



A new, simple, and effective variationally consistent mixed finite element formulation for the geometrically exact planar analysis of beams and frames

H. A. F. A. Santos¹

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Abstract

A robust and efficient mixed finite element formulation, applicable to both shear-rigid (Kirchhoff-type) and shear-flexible (Reissner-type) beam theories, is proposed for the geometrically exact planar analysis of beams and frames. The formulation is derived from a Hellinger–Reissner-type variational principle, in which the spatial components of the internal forces, bending moment, and cross-sectional rotation are adopted as primary unknowns. Equilibrium of the spatial internal forces is satisfied in strong form at the member level, while the assumed bending moment field is C^0 -continuous and satisfies equilibrium boundary conditions. The formulation is intrinsically free from shear- and membrane-locking phenomena, admits explicit closed-form expressions for both the residual vector and the consistent symmetric tangent matrix, and yields a highly sparse and banded global system with a reduced number of unknowns compared with conventional displacement-based formulations and previously proposed hybrid-mixed formulations. Displacement fields are recovered *a posteriori* by integrating the kinematic compatibility relations, producing globally C^0 -continuous fields that satisfy all prescribed kinematic boundary conditions. Nonlinear equilibrium paths are traced using an arc-length strategy with a linear extrapolation predictor, and critical points are identified through the determinant of a Schur-complement-based equivalent stiffness matrix that emerges naturally from the formulation. Numerical examples demonstrate the accuracy, convergence, and robustness of the proposed approach, including verification against analytical solutions under pure bending, assessment of shear-locking performance, and the simulation of complex nonlinear responses involving snap-through, snap-back, and symmetry-breaking bifurcation phenomena.

Keywords Beam and frame structures · Geometrically exact beam theory · Large displacements and rotations · Mixed finite element formulation · Shear locking · Stability analysis

1 Introduction

Beam-like structures appear across virtually every branch of engineering, reflecting their remarkable efficiency in transmitting loads while minimizing material usage. In civil and mechanical engineering, beams are the fundamental load-bearing components of bridges, towers, buildings, and machine frames, where accurate prediction of stiffness, load capacity, and stability is essential to safety and economy. In aerospace and astronautics, slender members form

the basis of aircraft wings, rotor blades, satellite booms, and deployable structures. Such components often combine extreme slenderness ratios with lightweight design, making them prone to large deflections, post-buckling behavior, and dynamic instabilities that cannot be described adequately by classical small-deformation models.

Similarly, in biomechanics, filaments such as DNA strands, actin fibers, and microtubules behave mechanically like beams on microscopic scales, where forces from molecular motors and viscous drag lead to substantial deformations and pronounced time-dependent behavior. In emerging technologies, such as soft robotics, energy-harvesting devices, and compliant mechanisms, beam-like structures are deliberately exploited for their ability to deform significantly and even snap-through to alternative equilibrium states, enabling new functions.

✉ H. A. F. A. Santos
hugo.santos@isef.pt

¹ Departamento de Engenharia Mecânica, UnIRE, Instituto Superior de Engenharia de Lisboa, Polytechnic University of Lisbon, Rua Conselheiro Emídio Navarro, 1, 1959-007 Lisboa, Portugal

Consequently, extensive research has focused on developing robust beam theories and associated numerical methods. Early research efforts concentrated on linear theories, such as the Euler-Bernoulli and Timoshenko beam models, owing to their ability to provide closed-form solutions for a wide range of engineering problems. Despite their widespread use, these classical beam theories are inherently limited. The Euler-Bernoulli theory neglects shear deformation and is valid only for slender beams under small rotations, while the Timoshenko model incorporates shear deformation but remains based on small-strain kinematics and, thus, cannot capture finite rotations or large deformations. These limitations become severe when analyzing slender members under complex loading, post-buckling phenomena, or dynamic scenarios involving rotations of arbitrary magnitude.

To overcome these shortcomings, geometrically exact beam theories were developed. Kirchhoff's beam theory constitutes the first model accounting for axial strain in which no assumptions are made regarding the magnitude of deformations. However, this theory does not account for shear deformation, being valid in practice only for thin beams (or rods). Reissner [1–3] extended Kirchhoff's rod model by incorporating shear deformation. Reissner's theory describes beams confined to the plane with arbitrarily large cross-sectional rotations, accommodating axial, shear, and bending deformations in a variationally consistent setting. In Reissner's theory, the equilibrium differential equations of cross-sectional forces and moments defined for an infinitesimal element of the deformed beam axis are taken as fundamental and a system of strain–displacement relations involving the translational and rotational strains is obtained from an appropriate form of the principle of virtual work. Subsequent developments for initially curved beams were presented in [4, 5]. Hyperelastic frames subject to non-conservative loads were considered in [6]. Extensions to incorporate elastic–plastic material behavior were proposed in [7].

Building on Reissner's framework, Simo and collaborators [8–11] developed a fully consistent theory by systematically deriving the kinematic measures and relations from the general three-dimensional continuum theory. This theory is known in the literature as the *geometrically exact Reissner–Simo beam theory*.

Over the following decades, research expanded along several fronts. One central line of research has been the design of finite element discretizations for large deformation analysis of beams, which has taken two major directions: one based on Kirchhoff's theory and the other on Reissner's theory. Even though Kirchhoff beams were used initially, Reissner beams were later favored. This choice was dictated by the higher-order smoothness that the finite element approximation of Kirchhoff beams imposes on the discretization space. For an excellent and detailed survey on geometrically exact

beam formulations and associated finite element discretizations, the interested reader is referred to [12].

Traditional displacement-based finite element formulations are derived from the principles of virtual work or total potential energy and rely on the interpolation of kinematic variables alone, namely displacements and rotations. In such formulations, stress resultants, such as axial forces, shear forces, and bending moments, are recovered only *a posteriori*, which may reduce accuracy. Moreover, shear-flexible beam elements with equal-order interpolation of transverse displacement and rotation are prone to shear locking, while both shear-rigid and shear-flexible elements may suffer from membrane locking in geometrically nonlinear analyses [13]. These numerical pathologies typically lead to overly stiff structural responses and poor convergence of the nonlinear solution process. Reduced or selective integration techniques are often employed to alleviate locking phenomena [14]; however, these techniques may introduce spurious zero-energy modes and compromise numerical stability.

An effective alternative is provided by mixed finite element formulations [14], which introduce additional independent fields, typically stress resultants such as axial force, shear force, and bending moment, alongside the kinematic variables. Mixed formulations are well known to improve accuracy, stability, and convergence, especially for slender structures where shear- and membrane-locking phenomena are most severe. Beyond classical locking phenomena, several authors have also observed that mixed finite elements exhibit superior robustness in geometrically nonlinear path-following analyses, allowing significantly larger load increments than displacement-based elements [15–17]. This behavior was first systematically investigated in [18], where it was shown that the reduced robustness of displacement formulations is linked to a phenomenon termed *extrapolation locking*. Unlike classical discretization locking, extrapolation locking affects the Newton iterative scheme rather than the finite element interpolation itself and typically arises in beam and shell problems characterized by high axial or membrane-to-bending stiffness ratios. In such cases, the tangent matrix evaluated at the current iterate may provide a poor approximation of the true Jacobian, resulting in slow convergence and severely restricted step sizes. Displacement-based formulations are particularly sensitive to this effect, whereas mixed formulations are essentially immune to it [19–21]. The origin and general implications of this enhanced robustness of mixed formulations in nonlinear analyses were further clarified in [22].

Several mixed finite element formulations have been developed within the framework of Reissner's geometrically exact beam theory. Early contributions include the formulations proposed in [4, 23], based on a modified Hu–Washizu variational principle. These approaches were later extended to account for elastic–plastic material behavior

[7] and initially curved beam geometries [24]. A particular subclass of mixed methods, known as *equilibrium-based formulations*, aims to satisfy equilibrium conditions in strong form, in contrast to classical displacement-based approaches where equilibrium is enforced only in weak form. However, the strong nonlinear coupling between kinematic variables and stress resultants in geometrically exact theories makes the construction of suitable equilibrium-based interpolations particularly challenging. An equilibrium-based hybrid-mixed formulation derived from a modified complementary energy principle was proposed in [25]. In that approach, the material components of the internal force resultants and bending moments, together with displacement and rotation fields, are treated as primary variables, while additional Lagrange multipliers associated with nodal displacements and rotations are introduced to enforce interelement equilibrium conditions. More recently, mixed Hu–Washizu-type [26] and Hellinger–Reissner-type [27] formulations have been successfully applied to the nonlinear dynamic analysis of planar beam structures.

In a three-dimensional setting, additional challenges related to the parameterization of spatial rotations arise. Within this context, mixed formulations for geometrically exact beams and frames have been proposed in [28, 29], where the equilibrium-based formulation originally developed for two-dimensional problems in [25] was extended to three dimensions. A different equilibrium-based finite element for initially curved beams was introduced in [30]. This formulation is grounded in a Hellinger–Reissner variational principle and requires the simultaneous approximation of both kinematic and static fields. However, the resulting internal forces and bending moments are discontinuous across interelement boundaries. More recently, an alternative equilibrium-based formulation was proposed in [31]. While this approach successfully addresses the geometric nonlinearity of three-dimensional beam problems, it leads to fully populated system matrices, thereby sacrificing the sparsity and small bandwidth properties that are characteristic of standard finite element discretizations and are essential for computational efficiency in large-scale applications. A four-field mixed finite element approximation for Kirchhoff beams based on Cartan’s moving frames was introduced in [32]. A mixed Petrov–Galerkin-type formulation based on a two-field Hellinger–Reissner variational principle involving both kinematic and static quantities was recently proposed in [33]. The formulation allows for interelement discontinuities of the internal force resultants and moments and, additionally, leads to a linearized problem governed by a non-symmetric tangent operator. Such non-symmetry introduces well-known computational drawbacks, including reduced robustness of Newton-type solvers and increased costs associated with linear system solution procedures.

The geometrically exact beam theory has been extended to account for increasingly complex physical phenomena, thereby considerably broadening its domain of applicability. Contemporary formulations incorporate, among others, contact and self-contact interactions [34], as well as fracture and failure mechanisms through the introduction of embedded strong discontinuities [35, 36]. These developments underscore the versatility of geometrically exact beam models and their relevance for the simulation of highly nonlinear structural behavior.

Beyond the finite element method, several alternative numerical approaches have been developed for the geometrically exact analysis of beams and frames, including finite difference schemes [37], hierarchical quadrature element methods [38], isogeometric formulations based on the higher-order smoothness of spline-based discretizations [39, 40], and virtual element methods [41, 42]. The virtual element framework, in particular, has demonstrated that locking-free beam discretizations can be constructed systematically, naturally accommodating higher-order approximations while remaining fully compatible with standard finite element implementations [41, 42].

The present work proposes a mixed finite element formulation for the geometrically exact planar analysis of beams and frames, derived from a variationally consistent Hellinger–Reissner principle and applicable to both shear-rigid (Kirchhoff-type) and shear-flexible (Reissner-type) beam theories. The formulation is constructed directly in terms of the spatial components of the internal forces and bending moments as independent statical variables, together with the cross-sectional rotation as the sole kinematic unknown. This choice distinguishes the proposed approach from existing hybrid-mixed formulations [25, 28], in which displacement fields and Lagrange multipliers are retained among the primary unknowns. By eliminating these variables entirely, the formulation attains a particularly compact discrete structure: the spatial components of the internal forces are represented by member-level parameters that satisfy force equilibrium in strong form, while the bending moment field satisfies the equilibrium boundary conditions and, in the absence of distributed loads, it also satisfies the moment equilibrium equation in strong form within each element. The resulting global system involves fewer unknowns than conventional displacement-based formulations, is intrinsically free from shear- and membrane-locking phenomena, and is governed by explicit closed-form expressions for both the residual vector and the consistent symmetric tangent matrix. Owing to the adopted discretization strategy, the tangent matrix exhibits a highly sparse and banded structure. Furthermore, the mixed discrete system gives rise to a Schur-complement-based equivalent stiffness matrix, providing a rigorous and unified framework for the detection and

classification of limit and bifurcation points along nonlinear equilibrium paths.

The remainder of the manuscript is organized as follows. Section 2 introduces the boundary-value problem of the geometrically exact beam theory and establishes the notation adopted throughout the work. The variational foundations of the proposed formulation are then developed in Sect. 3. The mixed finite element formulation is presented in detail in Sect. 4, including the discretization of the primary unknown fields and the *a posteriori* recovery of the secondary variables. Section 5 describes the consistent linearization of the extended discrete system, the adopted path-following strategy, and the procedures employed for the detection and continuation of limit and bifurcation points. Section 6 reports and discusses the numerical examples, with particular emphasis on comparisons between the proposed mixed formulation and a classical displacement-based formulation in terms of accuracy, convergence, computational efficiency, and robustness. Finally, concluding remarks and perspectives for future work are given in Sect. 7. Additional derivations and technical details are collected in the Appendix.

2 Boundary-value problem: Reissner's beam theory

The beam theory considered in this work was originally introduced by Reissner in 1972 for the analysis of plane beams undergoing elastic deformations [1]. Reissner's beam theory is geometrically exact, i.e., no assumptions are made regarding the magnitude of strains, displacements, or rotations. The formulation accounts for shear deformation and assumes that cross-sections, which are initially orthogonal to the beam axis, remain plane and undistorted throughout the deformation process.

The geometry of the beam cross-sections is assumed to be arbitrary but solid and constant along the beam axis. During deformation, the cross-sections are assumed to undergo rigid-body motions only, implying that warping effects are neglected. The beam geometry is described by its centroidal axis, denoted by $\mathcal{C}$, which is parameterized by the arc-length coordinate $s \in [0, L]$, where L denotes the length of the beam member in its reference configuration. The centroidal axis is decomposed into an interior domain Ω and a boundary $\Gamma = \Gamma_N \cup \Gamma_D = \{0, L\}$, where Γ_N and Γ_D represent the Neumann and Dirichlet boundaries, respectively, and satisfy $\Gamma_N \cap \Gamma_D = \emptyset$.

The undeformed configuration of the beam centroidal axis is described by the parametric functions $x(s)$ and $y(s)$. The orientation of the beam cross-section in the reference configuration is characterized by the angle $\theta_0(s)$, defined as the angle between the normal to the centroidal axis and the global y -axis, or equivalently, as the angle between the tangent to

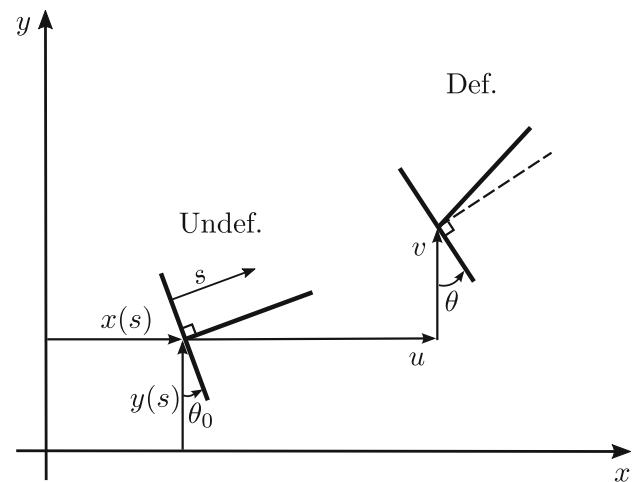


Fig. 1 Beam kinematics: undeformed and deformed configurations

the centroidal axis and the x -axis; see Fig. 1. Accordingly, the following geometric relations hold

$$x'(s) = \cos(\theta_0(s)), \quad (1a)$$

$$y'(s) = \sin(\theta_0(s)), \quad (1b)$$

where $(\cdot)'$ denotes differentiation of $(\cdot)$ with respect to the arc-length coordinate s .

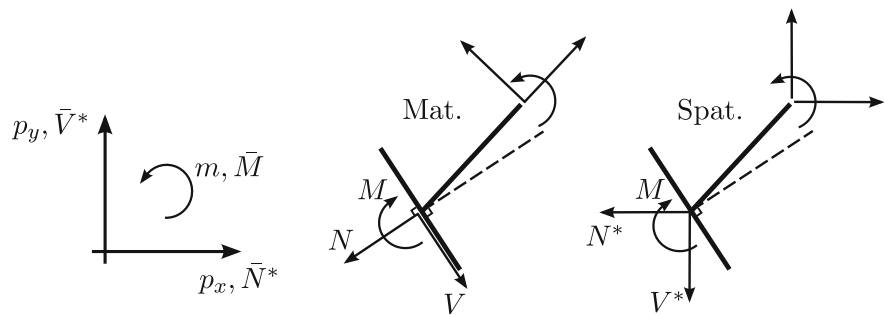
Upon deformation, a material point initially located at $(x(s), y(s))$ is displaced to $(x(s) + u(s), y(s) + v(s))$, where $u(s)$ and $v(s)$ denote the displacement components along the global x - and y -directions, respectively. The beam cross-sections, which are normal to the centroidal axis in the undeformed configuration, do not necessarily preserve this orthogonality after deformation. Their current orientation is therefore described by the rotation field $\theta(s)$, defined as the angle between the deformed cross-section and the global y -axis; see Fig. 1.

With respect to the internal force resultants, two alternative descriptions can be adopted. In the *spatial* description, the internal force components are expressed in the fixed global reference frame (x, y) and are denoted by (N^*, V^*) . In the *material* description, the same force components are expressed in a local reference frame attached to the beam centroidal axis and are denoted by (N, V) . The corresponding sign conventions for the internal force resultants and bending moment are defined in Fig. 2. The quantities N and V therefore represent the physical normal and shear forces acting on the beam cross-section.

In contrast, no distinction between spatial and material components is required for the bending moment, since the analysis is restricted to planar kinematics.

The beam is assumed to be subjected to distributed loads p_x and p_y , as well as a distributed bending moment m , acting along the domain Ω . The corresponding sign conventions for

Fig. 2 Beam statics: material and spatial internal force components and bending moment in deformed configuration



these external loads are defined in Fig. 2. In addition, concentrated forces $\bar{N}^*$ and $\bar{V}^*$, together with concentrated bending moments $\bar{M}$, may be applied on the Neumann boundary Γ_N . The same sign conventions are adopted for the concentrated loads. Prescribed displacements $\bar{u}$ and $\bar{v}$, together with prescribed rotations $\bar{\theta}$, are imposed on the Dirichlet boundary Γ_D . All applied loads are assumed to be conservative and to act along the centroidal axis of the beam.

According to Reissner's theory [1], the equilibrium differential equations of the beam in spatial form are given by

$$N^{*'} + p_x = 0 \quad \text{in } \Omega, \quad (2a)$$

$$V^{*'} + p_y = 0 \quad \text{in } \Omega, \quad (2b)$$

$$M' + (1 + \varepsilon)V - \gamma N + m = 0 \quad \text{in } \Omega, \quad (2c)$$

where ε and γ are the axial and shear strains as defined in Eq. (6).

The spatial and material internal force components are related through θ according to

$$N^* = N \cos \theta - V \sin \theta, \quad (3a)$$

$$V^* = N \sin \theta + V \cos \theta, \quad (3b)$$

or, equivalently, by the inverse transformation

$$N = N^* \cos \theta + V^* \sin \theta, \quad (4a)$$

$$V = V^* \cos \theta - N^* \sin \theta. \quad (4b)$$

On Γ_N , the following equilibrium (or Neumann) boundary conditions must hold

$$nN^* = \bar{N}^*, \quad (5a)$$

$$nV^* = \bar{V}^*, \quad (5b)$$

$$nM = \bar{M}, \quad (5c)$$

where

$$n = \begin{cases} 1, & \text{for } s = L \\ -1, & \text{for } s = 0. \end{cases}$$

The strain variables are linked to the configuration variables by means of the following compatibility differential equations

$$\varepsilon = (x' + u') \cos \theta + (y' + v') \sin \theta - 1 \quad \text{in } \Omega, \quad (6a)$$

$$\gamma = (y' + v') \cos \theta - (x' + u') \sin \theta \quad \text{in } \Omega, \quad (6b)$$

$$\kappa = \theta' - \theta_0' \quad \text{in } \Omega, \quad (6c)$$

where ε , γ , and κ denote the axial strain, shear strain, and curvature, respectively.

The associated compatibility (Dirichlet) boundary conditions, prescribed on Γ_D , are given by

$$u = \bar{u}, \quad (7a)$$

$$v = \bar{v}, \quad (7b)$$

$$\theta - \theta_0 = \bar{\theta}. \quad (7c)$$

As shown in [1], the compatibility equations can be derived from the well-known *Principle of Virtual Work* (PVW), which constitutes the weak form of the equilibrium equations given above.

The material response is assumed to be linear elastic. Accordingly, the generalized stress and strain measures are related through the following constitutive equations

$$\varepsilon = \frac{N}{EA} \quad \text{in } \Omega, \quad (8a)$$

$$\gamma = \frac{V}{GA_s} \quad \text{in } \Omega, \quad (8b)$$

$$\kappa = \frac{M}{EI} \quad \text{in } \Omega. \quad (8c)$$

The quantities EA , GA_s , and EI are the axial, shear, and bending stiffnesses of the beam cross-section. Here, E and G denote Young's modulus and the shear modulus, respectively, with $G = E/(2(1 + \nu))$, where ν is Poisson's ratio. The geometric properties A , A_s , and I correspond to the cross-sectional area, effective shear area, and second moment of area, respectively. The effective shear area is defined as $A_s = k_s A$, where k_s is the classical shear correction factor accounting for the non-uniform distribution of shear stresses over the cross-section.

It can be readily shown that the following energetic conjugacy relation holds

$$N\varepsilon + V\gamma = N^*\varepsilon^* + V^*\gamma^*, \quad (9)$$

where the spatial strain measures ε^* and γ^* are defined as

$$\varepsilon^* = x' + u' - \cos \theta, \quad (10a)$$

$$\gamma^* = y' + v' - \sin \theta. \quad (10b)$$

3 Variational setting

The variational structure of Reissner's geometrically exact beam theory can be established through the Principle of Stationary Total Potential Energy. To this end, let $\mathcal{U}$ denote the kinematically admissible space defined as

$$\mathcal{U} = \{(u, v, \theta) \in [H^1(\Omega)]^3 \mid u = \bar{u}, v = \bar{v}, (\theta - \theta_0) = \bar{\theta}, \text{ on } \Gamma_D\}, \quad (11)$$

where $H^1(\Omega)$ denotes the standard Sobolev space of square-integrable functions with square-integrable first derivatives on Ω . The total potential energy functional $\Pi_p : \mathcal{U} \rightarrow \mathbb{R}$ is defined as

$$\begin{aligned} \Pi_p(u, v, \theta) = & \int_0^L \frac{1}{2} (EI\kappa^2(\theta) \\ & + EA\varepsilon^2(u, v, \theta) + GA_s\gamma^2(u, v, \theta)) ds \\ & - \int_0^L (p_x u + p_y v + m(\theta - \theta_0)) ds \\ & - [\bar{N}^* u + \bar{V}^* v + \bar{M}(\theta - \theta_0)]_{\Gamma_N}. \end{aligned} \quad (12)$$

When the compatibility and constitutive relations are enforced as constraints, the stationarity condition of the functional (12) yields the equilibrium equations as its Euler-Lagrange equations. This classical variational statement is formally equivalent to the PVW.

By introducing suitable Lagrange multipliers, the equilibrium equations may be enforced directly at the variational level, leading to alternative but equivalent mixed variational principles. Several such formulations have been proposed in the literature; see, for example, [29].

In this work, the following Hellinger–Reissner-type functional $\Pi_{H-R} : \mathcal{S} \rightarrow \mathbb{R}$ is adopted:

$$\begin{aligned} \Pi_{H-R}(N^*, V^*, M, \theta) = & \int_0^L \left(\frac{M^2}{2EI} \right. \\ & + \frac{N^2(N^*, V^*, \theta)}{2EA} + \frac{V^2(N^*, V^*, \theta)}{2GA_s} \Big) ds \\ & + \int_0^L (N^*(\cos \theta - x') + V^*(\sin \theta - y')) \end{aligned}$$

$$\begin{aligned} & + M'(\theta - \theta_0) + m(\theta - \theta_0)) ds \\ & - [nN^*\bar{u} + nV^*\bar{v} + nM\bar{\theta}]_{\Gamma_D} \\ & - [(nM - \bar{M})(\theta - \theta_0)]_{\Gamma_N}, \end{aligned} \quad (13)$$

where $\mathcal{S} = \mathcal{F} \times H^1(\Omega) \times L^2(\Omega)$, with

$$\begin{aligned} \mathcal{F} = & \{(N^*, V^*) \in [H^1(\Omega)]^2 \mid \\ N^{*'} + p_x = 0, V^{*'} + p_y = 0 \text{ in } \Omega, \text{ and} \\ nN^* = \bar{N}^*, nV^* = \bar{V}^* \text{ on } \Gamma_N\}. \end{aligned} \quad (14)$$

where the distributed loads are assumed to satisfy $p_x, p_y \in L^2(\Omega)$, consistent with the $H^1(\Omega)$ regularity required for N^* and V^* .

This mixed variational formulation treats the spatial internal force components, bending moment, and cross-sectional rotation as independent primary fields. In a finite element setting, this permits the direct approximation of the static fields, rather than their recovery from a displacement interpolation, while allowing equilibrium to be incorporated explicitly into the approximation framework. Its full derivation is provided in the Appendix.

Under the subsidiary equilibrium conditions for the spatial internal force components, Eqs. (2a) and (2b), and the corresponding boundary conditions, Eqs. (5a) and (5b), an intermediate form of the first variation of Π_{H-R} reads

$$\begin{aligned} \delta \Pi_{H-R} = & \int_0^L \left(\frac{M}{EI} \delta M + \frac{N}{EA} \delta N + \frac{V}{GA_s} \delta V \right) ds \\ & + \int_0^L (\delta N^*(\cos \theta - x') - N^* \sin \theta \delta \theta \\ & + \delta V^*(\sin \theta - y') \\ & + V^* \cos \theta \delta \theta + \delta M'(\theta - \theta_0) + M' \delta \theta + m \delta \theta) ds \\ & - [n \delta N^* \bar{u} + n \delta V^* \bar{v} + n \delta M \bar{\theta}]_{\Gamma_D} \\ & - [n \delta M(\theta - \theta_0)]_{\Gamma_N} - [(nM - \bar{M}) \delta \theta]_{\Gamma_N} \\ & + \int_0^L (\delta N^{*'} u + \delta V^{*'} v) ds - [n \delta N^* u + n \delta V^* v]_{\Gamma_N}. \end{aligned} \quad (15)$$

The first variations of the spatial internal force components read

$$\delta N^* = \delta N \cos \theta - N \sin \theta \delta \theta - \delta V \sin \theta - V \cos \theta \delta \theta, \quad (16a)$$

$$\delta V^* = \delta N \sin \theta + N \cos \theta \delta \theta + \delta V \cos \theta - V \sin \theta \delta \theta. \quad (16b)$$

Finally, integration by parts of the terms $\delta N^{*'} u, \delta V^{*'} v$ and $\delta M'(\theta - \theta_0)$, substitution of Eq. (3), Eq. (8) and Eq. (16) into

Eq. (15), and rearrangement of terms lead to

$$\begin{aligned} \delta \Pi_{H-R} = & \int_0^L \left(\delta M (\kappa - \theta' + \theta'_0) \right. \\ & + \delta N (\varepsilon - (x' + u') \cos \theta \\ & - (y' + v') \sin \theta + 1) \\ & + \delta V (\gamma + (x' + u') \sin \theta - (y' + v') \cos \theta) \\ & \left. + \delta \theta (M' + (1 + \varepsilon)V - \gamma N + m) \right) ds \\ & + [n \delta N^* (u - \bar{u})]_{\Gamma_D} + [n \delta V^* (v - \bar{v})]_{\Gamma_D} \\ & + [n \delta M (\theta - \theta_0 - \bar{\theta})]_{\Gamma_D} - [(nM - \bar{M}) \delta \theta]_{\Gamma_N}. \end{aligned} \quad (17)$$

Since the admissible variations are arbitrary, the stationarity of Π_{H-R} yields the compatibility equations (6) and (7), together with the bending-moment equilibrium equations (2c) and (5c).

The proposed Hellinger–Reissner-type functional encompasses both shear-rigid (Kirchhoff-type) and shear-flexible (Reissner-type) beam theories within a unified variational framework. Shear-flexible behavior follows directly from the constitutive relation $\gamma = V/GA_s$, while the shear-rigid theory is recovered as the variational limit $1/GA_s \rightarrow 0$. In this limit, the shear strain vanishes identically and the shear force acts as a reactive internal variable enforcing the Kirchhoff constraint. Owing to the mixed variational structure of the formulation, this limiting process is mathematically consistent and does not induce shear-locking effects.

To the best of the author's knowledge, this particular variational formulation has not been previously reported in the literature and forms the foundation of the mixed finite element formulation presented in Sect. 4.

Building upon this variational structure, the consistent linearization of the formulation follows from the second variation of Eq. (13), which reads

$$\begin{aligned} \Delta \delta \Pi_{H-R} = & \int_0^L \left(\frac{\Delta M}{EI} \delta M + \frac{\Delta N}{EA} \delta N + \frac{\Delta V}{GA_s} \delta V \right) ds \\ & + \int_0^L \left(-\delta N^* \sin \theta \Delta \theta \right. \\ & - (\Delta N^* \sin \theta + N^* \cos \theta \Delta \theta) \delta \theta \\ & + \delta V^* \cos \theta \Delta \theta + (\Delta V^* \cos \theta - V^* \sin \theta \Delta \theta) \delta \theta \\ & + \delta M' \Delta \theta + \Delta M' \delta \theta \Big) ds \\ & - [n \delta M \Delta \theta]_{\Gamma_N} - [n \Delta M \delta \theta]_{\Gamma_N}, \end{aligned} \quad (18)$$

where the variations of the material internal force components are given by

$$\delta N = \delta N^* \cos \theta - N^* \sin \theta \delta \theta + \delta V^* \sin \theta + V^* \cos \theta \delta \theta, \quad (19a)$$

$$\delta V = \delta V^* \cos \theta - V^* \sin \theta \delta \theta - \delta N^* \sin \theta - N^* \cos \theta \delta \theta. \quad (19b)$$

The expressions for ΔN and ΔV follow from (19) by replacing the variational operator δ with the incremental operator Δ throughout.

The second variation (18) defines the bilinear form associated with the linearized problem at the continuous level. Upon discretization, it yields the consistent tangent matrix of the proposed mixed finite element formulation.

4 Mixed finite element formulation

A mixed finite element formulation is developed from the Hellinger–Reissner variational framework introduced in Sect. 3. The formulation adopts the spatial internal force components N^* and V^* , the bending moment M , and the cross-sectional rotation θ as the primary unknown fields. Displacement fields are treated as secondary variables and are recovered *a posteriori* from the solution of the mixed problem.

The discrete mixed finite element formulation is obtained by restricting the primary fields to suitable finite-dimensional approximation spaces within the Hellinger–Reissner functional (13). The choice of these approximation spaces follows directly from the mixed variational structure. The bending moment field is required to be C^0 -continuous along the beam axis. Although the bending moment interpolation need not satisfy the equilibrium boundary conditions (5c) *a priori*, it is chosen so that these conditions are enforced exactly. As a consequence, the corresponding boundary contribution in the Hellinger–Reissner functional (the last boundary term in (13)) vanishes identically and is therefore omitted from the discrete formulation.

The spatial internal force components are constructed to satisfy the equilibrium equations (2a) and (2b), together with the associated Neumann boundary conditions, in strong form. In contrast, the cross-sectional rotation field is allowed to be discontinuous across element interfaces, consistent with its role as a Lagrange multiplier associated with the weak enforcement of moment equilibrium.

The formulation is directly applicable to both shear-rigid (Kirchhoff-type) and shear-flexible (Reissner-type) beam models without modification of the approximation spaces.

4.1 Primary unknown fields: assumed approximations

Let the beam domain Ω be subdivided into n_{el} finite elements, i.e., $\Omega = \bigcup_{e=1}^{n_{el}} \Omega_e$. Throughout this section, the term *element* refers to an individual finite element Ω_e , whereas the term *member* denotes a structural segment bounded by

adjacent joints, load application points, or support locations, and which may comprise one or more finite elements. The finite-dimensional approximation of each field is denoted by a superscript $(\cdot)^h$, while element-level restrictions are indicated by a subscript $(\cdot)_e$.

The bending moment field is approximated by C^0 -continuous, piecewise-linear functions at the element level, whereas the cross-sectional rotation is approximated as a piecewise-constant field. The spatial internal force components are parameterized by member-level fields that are globally continuous over the entire beam member and are determined directly from the equilibrium equations.

Accordingly, the following finite-dimensional approximation spaces are introduced

$$\mathcal{M}^h \subset H^1(\Omega), \quad \Theta^h \subset L^2(\Omega), \quad \mathcal{F}^h \subset [H^1(\Omega)]^2, \quad (20)$$

where $\mathcal{M}^h$ denotes the space of C^0 -continuous piecewise-linear bending moment fields, Θ^h the space of piecewise-constant cross-sectional rotations, and $\mathcal{F}^h$ the space of continuous spatial internal force fields satisfying the equilibrium equations in strong form.

The bending moment field is approximated element-wise as

$$M^h|_{\Omega_e}(\xi) \equiv M_e^h(\xi) = M_{e,0} \left(1 - \frac{\xi}{L_e}\right) + M_{e,L} \frac{\xi}{L_e}, \quad (21)$$

where $\xi \in [0, L_e]$ denotes the local arc-length coordinate within element Ω_e , $M_{e,0}$ and $M_{e,L}$ denote the bending moments at the left and right ends of element Ω_e , respectively, and L_e is the element length. In the absence of externally applied concentrated moments, the nodal bending moments coincide across adjacent elements, yielding a globally C^0 -continuous bending moment field. When a concentrated moment is applied at an interelement boundary, the corresponding jump in the bending moment field is represented directly through the interelement equilibrium equations. In this case, the bending moments on the adjacent elements satisfy the appropriate jump condition induced by the applied moment. Likewise, discontinuities in the cross-sectional stiffness parameters are accommodated naturally by assigning different constitutive properties to neighboring elements. Since the constitutive relations are enforced element-wise, abrupt variations in flexural rigidity lead to discontinuities in the curvature field, whereas the bending moment field remains continuous, as dictated by the equilibrium conditions.

The cross-sectional rotation is approximated as a piecewise-constant field,

$$\theta^h|_{\Omega_e} \equiv \theta_e^h = \theta_e, \quad (22)$$

with a single rotational degree of freedom θ_e associated with each element.

The spatial internal force components are parameterized at the member level and are constructed to satisfy the equilibrium equations in strong form as follows

$$N^{*h}(s) = - \int p_x(s) ds + N_0^*, \quad (23a)$$

$$V^{*h}(s) = - \int p_y(s) ds + V_0^*, \quad (23b)$$

where N_0^* and V_0^* are integration constants representing the spatial internal force components of the beam member at $s = 0$. Consequently, N^{*h} and V^{*h} are globally continuous over the entire beam member and are parameterized independently of the number of finite elements used in the discretization.

With the adopted approximations, the moment equilibrium equation is satisfied in strong form in the absence of distributed loads, and in weak (variational) form in the presence of distributed loads.

4.2 Recovery of displacement fields

The displacement fields are recovered *a posteriori* through integration of the kinematic compatibility relations

$$u^h|_{\Omega_e} \equiv u_e^h(\xi) = \int ((1 + \varepsilon_e^h) \cos \theta_e^h - \gamma_e^h \sin \theta_e^h - x_e^{h'}) d\xi + u_{e,0}, \quad (24a)$$

$$v^h|_{\Omega_e} \equiv v_e^h(\xi) = \int ((1 + \varepsilon_e^h) \sin \theta_e^h + \gamma_e^h \cos \theta_e^h - y_e^{h'}) d\xi + v_{e,0}, \quad (24b)$$

where $u_{e,0}$ and $v_{e,0}$ are element-wise constants of integration representing rigid-body translations. These constants are uniquely determined by enforcing interelement displacement continuity, i.e., continuity of u^h and v^h , together with the prescribed displacement boundary conditions on Γ_D .

The axial strain ε_e^h and the shear strain γ_e^h are recovered from the constitutive relations (8) using the internal force fields (4). For simplicity, the geometric descriptors $x_e^{h'}$, $y_e^{h'}$, and the initial rotation $\theta_{0,e}^h$ are taken as piecewise-constant over each element. As a result, initially curved geometries are approximated by assemblies of straight beam segments, with the geometric accuracy of the approximation improving as the mesh is refined.

The recovery of the displacement fields is performed after the solution of the mixed problem and does not affect the solution procedure itself. The governing equilibrium problem is formulated entirely in terms of statical variables and rotations, and is therefore applicable to both statically determinate and indeterminate structures. In both cases, the

displacement recovery requires only two kinematic conditions, prescribing u and v at a single boundary point, to fix the translational constants of integration $u_{e,0}$ and $v_{e,0}$ and thereby suppress the translational rigid body modes. The rotational rigid body mode is suppressed separately through the prescribed rotation boundary condition enforced in the mixed problem. For statically indeterminate structures, where the prescribed kinematic boundary conditions exceed those required to suppress rigid-body motions, the additional constraints are enforced automatically through the variational equations of the Hellinger–Reissner formulation and therefore need not be imposed explicitly during the recovery procedure. The recovered displacement fields u^h and v^h are globally C^0 -continuous, while the cross-sectional rotation field, being approximated as piecewise-constant, is necessarily discontinuous across element interfaces.

4.3 Remarks on implementation and computational aspects

The following remarks summarize important implementation and computational aspects of the proposed mixed finite element formulation.

- **Retention of the element-level rotational degree of freedom**

Since the rotational degree of freedom is defined locally at the element level, it could, in principle, be eliminated by static condensation. In that case, the resulting element equations and the assembled global system would involve only force and moment (i.e., statical) degrees of freedom, and the corresponding tangent matrix would take the form of a flexibility-type matrix. Although this alternative formulation is theoretically attractive, it is not pursued here.

Retaining the rotational degrees of freedom leads to a more transparent formulation. The element residuals and their linearization can be expressed directly in terms of physically meaningful kinematic quantities, without introducing additional condensed matrices at the element level. In particular, the consistent tangent matrix can be assembled from closed-form expressions of the element-level derivatives with respect to θ_e , thereby avoiding the repeated construction and differentiation of element-level Schur complements. This substantially simplifies both the analytical derivation and the numerical implementation of the tangent matrix.

Keeping θ_e as an explicit unknown also facilitates nonlinear continuation strategies (including arc-length methods and bifurcation tracking), since the full Jacobian of the discrete equilibrium equations is directly available and remains symmetric. From a numerical standpoint, fully

condensed flexibility-type formulations may exhibit less favorable conditioning near limit points or in strongly nonlinear regimes, because the Schur complement can amplify round-off errors and sensitivities associated with near-singular tangent matrices. By contrast, retaining the rotational degrees of freedom preserves the intrinsic saddle-point structure of the mixed formulation and generally yields a better-conditioned Jacobian. This improves the robustness of Newton-type iterations and reduces sensitivity to step-size selection in continuation procedures.

Finally, the explicit availability of the rotation field is advantageous in post-processing and stability analysis, where local rotations play a central role in interpreting large-displacement and post-critical deformation patterns.

- **Comparison with classical displacement-based formulations**

Classical displacement-based formulations require C^0 -continuous interpolations of both displacements and rotations and typically rely on reduced or selective numerical integration to alleviate shear- and membrane-locking phenomena in the slender-beam limit. The simplest and most widely used representative of this class is the two-node beam element with three degrees of freedom per node, commonly evaluated using one-point Gauss quadrature [43]. While robust and easy to implement, such formulations introduce kinematic degrees of freedom at every node and lead to global systems with comparatively dense coupling patterns.

In contrast, the present formulation assigns the spatial internal force parameters N_0^* and V_0^* at the member level rather than at the element level. As a result, the total number of global unknowns associated with each beam member is substantially smaller than in standard displacement-based two-node formulations, which introduce the kinematic variables u , v , and θ at every node. Furthermore, the resulting global tangent matrix is symmetric and exhibits a pronounced block structure: the element-wise rotational variables θ_e appear only in diagonal blocks, while off-diagonal couplings arise exclusively through the interelement bending moments $M_{e,0}$ and $M_{e,L}$ and the member-level force parameters N_0^* and V_0^* . This structural property leads to a markedly sparser global system with a significantly narrower effective bandwidth than that of displacement-based formulations, in which all nodal displacement and rotation degrees of freedom are mutually coupled across adjacent elements. The resulting sparsity pattern reduces memory requirements and improves computational efficiency, particularly when combined with sparse direct solvers or large-scale iterative solution strategies.

5 Nonlinear solution strategy and stability analysis

5.1 Linearization and path-following

Setting the first-order variation of the discrete Hellinger–Reissner functional to zero, $\delta\Pi_{H-R}^h = 0$, yields a nonlinear algebraic system of the form

$$\mathbf{R}(\mathbf{d}, \lambda) = \mathbf{0}, \quad (25)$$

where $\mathbf{d}$ denotes the global vector collecting all statical and kinematic degrees of freedom, and λ denotes the load parameter. The residual equation (25) represents a set of N equations with $N + 1$ unknowns, namely the N degrees of freedom collected in $\mathbf{d}$ and the additional scalar parameter λ . The resulting system is therefore underdetermined and requires a supplementary path-following (continuation) condition to define the incremental solution along the equilibrium path.

The augmented discrete problem is written as

$$\mathbf{R}(\mathbf{d}, \lambda) = \mathbf{0}, \quad (26a)$$

$$g(\mathbf{d}, \lambda) = 0, \quad (26b)$$

in which g represents an arc-length constraint. Adopting a spherical measure based on the last converged configuration $(\mathbf{d}_0, \lambda_0)$ and on the prescribed increment $\Delta\ell$, the constraint is expressed as

$$g(\mathbf{d}, \lambda) = \sqrt{(\mathbf{d} - \mathbf{d}_0)^T (\mathbf{d} - \mathbf{d}_0) + (\lambda - \lambda_0)^2} - \Delta\ell. \quad (27)$$

For the Newton corrector stage, both the residual system (26a) and the continuation constraint (26b) are consistently linearized about the current iterate $(\mathbf{d}^i, \lambda^i)$. This yields the following incremental system

$$\begin{bmatrix} \mathbf{T}(\mathbf{d}^i, \lambda^i) & \mathbf{R}_{,\lambda}(\mathbf{d}^i, \lambda^i) \\ g_{,\mathbf{d}}^T(\mathbf{d}^i, \lambda^i) & g_{,\lambda}(\mathbf{d}^i, \lambda^i) \end{bmatrix} \begin{bmatrix} \Delta\mathbf{d} \\ \Delta\lambda \end{bmatrix} = - \begin{bmatrix} \mathbf{R}(\mathbf{d}^i, \lambda^i) \\ g(\mathbf{d}^i, \lambda^i) \end{bmatrix}, \quad (28)$$

where $\Delta\mathbf{d}$ and $\Delta\lambda$ denote the Newton corrections to the degrees of freedom and the load parameter, respectively. The operator $\mathbf{T} = \mathbf{R}_{,\mathbf{d}}$ denotes the global tangent matrix of the mixed formulation. It is obtained from the second variation of the discrete Hellinger–Reissner functional with respect to the primary unknowns $\mathbf{d}$ or, equivalently, from the direct finite element discretization of the variational expression given in Eq. (18). Owing to its variational origin, $\mathbf{T}$ is symmetric, as it corresponds to the Hessian of the underlying functional.

The explicit closed-form expressions for the element residual vector, $\mathbf{R}_e$, and the components of the element tangent matrix, $\mathbf{T}_e$, resulting from the adopted finite element approximations, are provided in the Appendix.

The derivatives of the arc-length constraint evaluated in (28) are

$$g_{,\mathbf{d}} = \frac{\mathbf{d} - \mathbf{d}_0}{\sqrt{(\mathbf{d} - \mathbf{d}_0)^T (\mathbf{d} - \mathbf{d}_0) + (\lambda - \lambda_0)^2}}, \quad (29a)$$

$$g_{,\lambda} = \frac{\lambda - \lambda_0}{\sqrt{(\mathbf{d} - \mathbf{d}_0)^T (\mathbf{d} - \mathbf{d}_0) + (\lambda - \lambda_0)^2}}, \quad (29b)$$

A key ingredient for efficiency and robustness is the predictor used to initialize each increment. Following Kadapa [44], a simple linear extrapolation based on the two most recent converged solutions is adopted. Denoting the converged configurations at steps $n - 1$ and n by $(\mathbf{d}_{n-1}, \lambda_{n-1})$ and $(\mathbf{d}_n, \lambda_n)$, the predictor for step $n + 1$ is

$$\mathbf{d}_{n+1}^{(1)} = (1 + \alpha) \mathbf{d}_n - \alpha \mathbf{d}_{n-1}, \quad (30a)$$

$$\lambda_{n+1}^{(1)} = (1 + \alpha) \lambda_n - \alpha \lambda_{n-1}, \quad (30b)$$

where $\alpha = \Delta\ell / \Delta\ell_n$ is the ratio between the new and the previous arc-length increments. The associated initial corrections are therefore

$$\Delta\mathbf{d}^{(1)} = \alpha (\mathbf{d}_n - \mathbf{d}_{n-1}), \quad (31a)$$

$$\Delta\lambda^{(1)} = \alpha (\lambda_n - \lambda_{n-1}). \quad (31b)$$

This extrapolation predictor follows the local trend of the equilibrium path and keeps the iterate close to the correct solution branch without requiring sign tests, determinant checks, or explicit monitoring of multiple roots.

The arc-length Newton procedure employed in the numerical analyses is thus summarized as:

1. compute the first configuration $(\mathbf{d}_1, \lambda_1)$ using simple load control;
2. for each step $n + 1$, evaluate the predictor $(\mathbf{d}_{n+1}^{(1)}, \lambda_{n+1}^{(1)})$ from (30);
3. solve the extended Newton system (28) until $\|\mathbf{R}\| < tol_R$ and $|g| < tol_g$, with tol_R , tol_g , and tol specified tolerances;
4. record the converged solution and adaptively adjust $\Delta\ell$ if convergence is slow.

5.2 Critical points

In nonlinear static stability analysis, a *critical point* on an equilibrium path corresponds to a configuration at which the solution loses local uniqueness, giving rise to either a limit point or a bifurcation point. Considering the nonlinear system of equations (25), a critical configuration $(\mathbf{d}_{cr}, \lambda_{cr})$ satisfies

$$\mathbf{R}(\mathbf{d}_{cr}, \lambda_{cr}) = \mathbf{0}, \quad (32)$$

together with the loss of regularity of the tangent matrix

$$\det(\mathbf{T}(\mathbf{d}_{\text{cr}}, \lambda_{\text{cr}})) = 0. \quad (33)$$

In mixed finite element formulations, the tangent matrix $\mathbf{T}$ arises from the second variation of a saddle-point functional and is therefore generally *indefinite*, even at stable equilibrium configurations. As a consequence, the sign of $\det(\mathbf{T})$ does not provide a reliable indicator of structural stability. In the absence of spurious zero-energy modes, stability assessment is therefore based on an appropriately *reduced* tangent matrix obtained by static condensation of the statical variables.

Let the vector of unknowns $\mathbf{d}$ be partitioned into statical variables $\mathbf{d}_s \in \mathbb{R}^{n_s}$ and kinematic variables $\mathbf{d}_k \in \mathbb{R}^{n_k}$ as

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}_s \\ \mathbf{d}_k \end{bmatrix}. \quad (34)$$

Accordingly, the tangent matrix $\mathbf{T} \in \mathbb{R}^{(n_s+n_k) \times (n_s+n_k)}$ admits the block partition

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_{ss} & \mathbf{T}_{sk} \\ \mathbf{T}_{ks} & \mathbf{T}_{kk} \end{bmatrix}. \quad (35)$$

The block $\mathbf{T}_{ss}$ associated with the statical degrees of freedom $\mathbf{d}_s$ is symmetric positive definite, which permits the elimination of $\mathbf{d}_s$ by static condensation. This property follows directly from the fact that $\mathbf{T}_{ss}$ is the Hessian of the discrete Hellinger–Reissner functional Π_{H-R}^h with respect to the statical variables. Since Π_{H-R}^h depends quadratically and strictly convexly on $\mathbf{d}_s$, the block $\mathbf{T}_{ss}$ is guaranteed to be symmetric positive definite, ensuring that the condensation procedure is well-posed and numerically stable. A detailed proof is provided in the Appendix.

The resulting reduced tangent matrix acting on the kinematic degrees of freedom is given by the *Schur complement* of the statical block,

$$\mathbf{K} = -(\mathbf{T}_{kk} - \mathbf{T}_{ks} \mathbf{T}_{ss}^{-1} \mathbf{T}_{sk}). \quad (36)$$

By construction, $\mathbf{K}$ governs the incremental response of the kinematic variables alone and thus represents the *equivalent stiffness matrix* of the condensed system. The global minus sign in Eq. (36) arises from the fact that the functional adopted in the present formulation, Π_{H-R} , differs by a sign from the original Hellinger–Reissner functional, Π_{H-R}^m , as discussed in the Appendix. While this sign reversal does not affect the stationarity conditions, it changes the sign of the associated second variation and, consequently, of the tangent matrix obtained after static condensation.

Since $\mathbf{T}_{ss}$ is invertible, the null spaces of $\mathbf{T}$ and $\mathbf{K}$ are in one-to-one correspondence: if $\mathbf{K}\boldsymbol{\phi}_k = \mathbf{0}$, then $\mathbf{T}\boldsymbol{\phi} = \mathbf{0}$ with

$$\boldsymbol{\phi} = \begin{bmatrix} \boldsymbol{\phi}_s \\ \boldsymbol{\phi}_k \end{bmatrix}, \quad \boldsymbol{\phi}_s = -\mathbf{T}_{ss}^{-1} \mathbf{T}_{sk} \boldsymbol{\phi}_k, \quad (37)$$

and conversely. Consequently, the tangent matrix $\mathbf{T}$ is singular if and only if the equivalent stiffness matrix $\mathbf{K}$ is singular, i.e.,

$$\det(\mathbf{T}) = 0 \iff \det(\mathbf{K}) = 0. \quad (38)$$

A critical point is therefore characterized by the singularity of the equivalent stiffness matrix,

$$\det(\mathbf{K}(\mathbf{d}_{k,\text{cr}}, \lambda_{\text{cr}})) = 0, \quad (39)$$

or, equivalently, by the existence of a non-trivial nullspace

$$\mathbf{K}(\mathbf{d}_{k,\text{cr}}, \lambda_{\text{cr}}) \boldsymbol{\phi}_k = \mathbf{0}, \quad \boldsymbol{\phi}_k \neq \mathbf{0}. \quad (40)$$

From the viewpoint of the condensed system, both limit points and bifurcation points are associated with such a loss of invertibility of $\mathbf{K}$. Denoting by $\mathbf{f}_k = -(\mathbf{R}_{k,\lambda} - \mathbf{T}_{ks} \mathbf{T}_{ss}^{-1} \mathbf{R}_{s,\lambda})$ the *reduced load vector* (the Schur-complement projection of the reference load onto the kinematic space), the two cases are distinguished as follows. At a *limit point*, the null vector $\boldsymbol{\phi}_k$ is generally not orthogonal to $\mathbf{f}_k$, and the loss of uniqueness corresponds to a turning point of the equilibrium path (snap-through or snap-back). At a *bifurcation point*, $\boldsymbol{\phi}_k$ is orthogonal to $\mathbf{f}_k$, indicating the emergence of an alternative equilibrium branch from the primary path.

Since the singularity of $\mathbf{K}$ is equivalent to the singularity of the full tangent matrix $\mathbf{T}$, the extended-system formulation may be expressed directly in terms of $\mathbf{T}$. For the accurate computation of critical points, the *method of extended systems* is employed [45–47]. The equilibrium equations (25) are augmented by the singularity condition

$$\mathbf{T}(\mathbf{d}, \lambda) \boldsymbol{\phi} = \mathbf{0}, \quad (41)$$

together with the normalization constraint

$$\boldsymbol{\phi}^T \boldsymbol{\phi} - 1 = 0, \quad (42)$$

yielding the extended system

$$\begin{bmatrix} \mathbf{R}(\mathbf{d}, \lambda) \\ \mathbf{T}(\mathbf{d}, \lambda) \boldsymbol{\phi} \\ \boldsymbol{\phi}^T \boldsymbol{\phi} - 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \end{bmatrix}. \quad (43)$$

Once (43) has been solved, the critical point is classified by checking the orthogonality condition

$$\boldsymbol{\phi}^T \mathbf{R}_{,\lambda} = 0, \quad (44)$$

which holds at bifurcation points but not at limit points. For branch switching, the normalized null vector $\boldsymbol{\phi}$ is used directly to initialize continuation along the bifurcating branch.

The activation of the extended system is triggered by monitoring the normalized determinant

$$\eta_K = \frac{\det(\mathbf{K})}{\det(\mathbf{K}_{ref})}, \quad (45)$$

where $\mathbf{K}_{ref}$ denotes the equivalent stiffness matrix evaluated at the undeformed configuration. As η_K approaches zero, the vicinity of a critical point is indicated. Since $\det(\mathbf{T}) = 0$ if and only if $\det(\mathbf{K}) = 0$ (see (38)), monitoring η_K along the equilibrium path is sufficient to detect the loss of regularity of the full tangent matrix. Moreover, because $\mathbf{K}$ represents the condensed kinematic stiffness of the system, its determinant provides a more meaningful indicator of stability than the determinant of the indefinite mixed tangent matrix $\mathbf{T}$.

Accurate initial guesses for the critical configuration are obtained using the extrapolated predictor proposed by Kadapa [44], which significantly enhances the robustness of the detection and continuation of both limit and bifurcation points.

6 Numerical results

The proposed mixed finite element formulation (MFEF) and the standard displacement-based two-node low-order Lagrange beam element formulation (DBFEF) [43], combined with Kadapa's arc-length path-following strategy [44], were implemented in Wolfram Mathematica[®]. The DBFEF employs 1-point Gaussian quadrature to alleviate locking, which integrates the bending term exactly but underintegrates the axial and shear terms.

The performance of the proposed mixed finite element formulation (MFEF) is assessed through a series of benchmark problems selected to examine its accuracy, convergence characteristics, computational efficiency, and robustness. The accuracy of the numerical solutions is quantified using the relative error for a generic quantity of interest q :

$$e_{rel}^{(n_{el})}(q) = \frac{|q^{(n_{el})} - q^{ref}|}{|q^{ref}|}, \quad (46)$$

where $q^{(n_{el})}$ is computed on the mesh with n_{el} elements and q^{ref} is the corresponding reference solution.

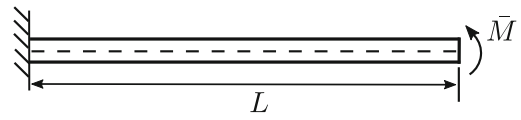


Fig. 3 Cantilever beam under pure bending

All examples consider prescribed (non-follower) external loads, whose directions remain fixed in the reference configuration. Geometric dimensions and material properties are in consistent units. In examples where cross-sectional geometry is explicitly defined, rectangular cross sections of width b and height h are assumed. For shear-flexible models, a shear correction factor $k_s = 5/6$ is applied throughout.

6.1 Pure bending of a cantilever beam

A cantilever beam subjected to a concentrated tip moment $\bar{M}$, as illustrated in Fig. 3, is first considered. The beam has length L and uniform bending stiffness EI . This problem constitutes a classical benchmark for geometrically exact beam formulations, since its response is characterized by constant curvature and a circular deformed configuration, admitting a closed-form solution. The problem therefore provides a direct verification of the ability of the proposed formulation to reproduce uniform-curvature states exactly.

The analytical solution is given by

$$M(s) = \bar{M}, \quad (47a)$$

$$\kappa(s) = \frac{\bar{M}}{EI}, \quad (47b)$$

$$\theta(s) = \kappa s, \quad (47c)$$

$$u(s) = \frac{1}{\kappa} \sin(\theta(s)) - s, \quad (47d)$$

$$v(s) = \frac{1}{\kappa} (1 - \cos(\theta(s))). \quad (47e)$$

The beam was analyzed using the proposed MFEF with meshes consisting of 2, 8, and 16 finite elements for the load ratios $\bar{M}L/EI \in \{\pi/4, \pi/2, \pi, 3\pi/2, 2\pi\}$. The bending moment distributions obtained for the various load levels are omitted for brevity, since all coincide with the exact constant solution $M(s) = \bar{M}$. The computed normalized tip-displacement ratios $u^h(L)/u(L)$ and $v^h(L)/v(L)$ are reported in Table 1. Excellent agreement with the analytical solution is observed for all load levels, with rapid convergence toward the exact response as the mesh is refined. Notably, the horizontal tip displacement is reproduced exactly for the cases $\bar{M}L/EI = \pi$ and $\bar{M}L/EI = 2\pi$, independently of the adopted mesh. In particular, the numerical solution accurately reproduces the closed circular configuration corresponding to $\bar{M}L/EI = 2\pi$, even for very coarse discretizations.

Table 1 Cantilever beam under pure bending: ratios between computed and analytical tip displacements

$\bar{M}L/EI$		$\pi/4$	$\pi/2$	π	$3\pi/2$	2π
$n_{el} = 2$	$u^h(L)/u(L)$	0.941704	0.954148	1.000000	1.048170	1.000000
$n_{el} = 8$		0.996372	0.997183	1.000000	1.002557	1.000000
$n_{el} = 16$		0.999093	0.999296	1.000000	1.000634	1.000000
Exact		1.000000	1.000000	1.000000	1.000000	1.000000
$n_{el} = 2$	$v^h(L)/v(L)$	1.006455	1.026172	1.110721	1.275163	Undef.*
$n_{el} = 8$		1.000402	1.001608	1.006455	1.014605	Undef.*
$n_{el} = 16$		1.000100	1.000402	1.001608	1.003624	Undef.*
Exact		1.000000	1.000000	1.000000	1.000000	Undef.*

*Undefined ratio (0/0). For $\bar{M}L/EI = 2\pi$, both the analytical and numerical vertical tip displacements are equal to zero

Table 2 Cantilever beam under transverse tip load: locking test results—tip displacements and normalized ratios for varying h/L

h/L	$u^h(L)$	$v^h(L)$	$u^h(L)/u^{\text{ref}}(L)$	$v^h(L)/v^{\text{ref}}(L)$
1/10	-2.41421×10^{-8}	-0.0002016	0.9968215	1.0000152
1/20	-1.53453×10^{-6}	-0.0016032	0.9968011	1.0000179
1/50	-0.0003740	-0.0249920	0.9967984	1.0000176
1/100	-0.0224230	-0.1923980	0.9970602	1.0001836
1/200	-0.3767520	-0.7075380	0.9994849	1.0017433

Figure 4 shows the deformed configurations obtained for increasing values of the applied moment on the 8-element mesh. Despite the coarse discretization, the computed solutions accurately reproduce the exact circular configurations and capture the continuous evolution of the beam from the undeformed state to the closed circular configuration corresponding to $\bar{M}L/EI = 2\pi$.

6.2 Cantilever beam under transverse tip load

This benchmark considers a shear-flexible cantilever beam subjected to a transverse tip load, as illustrated in Fig. 5, to assess the robustness of the MFEF. The analysis is structured in three parts: (i) the sensitivity of the MFEF to both shear- and membrane-locking is examined, (ii) the convergence behavior, accuracy, and computational efficiency of the formulation are evaluated through direct comparison with the DBFEF, and (iii) the numerical stability of the MFEF is investigated through an eigenvalue analysis of the equivalent stiffness matrix.

6.2.1 Shear- and membrane-locking test

The cantilever beam illustrated in Fig. 5 is first considered to assess the sensitivity of the proposed formulation to locking phenomena. The adopted numerical parameters are: $P = 1$, $L = 1$, $b = L/50$, $E = 10^9$, and $\nu = 1/3$.

The beam was discretized into 8 finite elements. The tip displacements $u^h(L)$ and $v^h(L)$, as well as the normalized ratios $u^h(L)/u^{\text{ref}}(L)$ and $v^h(L)/v^{\text{ref}}(L)$, were computed for

Table 3 Cantilever beam under transverse tip load: DOFs comparison between MFEF and DBFEF for uniform meshes

n_{el}	DOFs	
	MFEF	DBFEF
2	4	6
4	8	12
8	16	24
16	32	48
32	64	96
64	128	192
128	256	384

several height-to-length ratios h/L . The reference solutions $u^{\text{ref}}(L)$ and $v^{\text{ref}}(L)$ were obtained using the DBFEF with a highly refined mesh of 128 elements. The results are summarized in Table 2.

The computed tip displacements remain in excellent agreement with the reference solutions over the entire range of slenderness ratios considered. The absence of any deterioration in accuracy as h/L decreases provides strong numerical evidence that the proposed formulation is free from both shear- and membrane-locking.

6.2.2 Convergence, accuracy, and computational efficiency tests

The same cantilever beam is further analyzed to evaluate the convergence behavior, numerical accuracy, and computational efficiency of the MFEF relative to the DBFEF. The adopted numerical parameters are: $P = 10$, $L = 1$,

Fig. 4 Cantilever beam under pure bending: deformed configurations normalized by L computed with the MFEF on the 8-element mesh for load levels $\bar{M}L/EI$ ranging over $[0, 2\pi]$

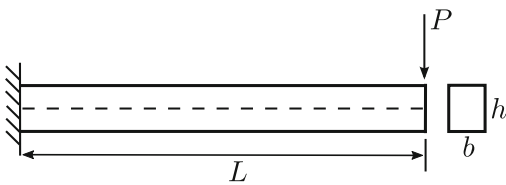
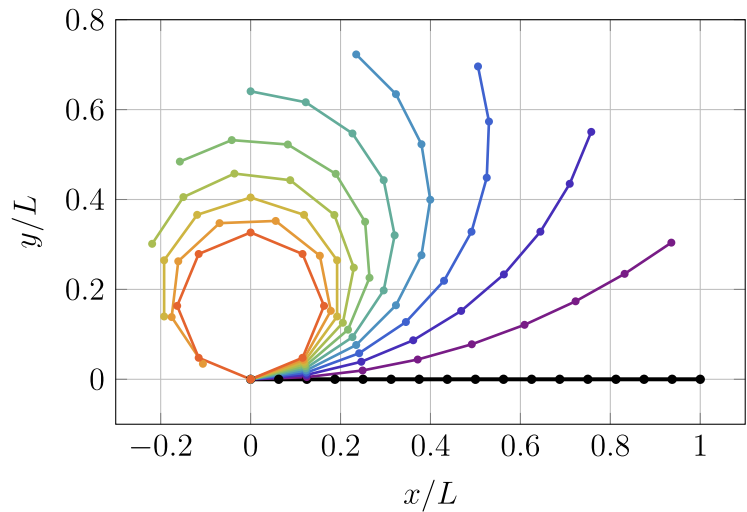


Fig. 5 Cantilever beam under transverse tip load: locking test

$b = L/20$, $h = L/20$, $E = 10^7$, and $\nu = 1/3$. The reference solutions are obtained using the DBFEF with a highly refined mesh of 256 elements, which requires 768 degrees of freedom (DOFs).

6.2.2.1 Discretization Convergence Behavior and Accuracy

Uniform meshes with $n_{el} = 2^n$ elements ($n = 1, 2, \dots, 7$) were employed. The DOFs required for each model are summarized in Table 3, highlighting that the DBFEF requires 50% more DOFs than the MFEF for the same mesh.

The relative errors of tip displacements, tip rotation, strain energy U (DBFEF), and complementary strain energy U_c (MFEF) were computed. The results, shown in Fig. 6, demonstrate that the MFEF converges monotonically toward the reference solution as the mesh is refined. The observed convergence rates are approximately quadratic and closely match those obtained with the DBFEF. For a given mesh, however, the MFEF consistently yields smaller errors, indicating superior accuracy per degree of freedom.

6.2.2.2 Path-Following Behavior and Computational Efficiency

To assess the path-following convergence characteristics of the two formulations, the cantilever beam was analyzed on the 8-element mesh using both the DBFEF and the MFEF. The total load $P = 10$ was applied in 10 equal increments, and the load-control Newton–Raphson scheme was adopted

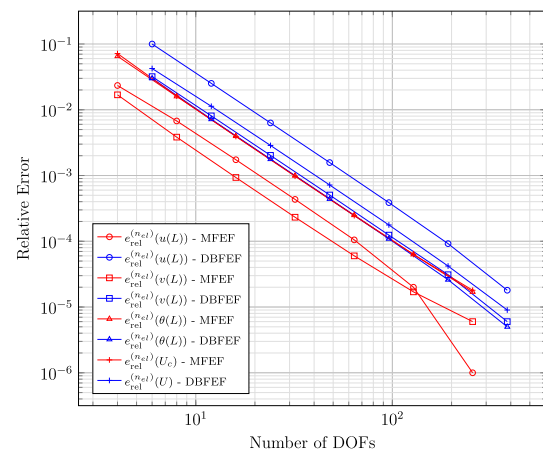


Fig. 6 Cantilever beam under transverse tip load: convergence and accuracy results

Table 4 Cantilever beam under transverse tip load: number of iterations required at each load step for convergence of the MFEF and DBFEF

Load Step	DBFEF (24 DOFs)	MFEF (16 DOFs)
1	4	3
2	3	2
3-10	4	2
Total	39	21

in both cases. An identical absolute tolerance of 10^{-8} was used to ensure a consistent comparison. The number of iterations required for convergence at each load step is reported in Table 4.

The mixed formulation demonstrates substantially better performance in the load-control Newton procedure, achieving convergence with 21 iterations as opposed to the 39 iterations required by the displacement-based model, which corresponds to a reduction of approximately 46% in the total

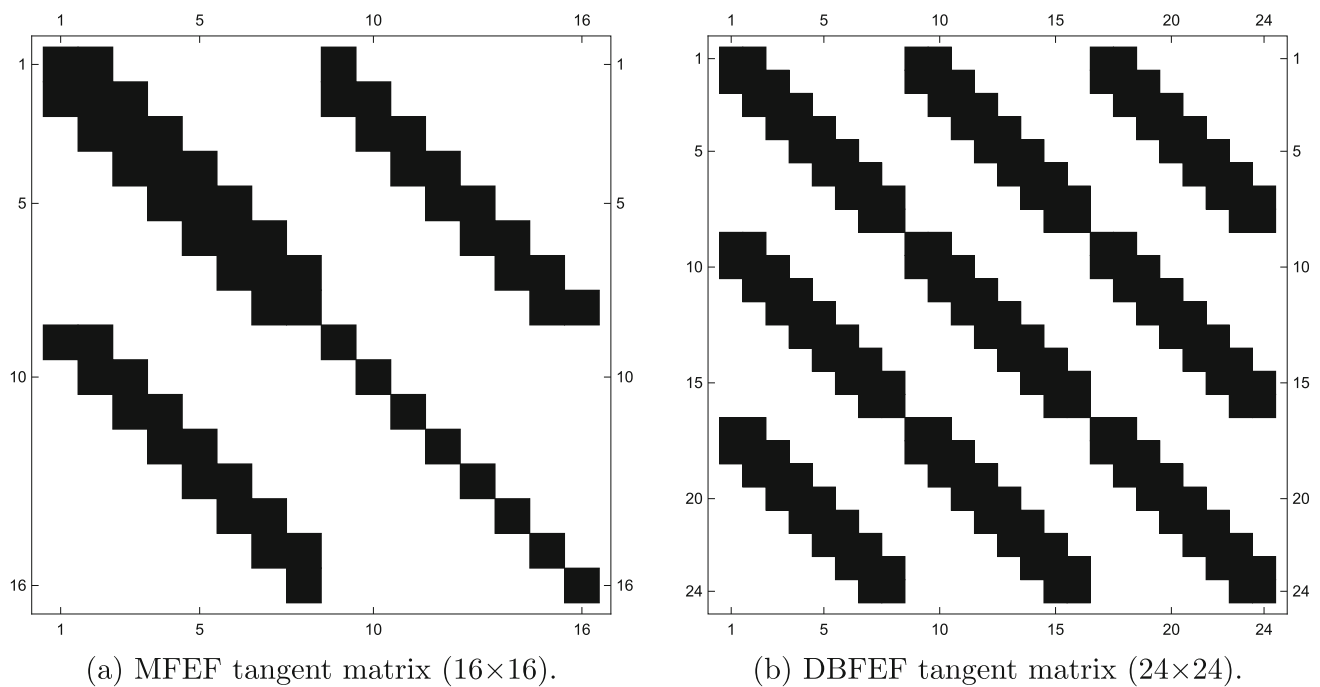


Fig. 7 Sparsity patterns of the global tangent matrices for an 8-element cantilever beam

number of Newton iterations. In addition to the overall reduction in iterations, the convergence response is more regular: the MFEF stabilizes to two iterations for nearly all subsequent increments, whereas the DBFEF consistently requires about four iterations, indicating a slower and less regular convergence behavior attributable to a stronger sensitivity to changes in the tangent stiffness. These results indicate a markedly higher robustness of the mixed formulation and are consistent with the trends reported in the literature and summarized in Sect. 1.

Although the detailed investigation of this aspect lies outside the main focus of the present study, the same trend was consistently verified in all other analyzed problems. The DBFEF systematically demands smaller continuation increments together with a higher number of Newton iterations to maintain convergence. As a result, the computational effort of the displacement-based approach, and hence its overall analysis time, grows considerably, particularly for refined meshes and demanding path-tracing tasks.

Finally, the algebraic structure of the global tangent matrix arising from the proposed MFEF is examined and compared with that of the DBFEF, with particular emphasis on their sparsity and bandwidth properties. Figures 7a, b compare the sparsity patterns of the global tangent matrices obtained with the MFEF and the DBFEF for an 8-element cantilever beam, where black squares denote nonzero matrix entries. Despite employing the same mesh, the MFEF is associated with a smaller global system (16 versus 24 DOFs), and the tangent matrix exhibits a highly structured sparse pattern. In contrast,

the DBFEF matrix displays a wider bandwidth and a higher density of nonzero entries, reflecting the full nodal coupling of displacement and rotation DOFs and the inherently stronger interelement coupling of the displacement-based formulation. Although the bandwidth of the DBFEF matrix depends on the adopted ordering of the degrees of freedom, the observed differences primarily reflect intrinsic coupling characteristics of the two formulations.

The differences observed in the sparsity patterns can be directly traced to the organization and physical nature of the DOFs adopted by each formulation. In the MFEF, the global tangent matrix is ordered such that the interelement bending moment DOFs appear first, followed by the element-local rotational DOFs. Note that, in this specific problem, no DOFs associated with internal forces are present. This ordering leads to the three dominant diagonal block bands visible in Fig. 7a: a central band associated with moment-moment coupling, flanked by two bands corresponding to moment-rotation interactions, with the remaining rotational DOFs contributing primarily to isolated diagonal entries.

In contrast, the DBFEF employs a field-wise ordering in which the rotational DOFs are followed by the axial and transverse displacements. The simultaneous coupling of these nodal kinematic variables across adjacent elements gives rise to the five diagonal block bands observed in Fig. 7b, each of larger dimension and higher density. Overall, the MFEF achieves a higher sparsity level, a narrower effective bandwidth, and a reduced system size relative to the DBFEF, which translates into lower memory requirements

and improved efficiency in the solution of the linearized equations.

6.2.3 Numerical stability test

The numerical stability of the proposed mixed finite element formulation is now investigated through the analysis of the cantilever beam problem using both 1- and 3-element meshes. The 1-element discretization comprises 2 global DOFs (one kinematic and one static), whereas the 3-element mesh comprises 6 global DOFs (three kinematic and three static). The structure is subjected to a total load of $P = 10$, applied incrementally in 10 equal load steps, and the non-linear equations are solved using a standard load-controlled Newton–Raphson scheme.

This benchmark problem is particularly well suited for evaluating numerical stability, since all equilibrium configurations along the loading path are known *a priori* to be physically stable. Accordingly, no loss of structural stability, limit points, or bifurcation phenomena are expected, and the equivalent stiffness matrix must remain positive definite throughout the analysis. Any occurrence of zero or negative eigenvalues of the equivalent stiffness matrix would therefore indicate numerical pathologies rather than genuine physical instabilities.

Table 5 reports the eigenvalues of the equivalent stiffness matrices associated with the 1- and 3-element discretizations, evaluated at each load step. In all cases, the eigenvalues remain strictly positive throughout the entire loading process. This confirms that: (i) no loss of structural stability occurs along the equilibrium path, in agreement with the known behavior of the benchmark problem; and (ii) the equivalent stiffness matrix remains positive definite throughout the analysis, indicating the absence of spurious zero-energy modes and the preservation of numerical stability even under extremely coarse discretizations.

6.3 Simply-supported beam under uniformly distributed transverse load

A simply-supported shear-flexible beam subjected to a uniformly distributed transverse load w is considered, as illustrated in Fig. 8. The geometric and material parameters adopted in the numerical simulations are: $L = 1$, $b = h = L/20$, $E = 10^6$, and $\nu = 1/3$.

The problem is analyzed using the DBFEF on a highly refined mesh of 128 elements (384 DOFs), while the proposed MFEF is employed on much coarser meshes comprising 2, 4, and 8 elements (4, 8, and 16 DOFs, respectively). The equilibrium paths of $\theta(L)$ and $u(L)$ are shown in Figs. 9 and 10, respectively. The MFEF solutions exhibit rapid and monotonic convergence toward the DBFEF reference solution as the mesh is refined. In particular, the equilibrium

Table 5 Cantilever beam under transverse tip load: eigenvalues of the equivalent stiffness matrix at each load step for the 1- and 3-element discretizations

Load step	$n_{el} = 1$	$n_{el} = 3$		
	Eig.	Eig. 1	Eig. 2	Eig. 3
1	15.688913	154.286581	46.896276	4.409817
2	15.878593	154.331424	46.959096	4.490612
3	16.188148	154.404243	47.060647	4.620883
4	16.608634	154.502538	47.196849	4.794970
5	17.129098	154.623404	47.363012	5.006400
6	17.737654	154.763874	47.554402	5.248688
7	18.422391	154.921158	47.766636	5.515879
8	19.172035	155.092762	47.995873	5.802812
9	19.976336	155.276544	48.238875	6.105187
10	20.826260	155.470694	48.492982	6.419520

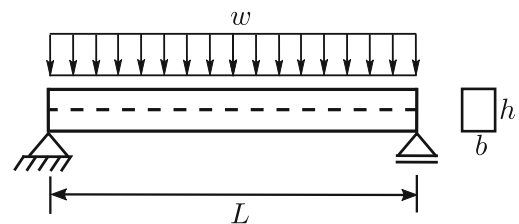


Fig. 8 Simply-supported beam under uniformly distributed transverse load

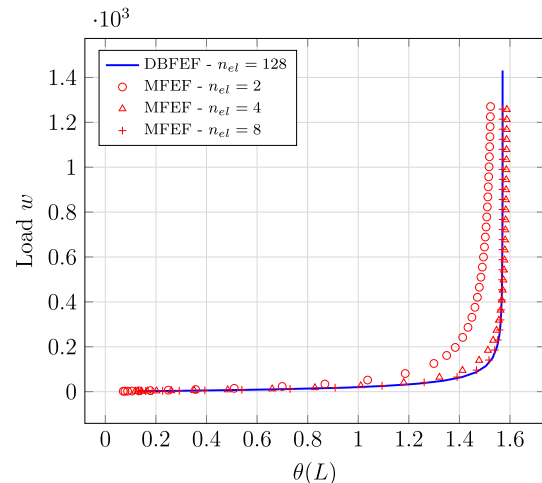


Fig. 9 Simply-supported beam under uniformly distributed transverse load: load-rotation curves for $\theta(L)$

paths obtained with the 8-element MFEF discretization are already in excellent quantitative agreement with those computed using the 128-element DBFEF model, despite the latter employing an order of magnitude more DOFs.

The deformed configurations of the beam computed with the MFEF on the 8-element mesh at selected load levels spanning $w \in [0, 1124.54]$, are shown in Fig. 11. The recov-

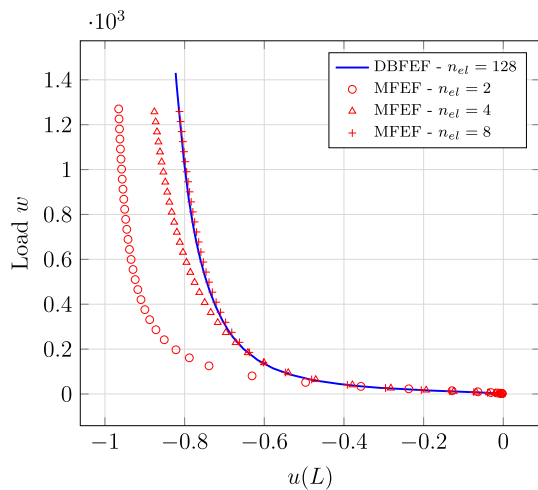


Fig. 10 Simply-supported beam under uniformly distributed transverse load: load–displacement curves for $u(L)$

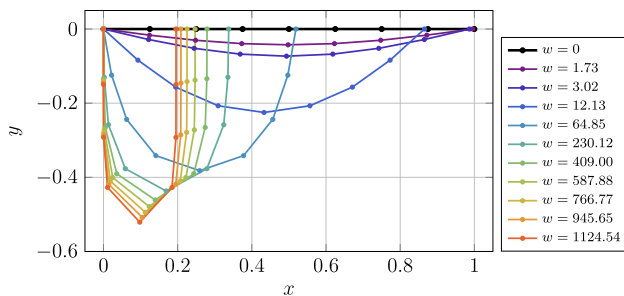


Fig. 11 Simply-supported beam under uniformly distributed transverse load: deformed configurations computed with the MFEF on the 8-element mesh for various load levels

ered displacement fields are C^0 -continuous and satisfy the prescribed kinematic boundary conditions of the problem, $v(0) = v(L) = 0$ and $u(0) = 0$, for all load levels considered.

The corresponding bending moment distributions are presented in Fig. 12. The distributions are C^0 -continuous and satisfy the equilibrium boundary conditions of the problem, namely $M(0) = 0$ and $M(L) = 0$. In contrast, the bending moment distributions recovered from the two-node displacement-based formulation employed herein are discontinuous across element interfaces and do not satisfy these boundary conditions, even on the finest mesh considered. This observation should not, however, be interpreted as a general limitation of displacement-based formulations. In geometrically linear beam analysis, higher-order formulations can satisfy equilibrium boundary conditions exactly. For example, the virtual element formulation of [41], based on a quadratic interpolation of the rotation field and a cubic interpolation of the transverse displacement, reproduces the exact nodal solution of the Timoshenko beam equations and therefore satisfies the corresponding equilibrium boundary conditions independently of mesh refinement. In the

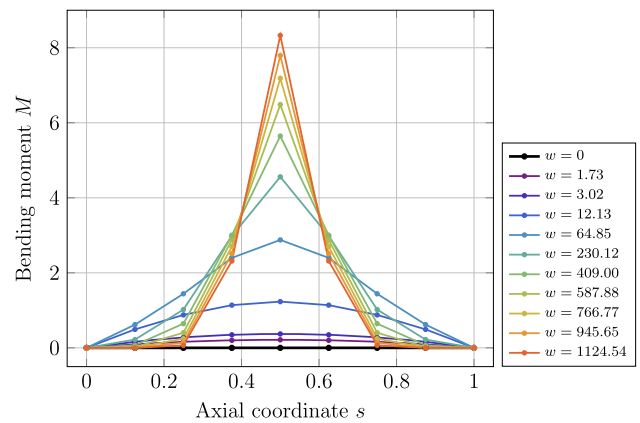


Fig. 12 Simply-supported beam under uniformly distributed transverse load: bending moment distributions corresponding to the deformed configurations shown in Fig. 11

geometrically exact setting, however, the strong nonlinear coupling among equilibrium, compatibility, and constitutive relations renders such exact satisfaction generally unachievable through displacement-based approximations, regardless of the order of the ansatz functions employed.

6.4 Clamped toggle frame

A two-member shear-flexible shallow frame with a rigid apex joint, as illustrated in Fig. 13, is considered. Both ends of the frame are clamped, and the structure is subjected to a vertically downward point load applied at the apex. This configuration is commonly referred to in the literature as a toggle frame, owing to its highly nonlinear behavior and characteristic snap-through response, in which the frame transitions abruptly from one equilibrium configuration to another under increasing load. Two distinct configurations are analyzed. First, the classical Williams toggle frame is analyzed. Subsequently, a modified Williams frame with increased height is analyzed. These two configurations exhibit markedly different nonlinear responses and therefore provide complementary benchmarks for assessing the path-following and stability-detection capabilities of the proposed formulation.

6.4.1 Clamped toggle frame according to Williams

The frame shown in Fig. 13 was studied analytically and experimentally by Williams [48] and has since served as a benchmark in numerous numerical studies. In his analytical treatment, Williams accounted for finite changes in geometry, the influence of axial forces on the flexural stiffness, and the flexural shortening of the members, yielding an approximate closed-form solution. Importantly, Williams' solution was

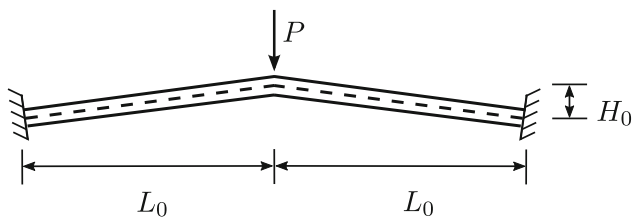


Fig. 13 Clamped toggle frame: geometric configuration and applied apex load

derived under the assumption that the members behave as shear-rigid.

Here, L_0 denotes the horizontal half-span of the frame and H_0 its initial apex height (see Fig. 13). The numerical parameters adopted are: $L_0 = 12.943$, $H_0 = 0.386$, $b = 0.753$, $h = 0.243$, $E = 10.3 \times 10^6$, and $\nu = 1/3$.

The frame was analyzed using the DBFEF on a 32-element-per-member mesh (189 DOFs) and the proposed MFEF with shear-flexible members on 4- and 16-element-per-member meshes (19 and 67 DOFs, respectively). The symmetry of the frame was not exploited to assess the performance of the formulation in a general setting.

The load–displacement curves for the vertical displacement of the frame apex are shown in Fig. 14. The results demonstrate excellent agreement between the proposed MFEF, the DBFEF reference solution, and the approximate analytical solution. The close agreement with Williams’ shear-rigid solution is physically expected, given the high slenderness of the members ($h/L \approx 1/53$, where $L = \sqrt{L_0^2 + H_0^2}$ is the actual member length), for which transverse shear deformation effects are negligible. The equilibrium paths exhibit a clear limit point, followed by snap-through behavior and post-peak softening. Notably, even for coarse MFEF discretizations, the overall nonlinear response (including the peak load, the snap-through transition, and the post-peak softening branch) is captured accurately. Increasing the number of elements per member further improves convergence toward the reference solution.

The load–reaction curves computed using the MFEF on the 16-element-per-member mesh are shown in Fig. 15, where $N^*(0)$, $V^*(0)$, and $M(0)$ denote the horizontal force, vertical force, and bending moment reactions at the left clamped support, respectively. These curves provide a complementary view of the snap-through instability in terms of boundary reactions. A distinct limit point is clearly identifiable, beyond which the equilibrium path becomes unstable. The separate evolution of the horizontal force, vertical force, and bending moment reactions highlights the strongly coupled nature of the boundary response near the instability.

Figure 16 shows the deformed configurations of the frame computed using the MFEF on the 16-element-per-member mesh at selected load levels spanning $P \in [0, 100.25]$.

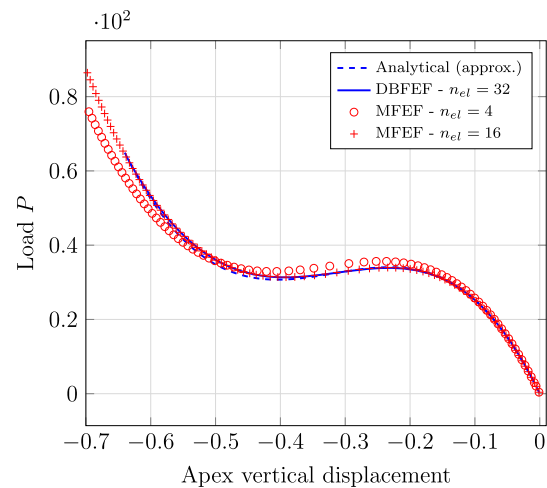


Fig. 14 Clamped toggle frame according to Williams: load–displacement curves for the vertical displacement of the frame apex

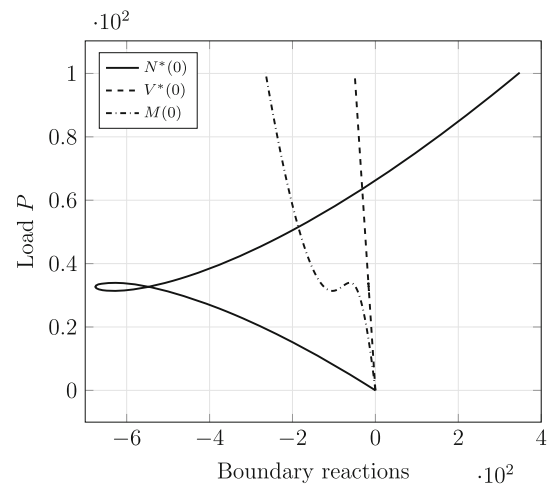


Fig. 15 Clamped toggle frame according to Williams: load–reaction curves computed using the MFEF on the 16-element-per-member mesh

The corresponding bending moment distributions are presented in Fig. 17. Both the displacement fields underlying the deformed configurations and the bending moment distributions are C^0 -continuous across element interfaces. Furthermore, the bending moments remain identically zero at the midpoint of the members across all load levels. This follows from the symmetry of the frame and loading: the apex rotation vanishes identically, as a result of the left–right symmetry of the deformed configurations about the vertical axis through the apex (see Fig. 16), so each member has zero rotation at both ends. The rotation field $\theta(s)$ within each member is therefore symmetric about its midpoint, making the curvature $\kappa = \theta'$ anti-symmetric and hence zero at the midpoint; by $M = EI\kappa$, the bending moment vanishes there for all load levels.

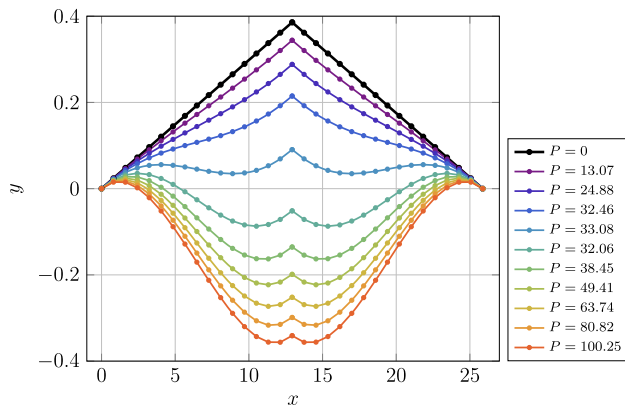


Fig. 16 Clamped toggle frame according to Williams: deformed configurations obtained for various load levels using the MFEF on the 16-element-per-member mesh

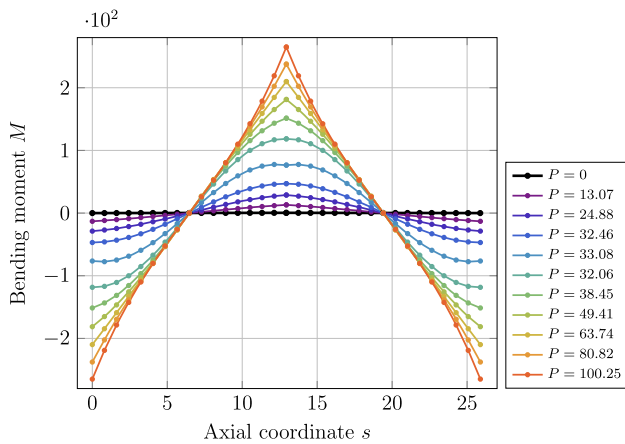


Fig. 17 Clamped toggle frame according to Williams: bending moment distributions corresponding to the deformed configurations shown in Fig. 16

6.4.2 Modified williams toggle frame with increased height

The configuration examined herein differs from the classical Williams frame only in that its total height is doubled, resulting in $H_0 = 0.772$, with all remaining geometric and material parameters unchanged. This geometric modification produces a markedly different nonlinear response. While the original Williams frame exhibits a pure snap-through mechanism, in which instability is triggered directly at a limit point, the modified frame develops a symmetry-breaking bifurcation prior to the onset of snap-through, leading to the emergence of unsymmetric deformation modes not present in the classical configuration. Consequently, this example complements the classical Williams frame by introducing a distinct instability mechanism and by enabling the investigation of bifurcation phenomena and post-bifurcation equilibrium paths in addition to the limit-point behavior examined previously.

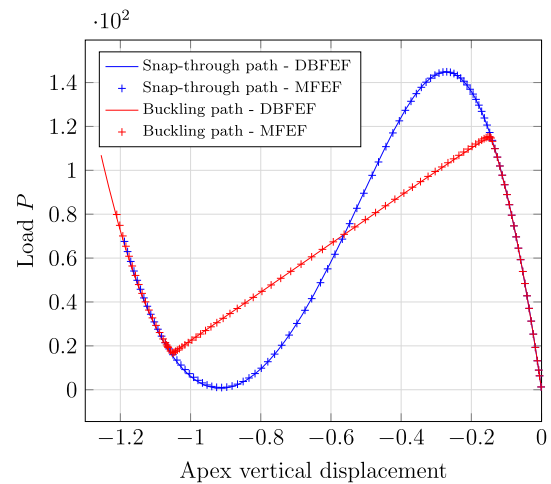


Fig. 18 Modified Williams toggle frame with increased height: load-displacement curves obtained for the vertical displacement of the apex using the 32-element-per-member mesh

The frame was analyzed using both the MFEF and the DBFEF on a 32-element-per-member mesh (131 and 189 DOFs for MFEF and DBFEF, respectively). The load-displacement curves for the vertical displacement of the apex are presented in Fig. 18, allowing a direct comparison of the two formulations. Excellent agreement is observed throughout the traced equilibrium paths. Initially, the frame deforms symmetrically along the fundamental equilibrium path; however, this symmetric deformation becomes unstable at a symmetry-breaking bifurcation, occurring shortly before the maximum limit point on the load-displacement curve. The transition between symmetric and unsymmetric deformation modes is further confirmed by the load-rotation curves at the frame apex computed using the MFEF, shown in Fig. 19. To capture the equilibrium paths associated with the unsymmetric deformation modes, a small perturbation in the form of an applied bending moment, $\bar{M} = -P/100$, was applied at the apex of the frame. Since the two post-bifurcation branches are mirror images of each other (the frame may buckle to either side of the vertical axis of symmetry), the specific sign adopted here is arbitrary; the opposite sign would trace the mirror-image branch and yield equivalent results.

The load-reaction curves at the left clamped support obtained using the MFEF are shown in Figs. 20 and 21. The horizontal force and bending moment reactions clearly distinguish between the snap-through and buckling paths, whereas the vertical force reaction remains almost insensitive to the transition from symmetric to unsymmetric deformation modes.

The deformed configurations of the frame obtained using the proposed MFEF with the small perturbation ($\bar{M} = -P/100$) at selected load levels spanning $P \in [0, 114.39]$,

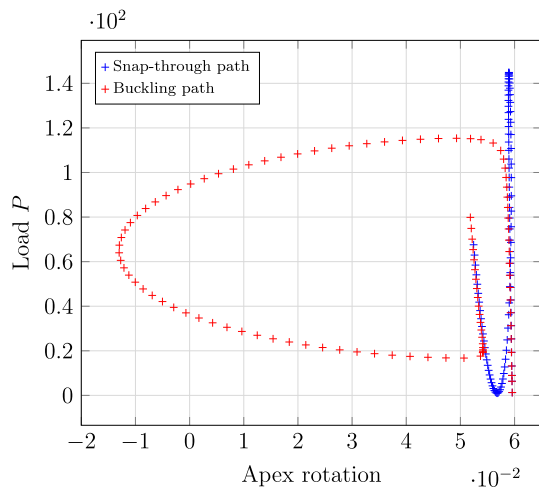


Fig. 19 Modified Williams toggle frame with increased height: load-rotation curves obtained for the apex using the MFEF on the 32-element-per-member mesh

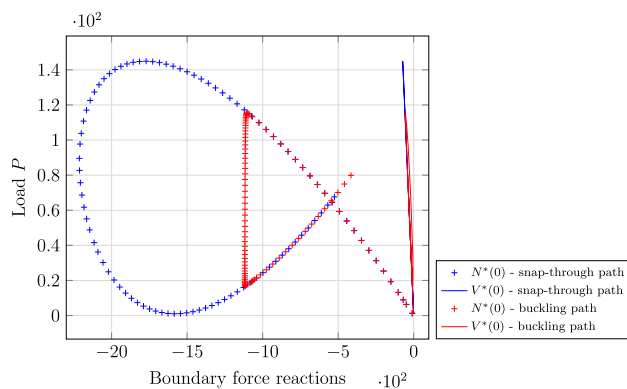


Fig. 20 Modified Williams toggle frame with increased height: load-reaction curves obtained for the horizontal and vertical forces at the left end of the frame using the MFEF on the 32-element-per-member mesh

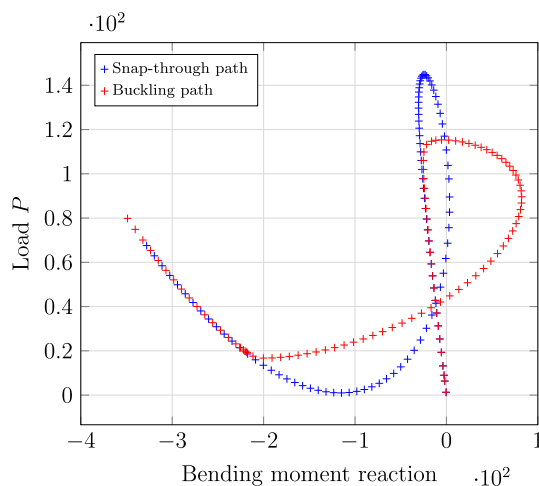


Fig. 21 Modified Williams toggle frame with increased height: load-reaction curves obtained for the bending moment at the left end of the frame using the MFEF on the 32-element-per-member mesh

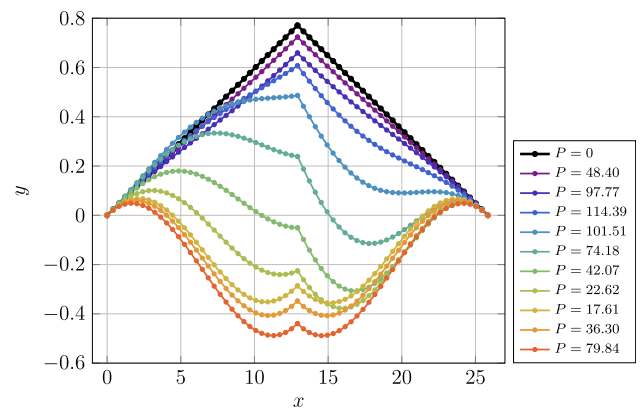
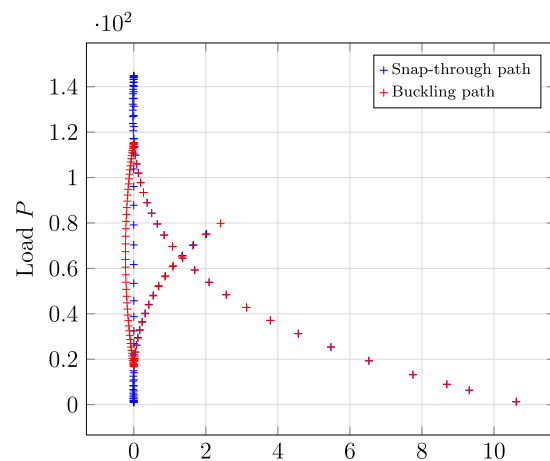


Fig. 22 Modified Williams toggle frame with increased height: deformed configurations corresponding to the buckling path obtained using the MFEF on the 32-element-per-member mesh



Normalized determinant of the equivalent stiffness matrix η_K

Fig. 23 Modified Williams toggle frame with increased height: normalized determinant of the equivalent stiffness matrix obtained using the MFEF on the 32-element-per-member mesh

are shown in Fig. 22. These results demonstrate (corroborating the preceding discussion) the transition from the initially symmetric deformation pattern to the unsymmetric (and unstable) configurations that arise beyond the symmetry-breaking bifurcation. Since the unsymmetric branch emerging from the bifurcation point is unstable, the structure cannot remain on this branch under infinitesimal perturbations and therefore transitions toward a stable equilibrium configuration. While some points along the traced paths correspond to stable equilibrium states, others represent unstable configurations, and yet others denote critical states that signal transitions between these regimes. This distinction is particularly evident in Fig. 23, which shows the normalized determinant η_K obtained for the 32-element-per-member discretization, clearly identifying the points associated with stability loss and bifurcation.

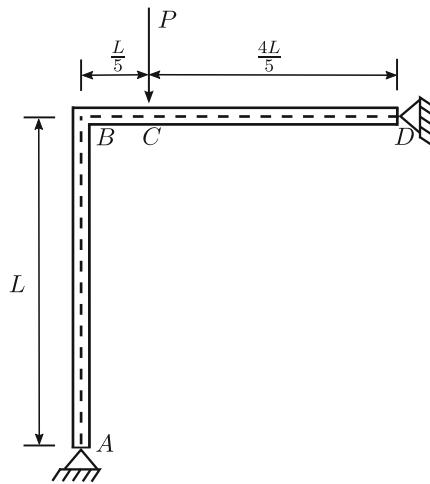


Fig. 24 Lee frame

6.5 Lee frame

The Lee frame, depicted in Fig. 24, is a statically indeterminate structure well known for its complex nonlinear behavior, including both snap-through and snap-back phenomena [49].

The following sectional stiffness parameters and geometric parameter are adopted: $EA = 4320$, $EI = 1440$, and $L = 120$. Both shear-flexible and shear-rigid cases are considered. In the shear-flexible case, the shear stiffness is taken as $GA_s = 1660$. While the MFEF is applicable to both shear-rigid and shear-flexible beam models, the DBFEF is restricted to the shear-flexible case.

The frame was analyzed using both formulations on a series of uniform meshes. For the MFEF, discretizations consisting of 5, 10, 20, 40, 80, and 160 elements per member were considered. The DBFEF was applied only to finer meshes, namely 300 and 360 elements per member. A convergence study focusing on the first and second limit loads of the frame was first carried out. The solutions obtained with the DBFEF using the mesh with 360 elements per member were taken

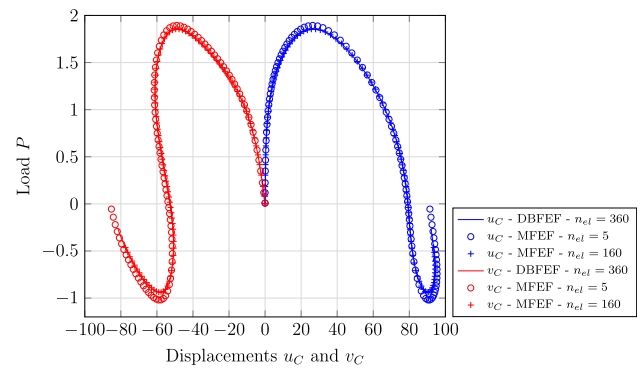


Fig. 25 Lee frame with shear-flexible members: load–displacement curves obtained for the horizontal and vertical displacements at point C

as reference values. Since the DBFEF is restricted to shear-flexible members, this reference applies to the shear-flexible case only; for the shear-rigid case, no independent reference is available, and convergence is assessed by inspecting the MFEF solutions obtained on progressively refined meshes. The numerical results produced by the two formulations are summarized in Table 6.

First, the shear-flexible models yield slightly lower critical loads than their shear-rigid counterparts, reflecting the influence of shear deformability on the global stiffness of the frame. Second, for both shear-rigid and shear-flexible cases, the MFEF exhibits monotonic convergence from the conservative side, yielding critical-load estimates whose absolute values exceed the converged values. This convergence pattern is analogous to that traditionally observed for standard displacement-based formulations. Finally, the results demonstrate that the MFEF provides highly accurate estimates of the critical loads even when relatively coarse discretizations are employed, underscoring its efficiency and robustness in capturing the nonlinear response of the structure.

The load–displacement responses obtained for the shear-flexible models at point C (the point at which the external

Table 6 Lee frame: convergence study of critical loads

	n_{el}	DOFs	Shear-rigid		Shear-flexible			
			$P_{cr}^{(1)}$	$P_{cr}^{(2)}$	$P_{cr}^{(1)}$	$P_{cr}^{(2)}$	$e_{rel}^{(n_{el})}(P_{cr}^{(1)})$	$e_{rel}^{(n_{el})}(P_{cr}^{(2)})$
MFEF	5	21	1.895009	−1.023074	1.893934	−1.021057	0.021168	0.086638
	10	41	1.865877	−0.961821	1.864845	−0.959945	0.005484	0.021601
	20	81	1.858247	−0.946531	1.857225	−0.944689	0.001376	0.005365
	40	161	1.856318	−0.942715	1.855299	−0.940881	0.000337	0.001313
	80	321	1.855834	−0.941761	1.854816	−0.939930	0.000077	0.000300
	160	641	1.855713	−0.941523	1.854695	−0.939692	0.000012	0.000047
DBFEF	300	1799			1.854682	−0.939663		
	360	2159			1.854674*	−0.939648*		

*Reference solutions for the shear-flexible case, obtained with the DBFEF on a 360-element-per-member mesh

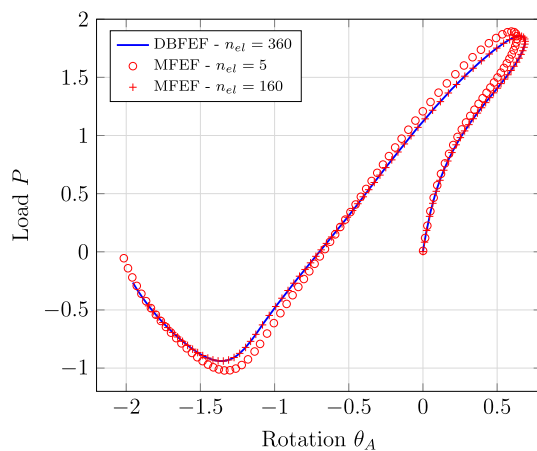


Fig. 26 Lee frame with shear-flexible members: load-rotation curves obtained for θ_A

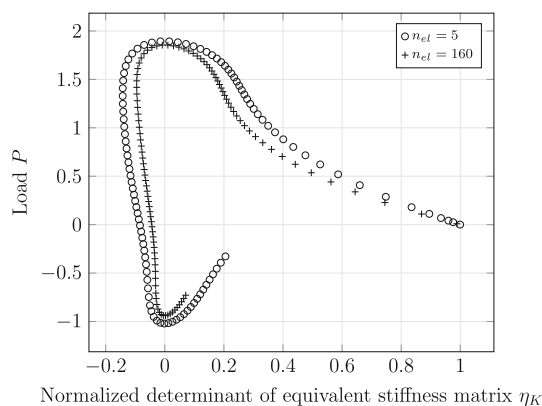


Fig. 27 Lee frame with shear-flexible members: normalized determinant η_K of the equivalent stiffness matrices of the MFEF discretized with 5 and 160 elements per member

load is applied) with the MFEF using the 5- and 160-element-per-member meshes (corresponding to 21 and 641 DOFs, respectively) are shown in Fig. 25, alongside the responses computed using the DBFEF on the 360-element-per-member mesh (2159 DOFs). Even the MFEF solutions obtained for the coarsest mesh closely match the reference solutions. Moreover, from a practical standpoint, the MFEF solutions for the finest mesh are indistinguishable from the DBFEF results.

Figure 26 presents the load-rotation curves at point A (the left pin support) for the shear-flexible model, obtained with the MFEF using the 5- and 160-element-per-member meshes alongside the DBFEF reference solution on the 360-element-per-member mesh. The same observations apply: even the coarsest MFEF discretization closely tracks the reference equilibrium path, and the finest MFEF solution is visually indistinguishable from the DBFEF result.

Figure 27 plots the normalized determinant η_K of the equivalent stiffness matrices as a function of the applied load,

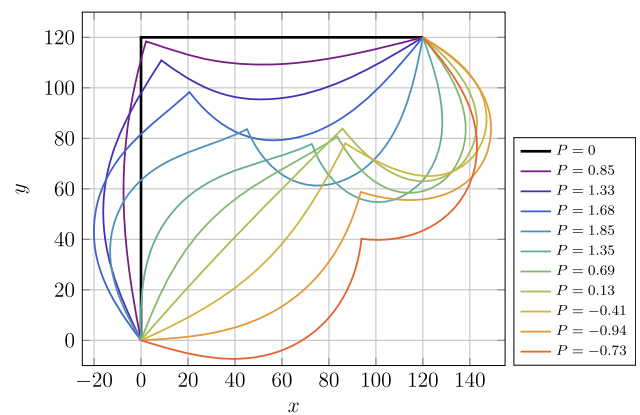


Fig. 28 Lee frame with shear-flexible members: deformed configurations obtained for various load levels using the MFEF on the 160-element-per-member mesh

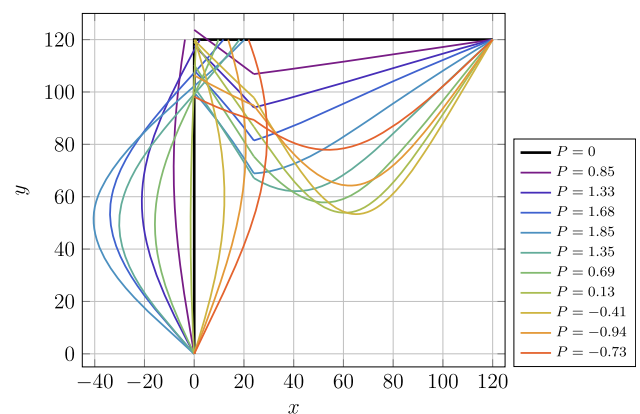


Fig. 29 Lee frame with shear-flexible members: bending moment distributions corresponding to the deformed configurations shown in Fig. 28 (bending moment values plotted at the same scale as the frame geometry)

obtained with the MFEF considering meshes of 5 and 160 elements per member. The results show that the limit loads coincide with sign changes of the normalized determinant. For the present problem, these sign changes are associated with transitions between stable and unstable equilibrium states. Figure 28 illustrates the deformed configurations of the frame for the shear-flexible model, computed using the MFEF with a 160-element-per-member discretization under selected load levels spanning $P \in [-0.94, 1.85]$. The displacement fields are C^0 -continuous and satisfy the kinematic boundary conditions, $u_A = u_D = v_A = v_D = 0$. The corresponding bending moment distributions are presented in Fig. 29. The bending moments are also C^0 -continuous and satisfy the equilibrium boundary conditions associated with the pin supports, namely $M_A = 0$ and $M_D = 0$.

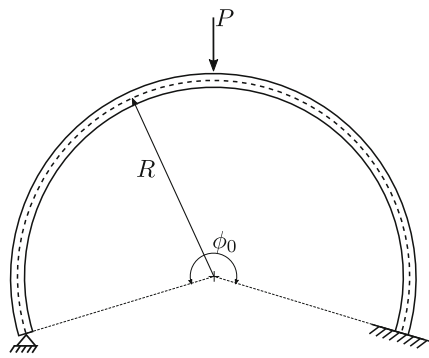


Fig. 30 Hinged-clamped 215-degree arch

Table 7 Hinged-clamped 215-degree arch: convergence of the critical load

	n_{el}	DOFs	$P_{cr} R^2 / EI$	$e_{rel}^{(n_{el})}(P_{cr})$
MFEF	8	18	9.810591	0.093355
	16	34	9.202712	0.025609
	32	66	9.032639	0.006655
	64	130	8.987988	0.001679
	128	258	8.976716	0.000423
	256	514	8.973892	0.000108
	512	1026	8.973185	0.000029
Ref. Sol. [50]			8.972922	
Sol. [51]			8.97	

6.6 Hinged-clamped 215-degree arch

This example considers a circular, shear-rigid arch with a central angle $\phi_0 = 215^\circ$ and radius $R = 100$, hinged at one end and clamped at the other, as shown in Fig. 30. A vertical point load P is applied at the crown of the arch. The cross-sectional stiffness parameters adopted in the simulations are: $EA = 10^8$ and $EI = 10^6$. The curved geometry of the arch is approximated by an assembly of straight finite elements through a piecewise-constant representation of the initial rotation field, as described in Sect. 4.2.

This benchmark problem was previously investigated in [50], where a finite-difference formulation coupled with a shooting method was employed to obtain a highly resolved estimate of the critical load of the arch. The converged value reported therein is adopted here as the reference solution for assessing the numerical accuracy of the proposed formulation. For further context and historical comparison, the classical prediction reported in [51], based on Euler's nonlinear theory of the inextensible curved elastica and computed numerically using a finite-difference scheme [52], is also included.

In the present work, the problem is solved using the proposed MFEF on a sequence of uniform meshes with $n_{el} = 2^n$

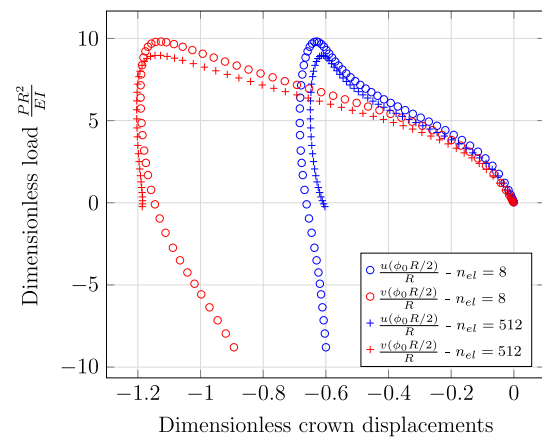


Fig. 31 Hinged-clamped 215-degree arch: load–displacement curves for the horizontal and vertical displacements at the crown of the arch obtained using the MFEF on the 8- and 512-element meshes

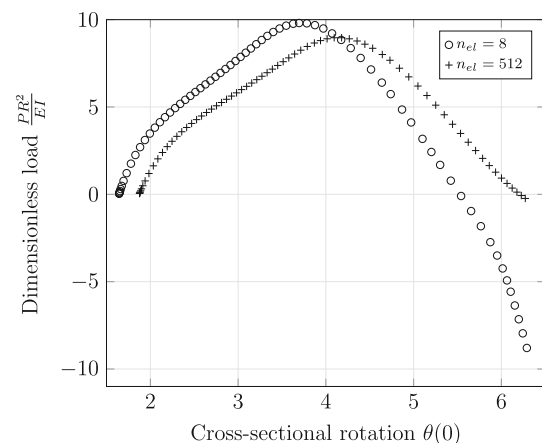


Fig. 32 Hinged-clamped 215-degree arch: load-rotation curves at the hinged support ($s = 0$) obtained using the MFEF on the 8- and 512-element meshes

elements, $n = 3, 4, \dots, 9$. The corresponding critical loads, P_{cr} , together with the associated numbers of DOFs and the relative errors $e_{rel}^{(n_{el})}(P_{cr})$, defined in Eq. (46), are reported in Table 7. All errors are evaluated with respect to the reference solution of [50]. The critical-load error decreases by approximately a factor of four under successive mesh refinement, indicating second-order convergence. The converged MFEF result is also in excellent agreement with the classical prediction of [51], the small difference being attributable to the limited decimal precision of that solution; the near-inextensibility of the arch ($EAR^2/EI = 10^6 \gg 1$) ensures that the extensible and inextensible critical loads are virtually identical.

The load–displacement responses at the crown of the arch, $s = \phi_0 R/2$, obtained using the MFEF are shown in Fig. 31. Results computed with a very coarse discretization ($n_{el} = 8$) and a highly refined mesh ($n_{el} = 512$) are compared. Despite the large disparity in mesh resolution, the response curves

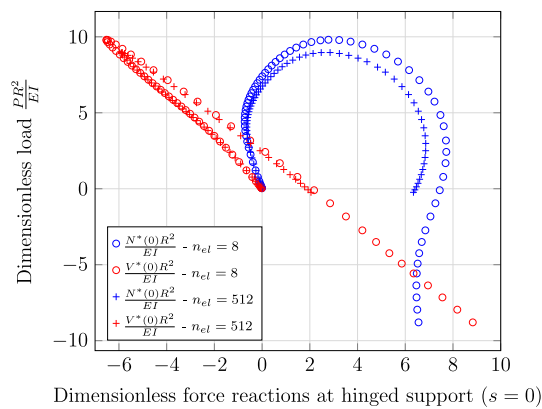


Fig. 33 Hinged-clamped 215-degree arch: load-reaction curves at the hinged support ($s = 0$) obtained using the MFEF on the 8- and 512-element meshes

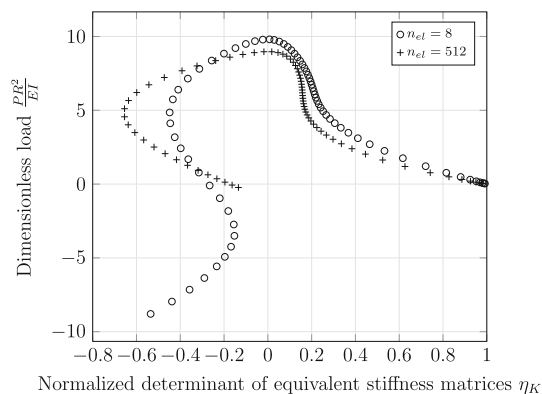


Fig. 34 Hinged-clamped 215-degree arch: normalized determinant of equivalent stiffness matrices obtained using the MFEF with 8 and 512 elements

exhibit excellent agreement over the entire loading range, including the post-critical regime. This highlights the ability of the proposed MFEF to accurately capture the global non-linear response of the arch with a remarkably small number of elements.

Figure 32 presents the load-rotation curves at the hinged support ($s = 0$) obtained with the MFEF using 8 and 512 elements. Due to the piecewise-constant representation of the initial rotation field, the geometric approximation of the curved centerline is relatively crude near the support for the 8-element discretization, which introduces a noticeable error in the computed support rotation. As a result, the corresponding load-rotation response deviates from the highly refined solution from the outset. This behavior contrasts with that observed for the crown displacements discussed previously, where excellent agreement between coarse and refined meshes was obtained over the entire loading range.

Figure 33 shows the load-reaction curves at the hinged support ($s = 0$) computed with the MFEF using 8 and 512 elements. The vertical reaction $V^*(0)$ obtained on the coarse mesh remains in very close agreement with the cor-

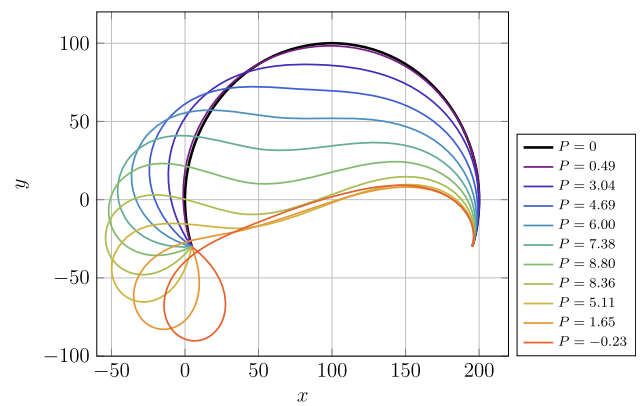


Fig. 35 Hinged-clamped 215-degree arch: deformed configurations obtained for various load levels using the MFEF on the 512-element mesh

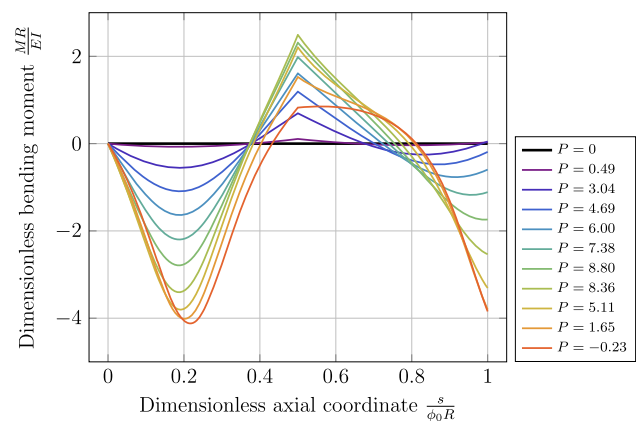


Fig. 36 Hinged-clamped 215-degree arch: bending moment distributions corresponding to the deformed configurations shown in Fig. 35

responding response computed on the highly refined mesh throughout the entire loading process, including the post-critical regime. In contrast, the horizontal reaction $N^*(0)$ computed with the 8-element discretization exhibits a progressive deviation from the reference 512-element solution as the load increases. This discrepancy can be attributed to the nature of the respective equilibrium conditions: the vertical reaction is governed predominantly by global vertical force equilibrium and is therefore comparatively insensitive to the geometric approximation error introduced by the coarse discretization, whereas the horizontal reaction is more sensitive to the geometric approximation of the curved centerline arising from the piecewise-constant representation of the initial rotation field, whose cumulative effect becomes increasingly pronounced as the deformations grow.

The evolution of the normalized determinant of the equivalent stiffness matrices, η_K , is shown in Fig. 34 as a function of the dimensionless load parameter PR^2/EI . The determinant remains strictly positive up to the critical load. Throughout this range, the equivalent stiffness matrix is positive definite and the traced equilibrium path is stable. At the critical point,

the determinant vanishes owing to the appearance of a zero eigenvalue. Beyond this point, one eigenvalue becomes negative, and the post-critical branch is unstable. The load level at which the determinant vanishes is noticeably higher for the 8-element discretization than for the 512-element mesh, consistent with the convergence data reported in Table 7.

Figure 35 illustrates the deformed configurations of the arch computed using the MFEF with the 512-element discretization at selected load levels, spanning both the pre-critical loading path and the post-critical regime, including configurations in which the applied load reverses in direction. The displacement fields are C^0 -continuous and satisfy the kinematic boundary conditions $u(0) = v(0) = 0$ at the hinged support and $u(\phi_0 R) = v(\phi_0 R) = 0$ at the clamped support. The corresponding bending moment distributions along the dimensionless axial coordinate $s/(\phi_0 R)$ are shown in Fig. 36. The bending moments are likewise C^0 -continuous and satisfy the equilibrium boundary conditions of the problem. In particular, the end moment vanishes at the hinged support, i.e., $M(0) = 0$.

7 Conclusions

This work has presented a mixed finite element formulation (MFEF) for the geometrically exact analysis of planar beams and frames, derived from a Hellinger–Reissner-type variational principle. The formulation adopts the spatial components of the internal forces, the bending moment field, and the cross-sectional rotation as primary unknowns, leading to a linearized problem governed by an explicit, symmetric, and banded tangent matrix. Applicable to both shear-rigid (Kirchhoff-type) and shear-flexible (Reissner-type) beam theories, it is intrinsically free from shear- and membrane-locking phenomena and yields C^0 -continuous bending moment fields that satisfy equilibrium boundary conditions, while the spatial internal force components satisfy equilibrium in strong form at the member level. Displacements are treated as secondary variables, recovered *a posteriori* by integrating the kinematic compatibility relations, producing globally C^0 -continuous displacement fields that satisfy all prescribed kinematic boundary conditions.

From an algebraic standpoint, the proposed mixed formulation yields a significantly smaller and sparser global system than the standard displacement-based formulation, owing to the use of member-level internal force parameters combined with element-local rotational variables, which limit interelement coupling and reduce the effective bandwidth of the tangent matrix. These properties translate into reduced memory requirements and potentially lower factorization costs, without compromising solution accuracy.

The MFEF delivers highly accurate results even on coarse meshes and enables reliable detection of stability transi-

tions through the determinant of the equivalent stiffness matrix, providing a consistent basis for the identification and classification of critical points. Moreover, the formulation has been observed to permit larger continuation increments and require fewer Newton iterations than the standard displacement-based formulation, reflecting a more robust nonlinear convergence behavior. When combined with the adopted arc-length path-following strategy, these properties result in reliable and efficient tracing of nonlinear equilibrium paths with reduced computational effort per load step.

Overall, the proposed MFEF combines accuracy, robustness, and computational efficiency, offering a reliable and attractive framework for the geometrically nonlinear analysis of planar beam and frame structures undergoing arbitrarily large displacements and rotations.

Perspectives for future work include extending the proposed formulation to dynamic settings, generalizing it to three-dimensional beam and frame structures, and applying it to beam models for functionally graded and layered composite members.

Appendix

Hellinger–Reissner-type energy functional

The total potential energy functional of the geometrically exact beam model is decomposed as

$$\Pi_p(u, v, \theta) = U(u, v, \theta) - W_{\text{ext}}(u, v, \theta), \quad (48)$$

where the strain energy is

$$U(u, v, \theta) = \int_0^L \frac{1}{2} (EI\kappa^2(\theta) + EA\varepsilon^2(u, v, \theta) + GA_s\gamma^2(u, v, \theta)) ds, \quad (49)$$

and the work of the external loads is

$$W_{\text{ext}}(u, v, \theta) = \int_0^L (p_x u + p_y v + m(\theta - \theta_0)) ds + [\bar{N}^* u + \bar{V}^* v + \bar{M}(\theta - \theta_0)]_{\Gamma_N}. \quad (50)$$

To derive a mixed variational formulation, the strain energy U is rewritten by introducing the stress resultants (M, N, V) as independent variables through a Legendre transformation. Each quadratic energy density admits the conjugate representation

$$\frac{EI\kappa^2}{2} = M\kappa - \frac{M^2}{2EI}, \quad (51a)$$

$$\frac{EA\varepsilon^2}{2} = N\varepsilon - \frac{N^2}{2EA}, \quad (51b)$$

$$\frac{GA_s \gamma^2}{2} = V\gamma - \frac{V^2}{2GA_s}. \quad (51c)$$

These identities hold for arbitrary values of the strain and stress measures; their stationarity with respect to each stress resultant recovers the corresponding constitutive relation in (8). Substituting (51) into (49) and collecting terms yields

$$U = \int_0^L (M\kappa + N\varepsilon + V\gamma) ds - U_c, \quad (52)$$

where the complementary strain energy is defined as

$$U_c = \int_0^L \left(\frac{M^2}{2EI} + \frac{N^2}{2EA} + \frac{V^2}{2GA_s} \right) ds. \quad (53)$$

The resulting decomposition is linear in the kinematic strain measures $(\kappa, \varepsilon, \gamma)$ and quadratic in the stress resultants (M, N, V) , the latter entering exclusively through U_c .

The force equilibrium equations (2a) and (2b), together with the corresponding Neumann boundary conditions (5a) and (5b), are incorporated into (48) as constraints via Lagrange multipliers, yielding the augmented functional

$$\begin{aligned} \Pi_{aug} = & \int_0^L \left(M\kappa - \frac{M^2}{2EI} + N\varepsilon - \frac{N^2}{2EA} \right. \\ & \left. + V\gamma - \frac{V^2}{2GA_s} \right) ds - W_{ext}(u, v, \theta) \\ & + \int_0^L ((N^{*'} + p_x)u + (V^{*'} + p_y)v) ds \\ & + [(\bar{N}^* - nN^*)u + (\bar{V}^* - nV^*)v]_{\Gamma_N}. \end{aligned} \quad (54)$$

The Hellinger–Reissner-type energy functional then follows by inserting the energetic conjugacy relations (9) and the compatibility equations (10) and (6c) into (54), integrating by parts the terms N^*u' , V^*v' , and $M(\theta' - \theta'_0)$, and enforcing the kinematic boundary conditions (7):

$$\begin{aligned} \Pi_{H-R}^m(N^*, V^*, M, \theta) = & - \int_0^L \left(\frac{M^2}{2EI} + \frac{N^2(N^*, V^*, \theta)}{2EA} \right. \\ & \left. + \frac{V^2(N^*, V^*, \theta)}{2GA_s} \right) ds \\ & - \int_0^L (N^*(\cos \theta - x') + V^*(\sin \theta - y')) \\ & + M'(\theta - \theta_0) + m(\theta - \theta_0)) ds \\ & + [nN^*\bar{u} + nV^*\bar{v} + nM\bar{\theta}]_{\Gamma_D} \\ & + [(nM - \bar{M})(\theta - \theta_0)]_{\Gamma_N}. \end{aligned} \quad (55)$$

For convenience (and consistency with the most common sign convention, which leads to a positive complementary strain energy) the sign-reversed functional $\Pi_{H-R} =$

$-\Pi_{H-R}^m$ is adopted in this work. The two functionals are variationally equivalent and yield identical stationarity conditions; however, the sign reversal induces a corresponding sign change in the second variation and, consequently, in the tangent matrix.

Element residual vector and tangent matrix

Let the element degrees of freedom be collected in the vector

$$\mathbf{d}_e = [N_0^* \ V_0^* \ M_{e,0} \ M_{e,L} \ \theta_e]^T, \quad (56)$$

where N_0^* and V_0^* are the member-level spatial force parameters, $M_{e,0}$ and $M_{e,L}$ the element bending moment parameters at the two end-points, and θ_e the element rotation parameter. This choice of degrees of freedom reflects directly the finite element approximations introduced in Sect. 4.

The element residual vector $\mathbf{R}_e$ is derived as the gradient of the discrete Hellinger–Reissner functional Π_{H-R}^h , restricted to element Ω_e , with respect to $\mathbf{d}_e$:

$$\mathbf{R}_e = \frac{\partial \Pi_{H-R}^h|_{\Omega_e}}{\partial \mathbf{d}_e}. \quad (57)$$

The integrals appearing in $\Pi_{H-R}^h|_{\Omega_e}$ can be evaluated analytically under the adopted finite element approximations, and the residual vector for a beam element of length L_e admits the closed-form expression

$$\mathbf{R}_e = \begin{bmatrix} \frac{L_e N_e \cos \theta_e}{EA} - \frac{L_e V_e \sin \theta_e}{GA_s} + L_e (\cos \theta_e - x'_e) \\ \frac{L_e N_e \sin \theta_e}{EA} + \frac{L_e V_e \cos \theta_e}{GA_s} + L_e (\sin \theta_e - y'_e) \\ \frac{L_e M_{e,0}}{3EI} + \frac{L_e M_{e,L}}{6EI} - (\theta_e - \theta_{0,e}) \\ \frac{L_e M_{e,0}}{6EI} + \frac{L_e M_{e,L}}{3EI} + (\theta_e - \theta_{0,e}) \\ \frac{L_e N_e V_e}{EA} - \frac{L_e N_e V_e}{GA_s} + L_e V_e + M_{e,L} - M_{e,0} \end{bmatrix}, \quad (58)$$

where $x'_e = \cos \theta_{0,e}$ and $y'_e = \sin \theta_{0,e}$, with $\theta_{0,e}$ the element-wise constant interpolation of θ_0 over Ω_e (see Sect. 4.2), and N_e, V_e are the element material force components, defined as

$$N_e = N_0^* \cos \theta_e + V_0^* \sin \theta_e, \quad (59a)$$

$$V_e = V_0^* \cos \theta_e - N_0^* \sin \theta_e. \quad (59b)$$

The element tangent matrix is then obtained by differentiating $\mathbf{R}_e$ with respect to $\mathbf{d}_e$, i.e.,

$$\mathbf{T}_e = \frac{\partial \mathbf{R}_e}{\partial \mathbf{d}_e}. \quad (60)$$

Owing to the underlying Hellinger–Reissner-type mixed variational principle, $\mathbf{T}_e$ is symmetric and incorporates both material and geometric contributions. The nonzero components of $\mathbf{T}_e$ then read

$$T_{11,e} = \frac{L_e \cos^2 \theta_e}{EA} + \frac{L_e \sin^2 \theta_e}{GA_s}, \quad (61)$$

$$T_{12,e} = \frac{L_e \sin \theta_e \cos \theta_e}{EA} - \frac{L_e \sin \theta_e \cos \theta_e}{GA_s}, \quad (62)$$

$$T_{15,e} = \frac{L_e(V_e \cos \theta_e - N_e \sin \theta_e)}{EA} - \frac{L_e(V_e \cos \theta_e - N_e \sin \theta_e)}{GA_s} - L_e \sin \theta_e, \quad (63)$$

$$T_{22,e} = \frac{L_e \sin^2 \theta_e}{EA} + \frac{L_e \cos^2 \theta_e}{GA_s}, \quad (64)$$

$$T_{25,e} = \frac{L_e(N_e \cos \theta_e + V_e \sin \theta_e)}{EA} - \frac{L_e(N_e \cos \theta_e + V_e \sin \theta_e)}{GA_s} + L_e \cos \theta_e, \quad (65)$$

$$T_{33,e} = \frac{L_e}{3EI}, \quad (66)$$

$$T_{34,e} = \frac{L_e}{6EI}, \quad (67)$$

$$T_{35,e} = -1, \quad (68)$$

$$T_{44,e} = \frac{L_e}{3EI}, \quad (69)$$

$$T_{45,e} = 1, \quad (70)$$

$$T_{55,e} = \frac{L_e(V_e^2 - N_e^2)}{EA} - \frac{L_e(V_e^2 - N_e^2)}{GA_s} - L_e N_e. \quad (71)$$

The stress-resultant-dependent geometric contributions are confined entirely to the row and column of $\mathbf{T}_e$ associated with the kinematic degree of freedom θ_e ; the remaining rows and columns are independent of the material internal forces N_e and V_e .

The coupling terms between the force and bending moment degrees of freedom vanish identically,

$$T_{13,e} = T_{14,e} = T_{23,e} = T_{24,e} = 0, \quad (72)$$

reflecting the complete decoupling between the spatial internal force and bending moment fields in the complementary strain energy.

For shear-rigid (Kirchhoff-type) elements, the above expressions reduce in the limit $1/GA_s \rightarrow 0$, thereby

eliminating all shear-dependent contributions. No further modifications to the formulation are required.

Static condensation and element equivalent stiffness matrix

To construct the element equivalent stiffness matrix, the element degrees of freedom are partitioned into *statical* and *kinematic* components according to

$$\mathbf{d}_e = \begin{bmatrix} \mathbf{d}_{s,e} \\ \mathbf{d}_{k,e} \end{bmatrix}, \quad \mathbf{d}_{s,e} \in \mathbb{R}^{n_{s,e}}, \quad \mathbf{d}_{k,e} \in \mathbb{R}^{n_{k,e}}. \quad (73)$$

Accordingly, the element tangent matrix admits the block decomposition

$$\mathbf{T}_e = \begin{bmatrix} \mathbf{T}_{ss,e} & \mathbf{T}_{sk,e} \\ \mathbf{T}_{ks,e} & \mathbf{T}_{kk,e} \end{bmatrix}, \quad (74)$$

with

$$\begin{aligned} \mathbf{T}_{ss,e} &\in \mathbb{R}^{n_{s,e} \times n_{s,e}}, & \mathbf{T}_{sk,e} &\in \mathbb{R}^{n_{s,e} \times n_{k,e}}, \\ \mathbf{T}_{ks,e} &\in \mathbb{R}^{n_{k,e} \times n_{s,e}}, & \mathbf{T}_{kk,e} &\in \mathbb{R}^{n_{k,e} \times n_{k,e}}. \end{aligned} \quad (75)$$

Elimination of the statical degrees of freedom via static condensation requires the block $\mathbf{T}_{ss,e}$ to be invertible. In the present formulation, the statical and kinematic variables are given by

$$\mathbf{d}_{s,e} = [N_0^* \ V_0^* \ M_{e,0} \ M_{e,L}]^T, \quad \mathbf{d}_{k,e} = \theta_e, \quad (76)$$

so that the statical block of the element tangent matrix assumes the block-diagonal structure

$$\mathbf{T}_{ss,e} = \begin{bmatrix} T_{11,e} & T_{12,e} & 0 & 0 \\ T_{12,e} & T_{22,e} & 0 & 0 \\ 0 & 0 & T_{33,e} & T_{34,e} \\ 0 & 0 & T_{34,e} & T_{44,e} \end{bmatrix}. \quad (77)$$

Invertibility is guaranteed if $\mathbf{T}_{ss,e}$ is *symmetric positive definite* (SPD), i.e.,

$$\mathbf{d}_{s,e}^T \mathbf{T}_{ss,e} \mathbf{d}_{s,e} > 0 \quad \forall \mathbf{d}_{s,e} \in \mathbb{R}^{n_{s,e}} \setminus \{0\}. \quad (78)$$

Since $\mathbf{T}_{ss,e}$ is block-diagonal and symmetric, positive definiteness is equivalent to that of each diagonal block. The first 2×2 block corresponds to the spatial internal force degrees of freedom (N_0^* , V_0^*), and the second to the bending moment degrees of freedom ($M_{e,0}$, $M_{e,L}$):

1. Internal force block

$$\mathbf{T}_{f,e} = \begin{bmatrix} T_{11,e} & T_{12,e} \\ T_{12,e} & T_{22,e} \end{bmatrix}.$$

Since $EA > 0$, $GA_s > 0$, and $L_e > 0$, the diagonal entries are strictly positive. Moreover, the determinant satisfies

$$\det \mathbf{T}_{f,e} = \frac{L_e^2}{EA GA_s} > 0,$$

which guarantees that $\mathbf{T}_{f,e}$ is SPD.

2. Bending moment block

$$\mathbf{T}_{m,e} = \begin{bmatrix} T_{33,e} & T_{34,e} \\ T_{34,e} & T_{44,e} \end{bmatrix}.$$

With $EI > 0$ and $L_e > 0$, the diagonal entries are strictly positive, and

$$\det \mathbf{T}_{m,e} = \frac{L_e^2}{12 E^2 I^2} > 0,$$

ensuring that $\mathbf{T}_{m,e}$ is SPD.

Since $\mathbf{T}_{ss,e}$ is block-diagonal with SPD blocks, it is itself symmetric positive definite and therefore invertible, guaranteeing that static condensation can be carried out. The equivalent stiffness associated with the kinematic degree of freedom θ_e equals the Schur complement of $\mathbf{T}_{ss,e}$ in $\mathbf{T}_e$, namely

$$K_e = \mathbf{T}_{kk,e} - \mathbf{T}_{ks,e} \mathbf{T}_{ss,e}^{-1} \mathbf{T}_{sk,e}. \quad (79)$$

Using the particular block structure of the present formulation, one obtains

$$K_e = T_{55,e} - [T_{15,e} \ T_{25,e}] \mathbf{T}_{f,e}^{-1} \begin{bmatrix} T_{15,e} \\ T_{25,e} \end{bmatrix} - \frac{12EI}{L_e}, \quad (80)$$

where the force-block inverse reads

$$\mathbf{T}_{f,e}^{-1} = \frac{EA GA_s}{L_e^2} \begin{bmatrix} T_{22,e} & -T_{12,e} \\ -T_{12,e} & T_{11,e} \end{bmatrix}, \quad (81)$$

and the bending moment contribution $12EI/L_e$ follows from $T_{35,e} = -1$, $T_{45,e} = 1$, and the inverse

$$\mathbf{T}_{m,e}^{-1} = \frac{EI}{L_e} \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}. \quad (82)$$

Since $\mathbf{d}_{k,e} = \theta_e$ is a scalar, K_e is a scalar quantity. All component expressions are provided in the preceding subsection, making (80) fully explicit at the element level.

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