fractal integral.

Definition 25 The fractal integral $I(\tau)$ is defined as follows:

$$I(\tau) = \int_{\mathfrak{C}(\vartheta_0-\tau,\vartheta_0+\tau)} g(\vartheta)d_F^{\alpha}\vartheta, \tag{74}$$

where $g(\vartheta)$ is a function defined as:

$$g(\vartheta) = d_\tau(\vartheta) = \begin{cases} \frac{1}{2\tau}, & -\tau < J(\vartheta) < \tau; \\ 0, & J(\vartheta) \leq -\tau \text{ or } J(\vartheta) \geq \tau. \end{cases} \tag{75}$$

Remark 5 We observe from Definition 25 that as τ approaches 0, the function $d_\tau(\vartheta)$ tends to 0 for $J(\vartheta) \neq 0$. However, for each non-zero τ, the fractal integral $I(\tau)$ evaluates to 1. Therefore, it follows that as τ tends to 0, the limit of $I(\tau)$ remains 1:

$$\mathbf{F}\_\underset{\tau\to 0}{lim}\, d_\tau(\vartheta) = 0, \quad J(\vartheta) \neq 0, \tag{76}$$

and

$$\mathbf{F}\_\underset{\tau\to 0}{lim}\, I(\tau) = 1 \tag{77}$$

Definition 26 The fractal delta Dirac function is a special function defined as follows:

$$\begin{aligned} \delta(J(\vartheta) - J(\vartheta_0)) = 0, \qquad J(\vartheta) \neq J(\vartheta_0), \\ \int_{\mathfrak{C}(-\infty,\infty)} \delta(J(\vartheta) - J(\vartheta_0)) d_F^\alpha \vartheta = 1, \qquad \vartheta \in \mathbf{F}. \end{aligned} \tag{78}$$

The fractal delta Dirac function is zero for all points ϑ where the fractal dimension $J(\vartheta)$ is not equal to the specified value $J(\vartheta_0)$. However, when integrated over the entire fractal set $\mathbf{F}$, it evaluates to 1.

Proposition 5 *The fractal Laplace transform of the function* $d_\tau(J(\vartheta) - J(\vartheta_0))$ *is given by:*

$$\mathcal{L}[d_\tau(J(\vartheta) - J(\vartheta_0))] = \frac{1}{J(s)\tau} \sinh(J(s)\tau) \exp(-J(s)J(\vartheta_0)) \tag{79}$$

This result shows the transformation of the fractal delta Dirac function, centered at $J(\vartheta_0)$, under the fractal Laplace transform operation. The resulting fractal Laplace transform involves a hyperbolic sine function and an exponential term, with the parameter τ determining the width of the distribution.

Proof To prove this proposition, we begin by evaluating the fractal Laplace transform of the function $d_\tau(J(\vartheta) - J(\vartheta_0))$. Using Definition 25, we have:

$$\begin{aligned} \mathcal{L}[d_\tau(J(\vartheta) - J(\vartheta_0))] &= \int_{\mathfrak{C}(0,\infty)} \exp(-J(s)J(\vartheta)) d_\tau(J(\vartheta) - J(\vartheta_0)) d_F^\alpha \vartheta \\ &= \int_{\mathfrak{C}(\vartheta_0-\tau,\vartheta_0+\tau)} \exp(-J(s)J(\vartheta)) d_\tau(J(\vartheta) - J(\vartheta_0)) d_F^\alpha \vartheta \end{aligned} \tag{80}$$

Substituting for $d_\tau(J(\vartheta) - J(\vartheta_0))$ from Eq.(75), we further simplify the expression:

$$\begin{aligned}\mathcal{L}[d_\tau(J(\vartheta)-J(\vartheta_0))] &= \frac{1}{2\tau}\int_{\mathfrak{C}(\vartheta_0-\tau,\vartheta_0+\tau)} \exp(-J(s)J(\vartheta))d_F^\alpha \\ &= -\frac{1}{2J(s)\tau}\exp(-J(s)J(\vartheta))\Big|_{\vartheta=\vartheta_0-\tau}^{\vartheta=\vartheta_0+\tau} \\ &= \frac{1}{2\tau J(s)}\exp(-J(s)J(\vartheta_0))(\exp(J(s)\tau)-\exp(-J(s)\tau)) \\ &= \frac{1}{J(s)\tau}\sinh(J(s)\tau)\exp(-J(s)J(\vartheta_0)).\end{aligned} \tag{81}$$

Simplifying further, we arrive at the result:

$$\mathcal{L}[d_\tau(J(\vartheta)-J(\vartheta_0))] = \frac{1}{J(s)\tau}\sinh(J(s)\tau)\exp(-J(s)J(\vartheta_0)). \tag{82}$$

This completes the proof. □

Remark 6 We observe from Corollary 5 that the fractal Laplace transform of the fractal delta Dirac function $\delta(J(\vartheta)-J(\vartheta_0))$ is given by:

$$\begin{aligned}\mathcal{L}[\delta(J(\vartheta)-J(\vartheta_0))] &= \mathbf{F}\_\underset{\tau\to 0}{lim}\ \mathcal{L}[d_\tau(J(\vartheta)-J(\vartheta_0))] \\ &= \exp(-J(s)J(\vartheta_0)),\end{aligned} \tag{83}$$

Similarly, the fractal Laplace transform of the fractal delta Dirac function $\delta(J(\vartheta))$ can be found as follows:

$$\mathcal{L}[\delta(J(\vartheta))] = \mathbf{F}\_\underset{\vartheta_0\to 0}{lim}\ \exp(-J(s)J(\vartheta_0)) = 1. \tag{84}$$

Thus, we conclude that the fractal Laplace transform of the fractal delta Dirac function is $\exp(-J(s)J(\vartheta_0))$ when $\vartheta \neq \vartheta_0$, and it is equal to 1 when $\vartheta = \vartheta_0$.

Remark 7 A function f that is $\mathbf{F}$-continuous, the integral of the fractal delta Dirac function $\delta(J(\vartheta)-J(\vartheta_0))$ multiplied by $f(\vartheta)$ over the fractal domain $\mathfrak{C}(-\infty,\infty)$ is equal to $f(\vartheta_0)$, which can be restated as follows:

$$\int_{\mathfrak{C}(-\infty,\infty)} \delta(J(\vartheta)-J(\vartheta_0))f(\vartheta)d_F^\alpha\vartheta = f(\vartheta_0), \quad \text{for } \vartheta_0\in\mathbf{F}, \tag{85}$$

where $f(\vartheta_0)$ is the value of the function f at the specific fractal point ϑ_0.

In simpler terms, the integral of the fractal delta Dirac function over the entire fractal domain picks out the value of the function f at the point ϑ_0, which is the same as saying that the integral evaluates to $f(\vartheta_0)$. We start from the left-hand side of Eq. (85), which is given by:

$$\int_{\mathfrak{C}(-\infty,\infty)} \delta(J(\vartheta)-J(\vartheta_0))f(\vartheta)d_F^\alpha\vartheta = \mathbf{F}\text{-}\lim_{\tau\to 0}\int \mathfrak{C}(-\infty,\infty)[d_\tau(J(\vartheta)-J(\vartheta_0))]f(\vartheta)d_F^\alpha\vartheta. \tag{86}$$

Using Definition 25, and the fractal mean value theorem for integrals, we can write:

$$\int_{\mathfrak{C}(-\infty,\infty)} [d_\tau(J(\vartheta)-J(\vartheta_0))]f(\vartheta)d_F^\alpha\vartheta = \frac{1}{2\tau}\int_{\mathfrak{C}(\vartheta_0-\tau,\vartheta_0+\tau)} f(\vartheta)d_F^\alpha\vartheta. \tag{87}$$

The fractal mean value theorem guarantees the existence of a point ϑ in the interval $(\vartheta_0-\tau,\vartheta_0+\tau)$ where the function $f(\vartheta)$ attains its average value over the interval. Therefore, we have $f(\vartheta)$ as the average value of $f(\vartheta)$ within the interval.

As τ approaches zero, the interval becomes infinitesimally small, and ϑ converges to ϑ_0. Thus, we can write $\vartheta\to\vartheta_0$ as $\tau\to 0$.

Taking the limit as τ tends to zero in the expression, we obtain:

$$\int_{\mathfrak{C}(-\infty,\infty)} \delta(J(\vartheta)-J(\vartheta_0))f(\vartheta)d_F^\alpha\vartheta = f(\vartheta_0). \tag{88}$$

Therefore, the integral of the fractal delta Dirac function over the entire fractal domain $\mathfrak{C}(-\infty,\infty)$ of a **F**-continuous function $f(\vartheta)$ is equal to the value of $f(\vartheta)$ at the point ϑ_0.

Proposition 6 *If both $F(s)=\mathcal{L}[f(\vartheta)]$ and $G(s)=\mathcal{L}[g(\vartheta)]$ exist for $J(s)>a\geq 0$, then their product $H(s)=F(s)G(s)$ also exists for $J(s)>a$. Consequently, we can find the inverse fractal Laplace transform of H(s) to obtain a new function $h(\vartheta)$.*

Specifically, the function $h(\vartheta)$ is defined as follows:

$$h(\vartheta)=\int_{\mathfrak{C}(0,\vartheta)} f(\vartheta-\tau)g(\tau)d_F^\alpha\tau = \int_{\mathfrak{C}(0,\vartheta)} f(\tau)g(\vartheta-\tau)d_F^\alpha\tau, \tag{89}$$

where $\vartheta\in\mathbf{F}$. Therefore, if the fractal Laplace transforms $F(s)$ and $G(s)$ exist for $J(s)>a\geq 0$, then their product $H(s)$ will also have a fractal Laplace transform, and its corresponding inverse fractal Laplace transform will be given by the function $h(\vartheta)$ defined in Eq. (89).

Proof To prove this proposition, let us start by considering the fractal Laplace transforms of the given functions. Suppose we have the fractal Laplace transform of $f(\vartheta)$ as $F(s)$ and the fractal Laplace transform of $g(\vartheta)$ as $G(s)$. Then, the product of $F(s)$ and $G(s)$ is given by:

$$F(s)G(s)=\int_{\mathfrak{C}(0,\infty)} \exp(-J(s)J(\zeta))f(\zeta)d_F^\alpha\zeta\int_{\mathfrak{C}(0,\infty)} \exp(-J(s)J(\eta))g(\eta)d_F^\alpha\eta. \tag{90}$$

Now, we can rearrange the integrals and use a change of variables to simplify the expression. First, let's introduce new variables $\zeta=\vartheta-\eta$ for fixed η. This gives us:

$$F(s)G(s) = \int_{\mathfrak{C}(0,\infty)} g(\eta) d_F^\alpha \eta \int_{\mathfrak{C}(0,\infty)} \exp(-J(s)(J(\eta) + J(\zeta))) f(\zeta) d_F^\alpha \zeta. \tag{91}$$

Next, we set $\eta = \tau$, and now the expression becomes:

$$F(s)G(s) = \int_{\mathfrak{C}(0,\infty)} g(\tau) d_F^\alpha \tau \int_{\mathfrak{C}(\tau,\infty)} \exp(-J(s)(J(\vartheta)) f(\vartheta - \tau) d_F^\alpha \vartheta. \tag{92}$$

We can now reverse the order of integration to get:

$$F(s)G(s) = \int_{\mathfrak{C}(0,\infty)} \exp(-J(s)(J(\vartheta))) d_F^\alpha \vartheta \int_{\mathfrak{C}(0,\vartheta)} f(\vartheta - \tau) g(\tau) d_F^\alpha \tau = \mathcal{L}[h(\vartheta)] \tag{93}$$

Thus, we have shown that the product of $F(s)$ and $G(s)$ can be expressed as the fractal Laplace transform of a new function $h(\vartheta)$, given by:

$$h(\vartheta) = \int_{\mathfrak{C}(0,\vartheta)} f(\vartheta - \tau) g(\tau) d_F^\alpha \tau = \int_{\mathfrak{C}(0,\vartheta)} f(\tau) g(\vartheta - \tau) d_F^\alpha \tau. \tag{94}$$

Therefore, when the fractal Laplace transforms $F(s)$ and $G(s)$ exist for $J(s) > a \geq 0$, the product $H(s) = F(s)G(s)$ also has a fractal Laplace transform, and it corresponds to the function $h(\vartheta)$ as given above. □

Example 11 Consider the fractal differential equation:

$$2(D_F^\alpha)^2 f(\vartheta) + D_F^\alpha f(\vartheta) + 2f(\vartheta) = \delta(J(\vartheta) - 5), \tag{95}$$

with the initial conditions:

$$f(0) = 0, \quad D_F^\alpha f(\vartheta)\Big|_{\vartheta=0} = 0. \tag{96}$$

To solve this example, we apply the fractal Laplace transform to both sides of Eq. (95) and use the initial conditions (96). Thus, we obtain:

$$(2J^2(s) + J(s) + 2)Y(s) = \exp(-5J(s)), \tag{97}$$

and by solving Eq. (97) for $Y(s)$, we have:

$$Y(s) = \frac{\exp(-5J(s))}{2J^2(s) + J(s) + 2} = \frac{\exp(-5J(s))}{2} \frac{1}{(J(s) + \frac{1}{4})^2 + \frac{15}{16}}. \tag{98}$$

Taking the inverse fractal Laplace transform, we obtain:

$$f(\vartheta) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{\exp(-5J(s))}{2}\frac{1}{(J(s)+\frac{1}{4})^2+\frac{15}{16}}\right]$$
$$= \frac{2}{\sqrt{15}}u_5(J(\vartheta))\exp(-\frac{(J(\vartheta)-5)}{4})\sin\frac{\sqrt{15}}{4}(J(\vartheta)-5) \tag{99}$$

which is the formal solution of Eq. (95). In Fig. 2, we have plotted the function $u_{0.6}(J(\vartheta))\exp(J(\vartheta))\sin(J(\vartheta))$.

4 Conclusion

This work has presented the fractal Laplace transform as a valuable tool for solving fractal differential equations with constant coefficients. By applying this transform, we can convert complex fractal differential equations into simpler algebraic forms, facilitating their solution. The inverse fractal Laplace transform has been instrumental in finding solutions to these transformed equations, providing valuable insights into the behavior of fractal systems. A table of important fractal Laplace transforms has been provided, offering practical utility when solving fractal differential equations in various applications. These transforms serve as a valuable resource for researchers in the field, enabling them to efficiently address challenges involving

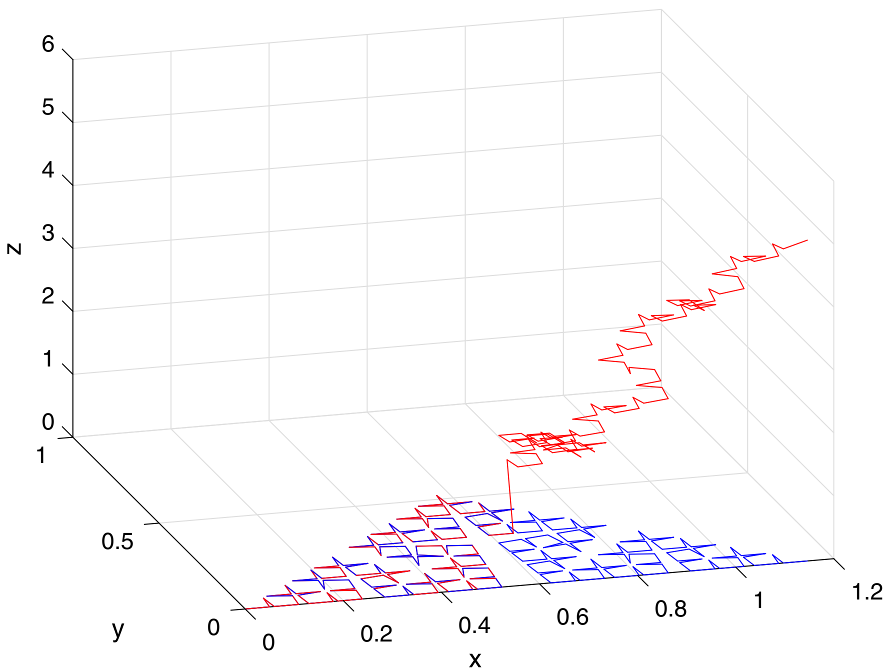

Fig. 2 Graph of the function $u_{0.6}(J(\vartheta))\exp(J(\vartheta))\sin(J(\vartheta))$

fractal dynamics. Furthermore, the introduction of fractal convolution integrals allows for the modeling of systems that depend not only on their present time but also on their past history. Such systems are commonly observed in diverse fields, including neutron transport, viscoelasticity, and population dynamics. This work enhances our understanding of fractal calculus and its applications in solving complex problems in the realm of fractal functions and systems. It opens new avenues for research and practical implementations in various scientific and engineering domains.

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Declarations

Conflict of interest The authors declare that they do not have any competing interests.

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