Reward-risk efficiency in proportional reinsurance with different risk measures

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Abstract. We have studied, in particular under normality of the implied random variables, the connections between different measures of risk such as the standard deviation, the W-ruin probability and the p-V@R. We discuss conditions granting the equivalence of these measures with respect to risk preference relations and the equivalence of dominance and efficiency of risk-reward criteria involving these measures. Then more specifically we applied these concepts to rigorously face the problem of finding the efficient set of de Finetti's variable quota share proportional reinsurance.

Keywords. Risk measures; Reward-risk efficiency; variable quota share proportional reinsurance; group correlation.

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1. Introduction

It has been recently recognized by leading scientists (Rubinstein 2006, Markowitz 2006) that B. de Finetti (de Finetti 1940) was, well before the celebrated papers by the Nobel Prize H. Markowitz (1952), (1956), (1959), the first to apply the mean variance approach in the theory of modern finance in order to solve a variable quota share proportional reinsurance problem.

A careful reading of de Finetti's paper reveals that in his paper there is much more to be found: the seed of a general quite modern reward risk approach to the theory of financial decisions under uncertainty, both in a single period and in a multiperiod framework.

Indeed de Finetti's initial idea was to work in a framework of mean-W ruin probability efficiency. This was coherent with the idea, widely shared at that time in insurance circles, that the W ruin probability (i.e. the probability of an insurance company to be bankrupt owing to a loss greater or equal to its free capital W), should be considered as the proper risk measure of any decision regarding insurance or reinsurance companies.

Nevertheless de Finetti recognized that the mean variance approach could turn out to be quite easier to manage for computational purposes. Then his strategy was at first to affirm the equivalence (at least concerning his specific reinsurance problem) between the mean-ruin probability and the mean-variance efficient sets, and only as a second step to look for solutions of the mean variance problem. In this second step he was able to obtain an extraordinary result, that is to develop an intuitive (let us say friendly) sequential procedure (which reveals to be nothing but the insurance counterpart of the celebrated critical line algorithm of Markowitz) to find the set of efficient retentions in a mean-variance framework. As said before the importance of this path breaking contribution gained recently top level recognition; on the contrary his key suggestion of the connection between ruin probability and variance as measures of risk has been almost completely neglected.

Here we wish to fill this gap and provide a careful analysis of the question at first in general and later with specific reference to the variable quota share reinsurance problem treated by de Finetti. To face the problem we will make large recourse to geometric considerations in a mean-standard deviation plane.

On the way we found some unexpected results. Indeed the old fashion ruin probability risk measure reveals to be a sort of dual of the modern V@R (Artzner *et al.* 1999). At the same time the ruin probability turns out to be the insurance version of the so called parametric shortfall measures, which express the probability of shortfalling a given target, a concept which goes back to Roy (Roy 1952). In turn, a uniform shortfall preference for X over Y, holding for any non positive target W (rather than for any real target), could be seen as a case of first order semistochastic dominance. First order semistochastic dominance has been introduced in Hadar-Russell (1969).

With reference to the de Finetti's problem and under his hypothesis, the main results we found concerning the efficient single period retention strategies are the following: a) the mean-W shortfall efficient set is robust to changes in W (it is the same for any W>0); b) the mean-W shortfall robust efficient set is the same as the mean standard deviation efficient one; c) the mean p V@R efficient set is only partially robust (it is the same only for an interval of p values lower than a given upper bound); d) in this interval is again the same as the mean standard deviation efficient one; e) enlarging the interval has the effect of cutting (a lower) part of the efficient frontier.

The plane of the paper is as follows: section 2 is devoted to a short recall of de Finetti's variable quota share proportional reinsurance problem and of the main results obtained concerning the efficient retentions. In section 3 a quick resume of a general approach to reward risk analysis in decision theory with a glance to applications to the proportional reinsurance problem is given. In section 4 with the help of a graphical interpretation in a mean standard deviation plane, we discuss the properties of the shortfall risk measure, with special attention to its robustness to changes in the W parameter. Section 5 discusses the connections and in particular the substitutability of standard deviation and shortfall rim based efficient sets. These concepts are applied in section 6 to give a rigorous proof of de Finetti's intuition of the equivalence between standard deviation and shortfall rim based efficient sets in the proportional reinsurance problem. In section 7 the idea of a duality between p V@R and W shortfall risk measures is introduced and shortly discussed. Section 8 offers fully (at the best of our knowledge) original results concerning the mean-p V@R efficient set in the proportional reinsurance problem. A short section 9 resumes the conclusions of the paper.

2. The variable quota share proportional reinsurance problem in de Finetti's approach

Let us briefly recall the essentials of a proportional reinsurance problem. An insurance company is faced with n risks (policies). The net profit, that is the difference between net premiums and losses, of these risks is described by a vector of random variables with expected values **m**>0, and a non-singular covariance matrix C. We denote the (i,j) entry of the matrix C as σ_{ij}. The diagonal elements are denoted also as $\sigma^2_i=\sigma_{ii}$, and the (i,j) entry may be alternatively denoted as $\sigma_{ij}=\rho_{ij}\sigma_i\sigma_j$ with ρ_{ij} the correlation coefficient.

The company has to choose a proportional reinsurance or retention strategy specified by a retention vector **x**. The retention is feasible if $0\leq\mathbf{x}\leq1$. By applying reinsurance on original terms, a retention **x** induces a random profit with expectation E= $\mathbf{x}^T \mathbf{m}$ and variance V=$\mathbf{x}^T C\mathbf{x}$. How to choose **x**?

De Finetti was the first to argue that an integrated reward risk approach should be applied to find good retention strategies. There was no doubt that the expectation of the random profit after retention should be the proper reward measure; as for the risk measure, both practice and theory of insurance make at that time use of the ruin probability, instead of the variance, most popular in other economic and statistical applications. De Finetti realized that the variance was more easily handled for computational purposes than the ruin probability. Then he decided to work in a mean-variance framework and was able to obtain his famous path breaking results. Yet previously he troubled about the equivalence of the two risk measures, and claimed that in his problem such equivalence was guaranteed at least with reference to the efficient mean-risk retentions in a single period problem. Finally he went back to the ruin probability measure in combination with the expectation to solve the optimal retention problem. His proposal was to select an upper bound of acceptable ruin probability and choose the point of the efficient mean variance set with the largest expectation conditional on such given upper bound. These ideas were ahead of time of most of the modern ideas of reward risk analysis in finance. Following these suggestions we will try here to gain new insights on the connections between different risk measures: at first in general and then with reference to the specific properties of the mean-risk efficient set of the proportional reinsurance problem.

3. A reward risk approach to reinsurance decisions

Let us give a systematic treatment of the rather informal approach suggested by de Finetti to face the variable quota share single period reinsurance problem. Let X^+ be a family of normal random variables (with strictly positive mean and standard deviation) completed with the constant 0, and W a positive constant. A reward risk approach to decisions under uncertainty for a set of problems involving such restrictions (including the variable quota share single period reinsurance problem) comes as follows.

For any X of X^+ define a reward measure (rem) and a risk measure (rim).

Given a reward measure and respectively a risk measure, for each pair X,Y of X^+ define in the usual way a weak reward preference (rep) and respectively a weak risk preference (rip).

Accordingly X is weakly reward preferred to Y if rem(X) is greater or equal to rem(Y), while X is weakly risk preferred to Y if rim(X) is lesser or equal than rim(Y).

On the basis of the given (rep,rip) couple corresponding to a given (rem,rim) couple, define the usual dominance relation: Y is dominated by X if X is weakly both rep and rip preferred to Y with at least a strong preference. Of course there may well be couples X,Y of variables such that no dominance holds.

Finally given the (rem,rim) couple, define a feasible X as (rem,rim) efficient iff there is no other feasible Y that dominates X. Given X^+ and the (rem,rim) measures we will denote by $X^*(X^+)$ the set of all the (rem,rim) efficient X.

The overwhelming important rem is the mean (expectation) m(X). As regards the rim, usually the standard deviation σ(X) (or its equivalent twin the variance) V(X) is used. Here we will use also the t(X;W) or shortly t_X measure, $t(X;W)=\sigma_X/(W+m_X)$ which is a parametric risk measure involving, besides relevant parameters of the X distribution, also a parameter W, exogeneous to the distribution.

The rationale for the choice of this measure is that, under our assumptions, the probability of shortfalling –W (or the equivalent, for continuous distributions like the normal one, probability of not beating –W), is smaller (equal) for X than for Y iff t(X;W) is smaller (equal) than t(Y;W). Indeed under normality $p(X \leq -W)=F_X(-W)=F_Z(-1/t_X)$, with F_Z the distribution function of the standard normal. It is natural to label this rim as *W shortfall* or also *W ruin*. Indeed not beating –W means $W+X\leq 0$, an event which could be seen as the single period ruin (coming from X) of an insurance company (or more generally of a decision maker) with initial free wealth W.

For the large part of the paper we concentrate on standard deviation and W shortfall as two alternative rim; in the final part of the paper we will show that a third rim (the well known V@R) is strongly connected (exhibits duality properties) with the shortfall measure.

4. A graphical representation in a mean standard deviation plane

Suppose each feasible random variable is associated to a point of the mean-standard deviation plane, so as X has coordinates (m_X,σ_X). It is well known that Y mean-standard deviation dominates X if it is weakly preferred both on mean ($m_Y \geq m_X$) and on standard deviation ($\sigma_Y \leq \sigma_X$) with at least one strict inequality. Graphically X is then efficient iff there are no other feasible Y dominating X, that is lying in the stripe defined by $m\geq m_X$, $0\leq\sigma \leq \sigma_X$.

Let us now consider W shortfall preferences for given couples (X,Y) and suppose the labelling is such that $\sigma_Y \geq \sigma_X$. Note that this way the point representative of X is confined to the stripe $m\geq 0$, $0\leq s\leq \ _Y$. In turn this stripe may be divided in 3 zones, labelled zone A, B and C respectively. In zone A $m_X \geq m_Y$ while the union of B and C is the rectangle in the stripe where: $0\leq m_X< m_Y$; moreover B is the part of this rectangle below the line connecting the origin with the point Y; C is the sector above this line, the B-C frontier (boundary) is just on the line.

The following results hold zone by zone.

R1. X in zone A: X is W shortfall preferred to Y for any W such that $W+m_X$ and a fortiori $W+m_Y$ are positive. Owing to the assumptions this is surely true for any positive W, so we will say that (in this zone) the W shortfall rim preference is robust (in the sense that it holds for any positive W) with respect to W.

Proof: immediate consequence of the fact that in this zone

$0<t(X;W)= \sigma_X/(W+m_X)< t(Y;W)= \sigma_Y/(W+m_Y)$. ■

Keeping account that X (in zone A) is mean preferred to Y, and robustly (for any W) and strictly shortfall preferred to Y, it follows that:

R2. any X in zone A mean-shortfall dominates Y; the result is robust to changes in W. ■

Things are not so obvious in the other zones. To understand what happens, let us consider the intersection of the straight line connecting X and Y with the horizontal axis.

Let us denote by $-W(X,Y)$ the abscissa of this intersection point. Note that $-W(X,Y)<0$ $(W>0)$ if X in zone C, $-W(X,Y)= 0$ for X on the B-C boundary, $-W(X,Y) > 0$ $(W<0)$ for X in zone B.

The following results hold under our assumptions:

R3. The straight line connecting X and Y is the W iso shortfall line for the value W(X,Y), that is the set of all points representing random variables with the same probability p of shortfalling W(X,Y).

Proof. $W^*=W(X,Y)$ is the solution of $\sigma_X/(W+m_X)= \sigma_Y/(W+m_Y)$ that is

$W^*=(\sigma_X\, m_Y - \sigma_Y\, m_X)/(\sigma_Y - \sigma_X)$.

Then $-(W^*+ m_X)/\sigma_X = ((m_X - m_Y)/(\sigma_Y - \sigma_X))= -(W^*+ m)/\sigma$ for any point on the W* iso shortfall line. ■

Result R3 implies that there is equal W shortfall preference between X and Y when W=W(X,Y).

R4. This probability is precisely

$p^*=p(W^*)=p_X(W(X,Y))=p_Y(W(X,Y))=F_Z((m_X - m_Y)/(\sigma_Y - \sigma_X))<1/2$.

Proof: it is $p^*=p(X\leq -W^*)=F_X(-W^*)= F_Z(-(W^*+ m_X)/\sigma_X)$. ■

Now choose $-W'<-W(X,Y)$ and consider the straight line connecting (-W',0) and X; let us call Y' the random variable (point) at the intersection of the straight line and the vertical line through Y. Alternatively choose $m_X> -W''> -W(X,Y)$ and consider the straight line connecting (-W",0) and X; let us call Y" the random variable (point) at the intersection of the straight line and the vertical line through Y; then

R5. the standard deviation of Y' is lower than that of Y. ■

R6 the standard deviation of Y" is greater than that of Y. ■

R7. In turn R5 implies that there is equal W shortfall preference between X and Y' when W=W'=W(X,Y') and strict W shortfall preference of Y' over Y (same mean and greater standard deviation than Y'); hence by transitivity of preferences strict W shortfall preference of X over Y when W=W'. ■

R8. Moreover R6 by the same logic implies that there is equal W shortfall preference between X and Y" when W=W"=W(X,Y") and hence strict W shortfall preference of Y over X (same mean and lower standard deviation than Y') when W=W". ■

After that for X in zone B it is:

R9. $-W(X,Y)\geq 0$ ■

Hence immediately for any $-W'<0$ ($W'>0$) R5 ad R7 imply:

R10. X is still W shortfall preferred to Y for any positive W, so still the W shortfall rim preference is robust with respect to W. ■

Keeping account that now Y is strictly mean preferred to X, it follows that:

R11. there is no dominance between X (in this zone) and Y. Still the result is robust to changes in W. ■

For X in zone C it is:

R12. $-W(X,Y)<0$. ■

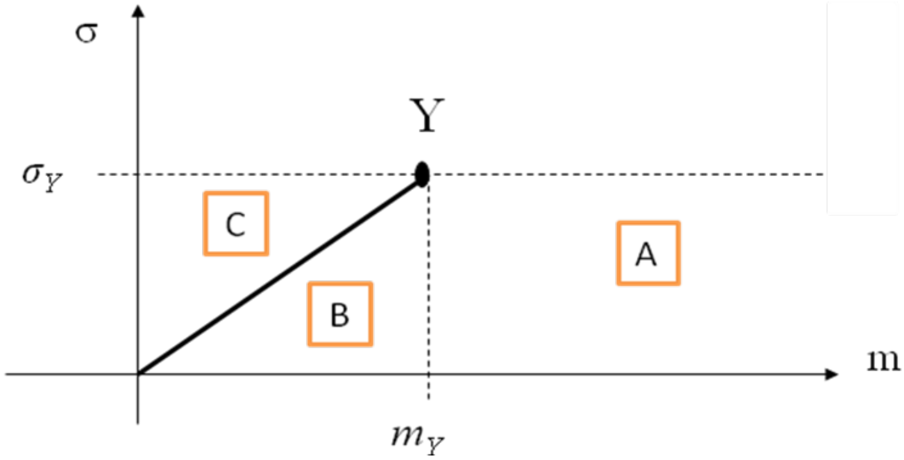
Then

R13. X is W-shortfall preferred to Y only for any $-W<-W(X,Y)$, that is $W>W(X,Y)$, whereas for $0<W<W(X,Y)$ on the contrary Y is W-shortfall preferred to X. For $W=W(X,Y)$, X and Y are indifferent. W shortfall preferences are no more robust: they depend on W. ■

R14. The same happens for dominance. Keeping account that Y is strictly mean preferred to X, on the interval ($0<W<=W(X,Y)$) X is dominated by Y. No dominance is found for $W>W(X,Y)$. This lack of robustness is stressed by the fact that W(X,Y) is X specific. ■

In conclusion it turns out that robust shortfall preferences and robust mean-shortfall dominance relations may be found only in zones A and B; in particular Y is robustly mean shortfall dominated by X in zone A, while no dominance holds between X, in zone B or on the BC frontier, and Y. It is interesting to note that in both zones the dominance relations are coherent with those found for the (surely robust) mean-standard deviation (or mean-variance) rem-rim. As we said, in zone C, W shortfall preferences are no more robust, which implies no robust mean shortfall dominance relations.

Graph 1: shortfall preference and mean-shortfall dominance relations.



Zona A: robust preference and dominance of X over Y.

Zona B: robust preference of X over Y; no dominance.

Zona C: neither robust preferences nor robust dominance.

5. The substitutability of standard deviation and shortfall rim based efficient sets

In what precedes we found that iff in the feasible set there are couples of points X,Y, such that the straight line connecting X and Y cuts the horizontal axis at a point of negative abscissa, there is no hope to have a robust shortfall rim preference relation, hence no robust agreement of the two rim (standard deviation and shortfall) preference relations and no agreed robust rem rim dominance relation between X and Y.

At first sight this sounds as an unpleasant negative result in the substitutability of the two rim measures kept in consideration. But luckily the robustness and the substitutability may be recovered (at least under convenient conditions) at the level of the efficient set. In detail under proper conditions the rem,rim efficient set of a problem could be the same (robust) set under both rim.

We will show that the following fundamental result holds:

T1. Suppose the mean-standard deviation efficient set (frontier) $X^*(X^+)$ of (X^+) includes the origin and is a convex curve connecting the origin (0,0) with the point of highest expectation P_{ME}(max m, σ(max m)), then X^* is surely also the mean-shortfall robust (for any W>0) efficient set.

Proof.

Let us at first prove that a mean standard deviation efficient point is surely also mean-shortfall efficient in a robust sense. Let X be a point of the mean standard deviation efficient frontier, and consider any other point Y of this frontier with greater mean (and of course greater standard deviation). By the convexity of the frontier the straight line connecting Y to the origin lies above the point X, which then belongs to the B zone (of course Y specific). Hence by R11 X is not dominated by any other point Y (with greater mean) of the frontier. Note that, implying the B zone, this is a robust argument. On the other side of course X is surely not dominated by any point Z of the frontier with lower mean. Hence there is no mean-shortfall dominance involving points of the mean-standard efficient frontier. Moreover it is easy to check that if X is not dominated by any efficient frontier point Y (with greater mean), it a fortiori cannot be dominated either by any other internal point Y', with the same standard deviation of Y, or by not efficient points Y" with standard deviation greater than σ (max m)).

To check this result think that the straight line connecting Y' (or Y") with the origin lies surely above the straight line connecting Y with the origin. Hence a fortiori X lies in the B zone also of these Y's. To sum up, a point on the mean-standard deviation efficient frontier cannot be mean-shortfall dominated, and this is a robust conclusion with respect to W.

It remains to show that a mean-standard deviation inefficient point is surely also mean-shortfall inefficient in a robust sense. Let Y' be a mean-standard deviation inefficient point, then by hypothesis, there is surely a feasible point X in the A zone of Y'. Previously we argued that Y' is mean-shortfall dominated by such X; and once again this (implying the A zone) is a robust result. ■

6. The coincidence of mean-standard deviation and mean-shortfall efficient sets in de Finetti's variable quota share proportional reinsurance problem

Now we show that the hypothesis of the T1 are satisfied by de Finetti's classical problem of variable quota share proportional reinsurance on original terms. Precisely we will show that the mean-standard deviation efficient set (frontier) $X^*(X^+)$ of de Finetti's problem is a continuous convex increasing curve, connecting the origin with the point of largest expectation P_{ME}, which is the point of full retention.

To obtain this result we exploit the properties of the mean variance efficient frontier of that problem, properties guessed by de Finetti (without giving a formal proof), recently rigorously precised for the general case by Pressacco-Serafini (2007 sect. 6-7) and discussed in the particular case of group correlation by Pressacco, Serafini, Ziani (2011 sect 5). Let us quickly recall these properties.

In a mean variance plane the mean-variance rem-rim efficient set is a continuous convex union of parabolae. The set starts from the origin with zero derivative, and is almost everywhere differentiable (that is without kinks at the connection points, except possibly in a finite very small number of such points, corresponding to the vertices of the feasible unitary hypercube in the space of retentions). The first derivative is continuously increasing (except at kink points where the derivative has an upward jump).

Going now from the mean variance space to the mean standard deviation one let us denote the equation of any arc of the efficient standard deviation set by

$$\sigma = V^{1/2} = (a_k m^2 + b_k m + c_k)^{1/2}$$

Remark. The triplet a_k, b_k, c_k is arc specific and in particular the first arc going out from the origin has b=c=0 i.e. it is $\sigma = m \cdot a_k^{1/2}$.

Computing derivatives we have:

$$\partial\sigma/\partial m = (\partial V/\partial m) / 2\sigma$$

and

$$\partial^2\sigma/\partial m^2 = [2\cdot(\partial^2 V/\partial m^2)\cdot\sigma - (\partial V/\partial m)^2 / \sigma] / 2V$$

so

$$\partial^2\sigma/\partial m^2 = [2\cdot(\partial^2 V/\partial m^2)\cdot V - (\partial V/\partial m)^2] / 2\cdot V\cdot\sigma$$

Now the sign of the second derivative is that of the numerator, whose value is

$$2\cdot 2a_k\,(a_k m^2+b_k m+c_k) - (2a_k m+b_k)^2 = 4a_k c_k - b_k^2$$

that is the non negative (indeed positive except for the first parabola going out from the origin) vertical coordinate (variance) of the vertex of the corresponding parabola. This guarantees the convexity of each arc of the efficient set. Moreover it is straightforward to check that the convexity is granted also at the connection points. This is obvious by the growth of the first derivative if there is differentiability; if there is a kink we exploit the positive jump of the first derivative. ■

Thus we have given a rigorous proof of the following key de Finetti's intuition:

T2: in the de Finetti's variable quota share proportional reinsurance problem, the set of mean-standard deviation efficient retentions is coincident with the set of mean-shortfall efficient retentions. The result is robust, that is holds for any $W>0$.

7. The p V@R as a risk measure and its duality with the W shortfall measure

Let X be a continuous normal random variable with positive mean; for any $0<p<1$, the p V@R of X, $V@R_X(p)$ is implicitly defined as the opposite of the worst result that may happen to X with confidence $q=1-p$, or more formally the solution of $P(X\leq -V@R)=p$. Explicitly

$$V@R_X(p) = -F_X^{-1}(p)$$

In what follows we will often assume $p\leq ½$.

Under normality:

$p=P(X\leq -V@R)=P((X-m_X)/\ \sigma_X) \leq -(V@R_X(p) + m_X)/\ \sigma_X)=F_Z(-1/\square_X)$ or $F_Z^{-1}(p)=(-1/\square_X)$ where

$\square_X = \sigma_X /(V@R_X(p) + m_X)$.

It is convenient to write $-c(p)= F_Z^{-1}(p)$ so that $c=(V@R_X(p) + m_X)/\ \sigma_X$ and it is

$$V@R_X(p)=c(p)\,\sigma_X - m_X$$

Remark. The sophisticated pV@R measure could be seen as a parametric risk preference measure which reduces under normality to the naive index $c\sigma$-m, opposite of the naive reward preference index m-$c\sigma$. Connections between this parametric reward-preference index and stochastic dominance have been introduced and deeply discussed by Ogryczak-Ruszczynski (1999), (2002).

Remark. As $0<p\leq 1/2$, $-c(p)$ grows from $-\infty$ to zero or c goes from $+\infty$ to zero.

A comparison between $\square_X$ and $t_X=\sigma_X/(W+m_X)$ reveals immediately the strong connection between p V@R and W shortfall and makes clear that V@R may be seen also as the free capital needed to keep the shortfall probability bounded at the desired level p.

The following duality results hold:

T3. The $pV@R_X$ is just the inverse function of the W shortfall (ruin probability) function p_X (W).

Proof: define for $-m_X\leq W<+\infty$ the function $p_X(W)=P(X+W\leq 0)=F_X(-W)=G(W)$. As a function of W, p is monotone decreasing with the maximum value ½ at $W=- m_X$ and goes to the limit zero as W goes to $+\infty$. Then it admits inverse and the inverse, defined for any $0<p\leq 1/2$, is:

$$G^{-1}(G(W))=W = -F_X^{-1}\, F_X(-W)= -(-W).$$

On the other side $F_X(-W)=p$ and $V@R_X(p)=- F_X^{-1}(p)$. ■

More explicitly the duality relation is expressed as follows:

R15. In the intervals $-m_X\leq W<+\infty$ and $0<p\leq 1/2$:

$$p^*=p_X (W^*) \Leftrightarrow W^*= V@R_X (p^*);$$

verbally if p* is the W* shortfall probability of X, then W* is the p* V@R of X and viceversa. ■

Let us concentrate now on duality properties of preference relations between couples of r.v. X,Y (with the usual labelling $\sigma_Y \geq \sigma_X$ and with $m_Y \geq m_X$, otherwise, if $m_Y \leq m_X$, it is immediate to check that X is p-V@R preferred to Y for any p).

To understand results we will make recourse once again to a graphical representation in the mean-standard deviation plane.

Recall that for $p<1/2$, $c(p)=- F_Z^{-1}(p)$ is strictly positive. After that $\sigma=\sigma_X + c^{-1}(p)\,(m-m_X)$ is the equation of the p-X isoV@R, that is the equation of the line on which all r.v. sharing the same p V@R of X are found.

Remark. The slope of the line is $c^{-1}(p)$, strictly positive; r.v. lying on each parallel line above the p-X isoV@R have obviously a common greater p-V@R than X, r.v. lying on each parallel line below the X isoV@R have common lower p-V@R than X. Hence for a given $0<p<1/2$, Y is p V@R strictly preferred to X iff it lies below the X isoV@R.

R16. If the slope of the isoV@R is $c^{-1}(p) = (\sigma_Y - \sigma_X)/(m_Y - m_X)$, X and Y are p-V@R indifferent. For a greater slope Y is preferred; for a lower slope X is preferred. ■

Note that as –c is increasing with p, the same holds for $c^{-1}(p)$, revealing that the slope of the X isoV@R is increasing with p. Let p*(X,Y) such that $c^{-1}(p^*) = (\sigma_Y - \sigma_X)/(m_Y - m_X)$, that is concretely $p^*(X,Y) = F_Z[(m_X - m_Y)/(\sigma_Y - \sigma_X)]$then the following result hold:

T4. For 0<p<p*, X is p-V@R preferred to Y, for p*<p<1/2 Y is p-V@R preferred to X.

Proof. Exploit R3 and R4 of section 4. ■

Suppose now X is in zone B, then the straight line connecting X and Y cuts the x axis at a point of positive mean. We know that in this case, X is robustly (for any W) shortfall preferred to Y. For any 0<p<p* X is V@R preferred, and it is easy to check that p* is greater than $F_Y(0)$.

Remark. This reveals that p V@R preferences are not completely robust (they do not hold for any p, but only for any p up to p*). Anyway for any $0<p<\max(p^*_X, p^*_Y)$ and any W>0, risk preferences between X and Y are the same for both measures (W shortfall and p V@R).

For X in zone C, then the straight line connecting X and Y cuts the x axis at a point of negative mean –W*, which is the value such that X and Y are W-shortfall indifferent (have the same probability p* of shortfalling –W). For p<p*, X is V@R preferred, for p>p* (but lower than ½) Y is V@R preferred. On the other side, as shown in **R13**, for 0<W<W*, Y is shortfall preferred, whereas for W>W* X is preferred.

Then for X in zone C:

R 17. There is perfectly duality in W shortfall and p V@R preferences. ■

8. Efficiency in mean p V@R in de Finetti's problem

Recall that in de Finetti's problem there is perfect coincidence between the efficient mean-standard deviation set and the mean-shortfall set of retentions (cfr. T2). In the mean-standard deviation space the efficient set is a continuous convex set with a first linear part (let us say the first line) connecting the origin 0 with a point $Q(m, a^{1/2} m)$. As explained in the remark of section 6, a is the positive coefficient of the quadratic term of the first arc of parabola whose equation in the mean-variance plane is $V = a \cdot m^2$.

Now, the basis of the fundamental theorem concerning the coincidence, in the standard de Finetti's proportional reinsurance problem, of the mean V@R efficient set and the efficient mean - rim set of the other two risk measures (standard deviation and shortfall) comes from:

T5. To avoid dominance of the point Q over the other points of the first line it is necessary (see remark in section 7) that p is fixed at a level such that the slope of the p iso V@R lines is not greater than the slope $a^{-1/2}$ of the line. Precisely the upper bound is $p^*=F_Z(-a^{1/2})$.

Proof. Immediate by R4 of section 4. ■

Corollary: This is the upper bound of the probabilities granting no mean p-V@R dominance between points of the whole efficient mean standard deviation frontier.

Proof. Exploit the convexity of the remaining part of the efficient frontier, to be sure that no dominance may turn out involving other points of the mean-standard deviation efficient frontier. ■

Keeping account that of course internal points, or frontier not efficient points, are surely dominated by some point of the frontier for any (positive) slope of the iso V@R line, so that the efficiency of such points is excluded the following fundamental theorem is obtained.

T6. For any $0<p<p^*=F_Z(-a^{1/2})$, the efficient mean pV@R set of the de Finetti's proportional reinsurance problem is the same as the mean-standard deviation and the mean-shortfall efficient set. ■

Moreover, it is clear that if we wish to increase the probability upper bound, this may be done at the cost of cutting (a lower) part of the efficient frontier. Precisely:

T7. Given a point $Q(m, \sigma(m))$ on the efficient frontier, if the upper bound p* is fixed at a level $p^*(Q)$ such that the slope of the p iso V@R lines is that of the first derivative of the frontier at Q, the efficient mean V@R set (robust for any $0<p<p^*(Q)$) is the subset of the original mean standard deviation efficient set given by its upper part (from Q to the point of largest expectation). We obtain $p^*(Q) = F_Z[-2\sigma/(\partial V/\partial m)]$, with σ and $\partial V/\partial m$ computed at Q. In case of kink points the upper bound is found by the slope of the right derivative.

9. Conclusions

We have studied, in particular under normality of the implied random variables, the connections between different measures of risk such as the standard deviation, the W-ruin probability and the p-V@R. We discuss conditions granting the equivalence of these measures with respect to risk preference relations and the equivalence of dominance and efficiency of risk-reward criteria involving these measures.

Then more specifically we applied these concepts to rigorously face the problem of finding the efficient set of de Finetti's variable quota share proportional reinsurance. Under the very same assumptions of the original de Finetti's paper we have found that: a) the mean-W shortfall efficient set is robust to changes in W (it is the same for any W>0); b) the mean-W shortfall robust efficient set is the same as the mean standard deviation efficient one; c) the mean p V@R efficient set is robust for p values in the interval $(0,p^*=F_Z(-a^{1/2}))$ and in this case is again the same as the mean standard deviation efficient one; d) enlarging the interval implies cutting (a lower) part of the efficient frontier.

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