Wave Motion 123 (2023) 103232

Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wamot

Velocity and energy of periodic travelling interfacial waves between two bounded fluids

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ARTICLE INFO

Keywords:
Interfacial waves
Two-layer fluid
Velocity of propagation
Wave energy

ABSTRACT

For a periodic travelling irrotational wave propagating at the interface between two homogeneous, incompressible and inviscid fluids bounded by horizontal planes, we generalise the Stokes definitions for the velocity of the wave propagation. Under certain conditions imposed on the horizontal velocity of the motion at the interface and supposing that the horizontal components of the velocity in each layer never reach the wave speed, we prove that the mean horizontal velocity of propagation of the wave is greater than the generalised mean horizontal velocity of the mass of the fluid. We show that, for interfacial waves of small amplitude, the excess kinetic and potential energy of the fluid have the same magnitude, but different signs, and for the nonlinear setting, we prove that the excess kinetic energy is negative.

1. Introduction

The study of general periodic water waves, propagating at a constant velocity, was initiated by Stokes [1] in the 19th century, obtaining approximate solutions for a system of nonlinear equations of motion, when the amplitude of the wave is sufficiently small. Since then the study of waves of this type has progressed quite comprehensively and thoroughly. The existence of Stokes waves was proved in the 1920s for small amplitude waves by Levi-Civita [2] and Nekrasov [3], using a hodograph transformation to map the unknown domain into a fixed rectangle where the variable is the potential of the flow. The existence of Stokes waves for large amplitude waves was proved in the 1960s by Krasovskiĭ [4,5]. Over the past decades, a rigorous mathematical theory on this subject has been developed (see Toland [6], Sun [7], Strauss [8], A. Constantin [9–11], and Henry [12]).

In 1848, Stokes proposed two definitions for the velocity of propagation of the wave [1]. In the first one the author defined the velocity of propagation to be the *velocity with which the wave form is propagated in space, when the mean horizontal velocity at each point of space occupied by the fluid is zero*. In the second one Stokes defined the velocity of propagation *to be the velocity of propagation of the wave form in space, when the mean horizontal velocity of the mass of fluid comprised between two very distant planes perpendicular to the axis of x is zero*. A. Constantin [10], in 2013, proved, for the case of a homogeneous fluid in potential flow, that the first Stokes definition for velocity of propagation of the wave is greater than the second one. Later, in 2021, Henry [12], in the same framework, computed the excess potential and kinetic energy densities (per unit horizontal area) and, based on Constantin's results, proved that the excess potential energy density is positive and that the excess kinetic energy density is negative.

In the more recent years, the interest in the conversion of wave energy has increased to a level that it is now a field onto itself (see for instance the book by Babarit [13] for an overview on this subject). This sparks evidently an increased necessity of understanding

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https://doi.org/10.1016/j.wavemoti.2023.103232
Received 5 July 2023; Received in revised form 15 September 2023; Accepted 3 October 2023
Available online 12 October 2023

the energy properties and relations beyond the linear water wave approximations, which are very limited in what concerns the real world wave phenomena. For example, the usual assumption of linear water wave theory demands that the wave amplitude is small regarding both the water depth and the wavelength, cf. Lamb [14]. The inherent difficulties in analysing wave phenomena beyond the linear approximation still prevent the straightforward use of procedures to obtain meaningful results in that domain.

Energy in the oceans is not only carried by the surface waves, but also by internal waves. These waves occur in density-stratified fluids, when the force of gravity acts vertically on the displaced fluid, pulling it back to the mean, equilibrium position. Most of the infrastructure to generate energy from waves is based near the coast, where the interaction of masses of water of different densities and temperatures will provide for the existence of discriminated layers, with the accompanying interfaces, where the wave motion will certainly play a role in carrying energy.

The existence of steady periodic travelling interfacial waves between two bounded fluids was first proved by Kotchine [15], where the case of the fluids of infinite extent was also considered. Kotchine demonstrated the existence of a solution for the problem when the amplitude of the interfacial waves is small enough, generalising the works by Levi-Civita [2] and Struik [16].

For the case of progressive interfacial waves of large amplitude, Hoyler [17] calculated the height of the highest wave (applying the criterion that the highest wave will have the horizontal fluid speed equal to the phase speed at some point in the fluid) and computed the kinetic and potential energies, the momentum flux and the energy flux of the waves. Meiron and Saffman [18] were the first to establish the existence of overhanging interfacial waves of large amplitude.

The existence of internal solitary waves between two fluids bounded by rigid horizontal planes has been proved by Amick and Turner [19]. In 2001, Sun [7] derived an integral formulation for two-dimensional periodic travelling gravity waves in two fluids without boundaries and, using the global bifurcation theory, proved the existence of such waves of small and large amplitude. More recently, Maklakov and Sharipov [20] studied the configuration of limiting wave profiles propagating at the interface between two unbounded fluids and Guan et al. [21], using an integral-equation method coupled with Fourier expansions of the unknown functions, obtained highly accurate solutions in a framework of a two-layer fluid of finite depth.

Aleman and A. Constantin [22], in 2019, making use of complex function theory, proved some results on the vertical attenuation of the velocity beneath a spatially periodic wave which propagates at the surface of a water flow of constant vorticity over a flat bed, showing in particular that the decay in depth is exponential and that at greater depths the flow pattern is close to a pure current state. Roberti [23] adapted the methods developed by Aleman and A. Constantin for irrotational periodic water waves, improving the estimate of the velocity decay in depth.

In 2022, O. Constantin [24] proved the logarithmic convexity of certain flow quantities associated with irrotational periodic travelling waves that propagate at the surface of water over a flat bed and, based on these results, observed that the kinetic energy and the time-period of the particle paths are larger near the surface and reduce with increasing depth. O. Constantin and Persson [25], using methods from complex analysis, obtained qualitative results on the monotonicity of horizontal averages with depth and on the location of the extrema for the kinetic energy of two-dimensional irrotational water flows, finding inter alia that the kinetic energy with respect to the moving frame decreases on the surface with increasing elevation and that the minimum of this kinetic energy is attained at the wave crest. Henry [26] studied the excess kinetic and potential energies for exact nonlinear equatorial water waves and proved that, for negative wavespeeds, the excess kinetic energy density is always negative, whereas the excess potential energy density is always positive.

In this paper, we focus our attention in studying the velocity and the energy of travelling periodic waves propagating at the interface between two fluid layers, extending the results of A. Constantin [10] and Henry [12]. Departing from a homogeneous fluid, we assume that the fluid occupies two homogeneous layers bounded by a rigid lid and suppose that the flow is layerwise irrotational.

The two-layer model is a well-accepted first approximation in many situations, e.g. in estuaries, underground waters and caves, fjords, channelled waters, as further environments where this analysis may prove of relevance. The two-layer model is also the simplest approximation to a continuously stratified fluid, where the density is in general dependent on position and time. The rigid-lid approximation is usually used to replace the free surface in the study of density-stratified fluids (see Bona et al. [27], Saha and Bora [28], Cal et al. [29]) when the displacements of the surface are much smaller than the interface displacements.

To overcome the fact that, for the case of interfacial periodic water waves, the pressure is not constant at the interface, we impose additional monotonic conditions on the horizontal velocity of the motion at the interface and suppose that the horizontal fluid velocity throughout the fluid never reaches the wave speed c. Under these assumptions, we were able to prove that the mean horizontal velocity of propagation of the wave is greater than the generalised mean horizontal velocity of the mass of the fluid and, therefore, to characterise the excess potential and kinetic energy densities of the fluid in the linear and non-linear settings.

The paper is organised as follows. In Section 2, we introduce the notation and present the equations of motion for a fluid bounded from below by a flat bottom and from above by a rigid lid, consisting of two homogeneous, incompressible, inviscid fluids, with constant densities, and irrotational. Along the interface between the two fluid layers, conditions on the pressure and on the velocity field are imposed. Since the fluid is layerwise incompressible, we introduce, up to a constant, in each layer, a stream function which allows us to rewrite the previous equations of motion as an elliptic equation in each layer with nonlinear boundary conditions. Taking into account the layerwise irrotationality of the fluid, we introduce a velocity potential in each layer and, making use of the hodograph change of variables, the problem is transformed into a nonlinear boundary problem in a fixed rectangular domain. In Section 3, we generalise Stokes' definitions of the velocity of propagation of the wave for the present layerwise framework and prove that the mean horizontal velocity of propagation of the wave is greater than the generalised mean horizontal velocity of the mass of the fluid. In Section 4, under the linear water wave assumptions, we introduce the linearised equations of motion for a two-layer fluid bounded from below by a flat bottom and from above by a rigid lid and present the solution for this problem. We show that the excess kinetic and potential energy of the fluid per horizontal unit area have the same magnitude, but different signs. Finally, for nonlinear water waves we prove that the excess potential energy density is positive, whereas the kinetic is negative and present a formula for the total excess energy.

2. Formulation of the problem

2.1. Equations of motion

Consider an incompressible inviscid heavy fluid occupying two homogeneous immiscible layers, one on top of the other, with constant densities (see Fig. 1). For gravitational stability, assume that the density of the lower layer ρ_2 is greater than that of the upper layer ρ_1. The fluid domain extends to infinity in the horizontal direction, being bounded from below by a flat bottom and from above by a rigid lid. Moreover, suppose that the flow is layerwise irrotational and that the fluids are immiscible. We fix a Cartesian coordinate system $(x, y) \in \mathbb{R}^2$ in such a way that the x-axis is the direction of wave propagation with the y-axis pointing upward. Let $(x, t) \mapsto \eta(x, t)$ be the smooth function of period λ, even with respect to the spatial variable x, representing the interface between the two fluid layers, satisfying the following conditions

$$\partial_x \eta(x, t) < 0 \qquad \forall x \in (0, \lambda/2),$$

and

$$\int_0^\lambda \eta(x, t)\,\mathrm{d}x = 0\,, \tag{1}$$

for all $t \geq 0$. Since we are interested in travelling waves propagating at speed $c > 0$, we consider that the function representing the interface takes the form $\eta(x, t) = \eta(x - ct)$. Thus, in the new moving reference frame with speed c, the stationary subset representing the upper fluid layer is denoted by Ω_1 and defined through

$$\Omega_1 = \{(x, y) \in \mathbb{R}^2 \,:\, \eta(x) < y < h_1\},$$

and the let Ω_2 be the stationary subset denoting the lower fluid layer and defined by

$$\Omega_2 = \{(x, y) \in \mathbb{R}^2 \,:\, -h_2 < y < \eta(x)\},$$

where h_1 and h_2 are positive constants denoting the thickness of the upper and of the lower layer, respectively. Within each fluid domain Ω_j, $j = 1, 2$, we introduce the velocity field $(u^{(j)}, v^{(j)})$ and write the mass conservation equation

$$u_x^{(j)} + v_y^{(j)} = 0\,, \tag{2}$$

and the Euler equations

$$(u^{(j)} - c)u_x^{(j)} + v^{(j)}u_y^{(j)} = -\frac{P_x^{(j)}}{\rho_j}\,, \tag{3a}$$

$$(u^{(j)} - c)v_x^{(j)} + v^{(j)}v_y^{(j)} = -\frac{P_y^{(j)}}{\rho_j} - g\,, \tag{3b}$$

where $P^{(j)}$ denotes the internal fluid pressure in each layer and g the gravitational acceleration at the surface of the Earth. Moreover, we denote the rigid lid and the flat bottom, respectively, by

$$\Gamma_1 = \left\{(x, y) \in \mathbb{R}^2 \;:\; y = h_1\right\} \quad \text{and} \quad \Gamma_2 = \left\{(x, y) \in \mathbb{R}^2 \;:\; y = -h_2\right\},$$

and the stationary interface between the two fluid layers by

$$\Gamma = \left\{(x, y) \in \mathbb{R}^2 \;:\; y = \eta(x)\right\}\,.$$

At the interface Γ, the transmission boundary conditions read as

$$v^{(j)} = (u^{(j)} - c)\,\eta_x\,, \quad j = 1, 2 \tag{4a}$$

$$P^{(1)} = P^{(2)}\,, \tag{4b}$$

and express the continuity of the pressure and the normal velocity across the interface between the fluid layers. On the rigid lid and on the flat bottom, we have the Neumann boundary condition

$$v^{(j)} = 0 \quad \text{on} \quad \Gamma_j\,, \quad j = 1, 2\,. \tag{5}$$

Since we assume that the fluid is layerwise irrotational, we have

$$u_y^{(j)} = v_x^{(j)} \quad \text{in} \quad \Omega_j\,, \quad j = 1, 2\,. \tag{6}$$

In this paper we are concerned about smooth interfacial waves, i.e., solutions of the Eqs. (2)–(6), assuming that $u^{(j)}, v^{(j)}$ and $P^{(j)}$ are smooth functions of period λ in the x–variable, with the functions $u^{(j)}, P^{(j)}$ even and $v^{(j)}$ odd in the x-variable. Moreover, we

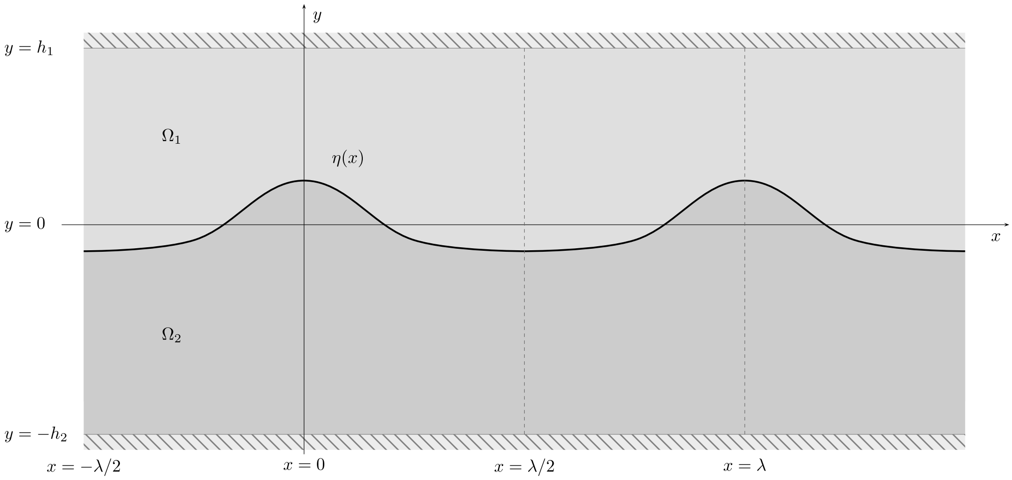

Fig. 1. The fluid domain.

impose that the values of the horizontal fluid velocity $u^{(j)}$ in each fluid layer never reach the wave speed c. In our framework, this is captured by imposing the condition

$$u^{(j)} - c < 0 \quad \text{in} \quad \overline{\Omega_j}\,, \quad j = 1,2\,. \tag{7}$$

At the flat bottom and at the rigid lid, we assume that there is no constant underlying current, i.e.,

$$\int_0^\lambda u^{(1)}(x,h_1)\,\mathrm{d}x = 0 \quad \text{and} \quad \int_0^\lambda u^{(2)}(x,-h_2)\,\mathrm{d}x = 0 \tag{8}$$

and, along the interface, we suppose that $u^{(1)}$ is strictly increasing and $u^{(2)}$ strictly decreasing from crest to trough, i.e., for all $x \in (0, \lambda/2)$,

$$\partial_x u^{(1)}(x,\eta(x)) > 0 \qquad \text{and} \qquad \partial_x u^{(2)}(x,\eta(x)) < 0\,. \tag{9}$$

2.2. Formulation of the problem using a stream function

Eqs. (2)–(6) can now be rewritten in a more suitable way. Since the fluid is layerwise incompressible, we can introduce, up to a constant, in each layer, the stream function $\psi^{(j)}$ satisfying

$$\psi^{(j)}_y = u^{(j)} - c \quad \text{and} \quad \psi^{(j)}_x = -v^{(j)}\,, \quad j = 1,2.$$

Moreover, Neumann boundary condition (5) and Eq. (4a) imply that the stream functions $\psi^{(j)}$, $j = 1,2$, are constant at the rigid surfaces Γ_1, Γ_2 and at the interface Γ. Setting $\psi^{(1)} = \psi^{(2)} = 0$ at the interface, we can write the stream functions as

$$\psi^{(1)}(x,y) = -\frac{m_1}{\rho_1} - \int_y^{h_1} \left(u^{(1)}(x,s) - c\right)\mathrm{d}s\,, \tag{10a}$$

$$\psi^{(2)}(x,y) = \frac{m_2}{\rho_2} + \int_{-h_2}^{y} \left(u^{(2)}(x,s) - c\right)\mathrm{d}s\,, \tag{10b}$$

where

$$m_1 = -\rho_1 \int_{\eta(x)}^{h_1} \left(u^{(1)}(x,s) - c\right)\mathrm{d}s \quad \text{and} \quad m_2 = -\rho_2 \int_{-h_2}^{\eta(x)} \left(u^{(2)}(x,s) - c\right)\mathrm{d}s\,, \tag{11}$$

denote the mass flux of the fluid motion in each layer. These quantities are positive, cf. (7), and invariant of the flow. Finally, from the Eqs. (3), we arrive in each fluid layer at Bernoulli's law, which states that the total energy

$$E_j = \rho_j \frac{(u^{(j)} - c)^2 + (v^{(j)})^2}{2} + \rho_j g(y + h_2) + P^{(j)}\,, \quad j = 1,2$$

is constant in the fluid domain Ω_j and, from the boundary condition (4b) at the interface Γ, it follows that

$$\frac{\rho_2|\nabla\psi^{(2)}|^2 - \rho_1|\nabla\psi^{(1)}|^2}{2} + (\rho_2 - \rho_1)g(y + h_2) = E_2 - E_1 \quad \text{on} \quad y = \eta(x).$$

The equations of motion (2)–(6) can now be rewritten as the following free boundary problem

$$\Delta\psi^{(j)} = 0 \quad \text{in} \quad \Omega_j\,, \quad j = 1,2\,, \tag{12a}$$

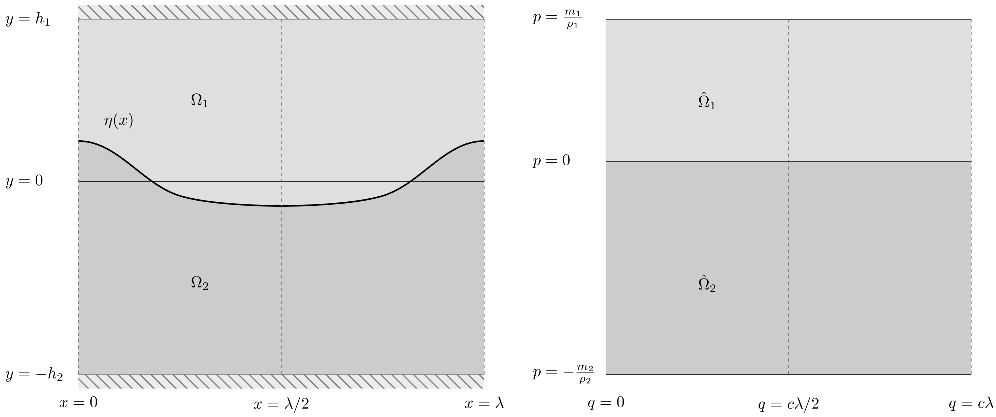

Fig. 2. Hodograph change of variables.

$$\psi^{(1)} = -\frac{m_1}{\rho_1} \quad \text{on} \quad \Gamma_1 , \tag{12b}$$

$$\psi^{(2)} = \frac{m_2}{\rho_2} \quad \text{on} \quad \Gamma_2 , \tag{12c}$$

$$\psi^{(j)} = 0 \quad \text{on} \quad y = \eta(x) , \quad j = 1, 2 , \tag{12d}$$

$$\frac{\rho_2 |\nabla \psi^{(2)}|^2 - \rho_1 |\nabla \psi^{(1)}|^2}{2} + (\rho_2 - \rho_1) g(y + h_2) = E \quad \text{on} \quad y = \eta(x) , \tag{12e}$$

where $E = E_2 - E_1$ is a constant.

2.3. Hodograph transformation

The layerwise irrotationality of the fluid guarantees the existence of a velocity potential $\phi^{(j)}$ in each layer defined, up to a constant, by

$$\phi^{(j)}_x = u^{(j)} - c \quad \text{and} \quad \phi^{(j)}_y = v^{(j)} , \quad j = 1, 2.$$

Fixing $\phi^{(1)} = \phi^{(2)} = 0$ at the line $x = 0$, we get

$$\phi^{(1)}(x, y) = \int_0^x \left(u^{(1)}(l, h_1) - c \right) \mathrm{d}l - \int_y^{h_1} v^{(1)}(x, s) \, \mathrm{d}s , \tag{13a}$$

$$\phi^{(2)}(x, y) = \int_0^x \left(u^{(2)}(l, -h_2) - c \right) \mathrm{d}l + \int_{-h_2}^{y} v^{(2)}(x, s) \, \mathrm{d}s . \tag{13b}$$

Since $\phi^{(j)}$ and $\psi^{(j)}$ satisfy Cauchy–Riemann equations in the connected open set Ω_j

$$\begin{cases} \phi^{(j)}_x &= \psi^{(j)}_y \\ \phi^{(j)}_y &= -\psi^{(j)}_x \end{cases} ,$$

the function $(x, y) \mapsto \phi(x, y)^{(j)} + \mathrm{i}\psi(x, y)^{(j)}$ is analytic and the velocity potential $\phi^{(j)}$ is harmonic in Ω_j, $j = 1, 2$. We can now introduce the conformal map $H : \Omega_1 \cup \Omega_2 \to \hat{\Omega}_1 \cup \hat{\Omega}_2$, where $(q, p) \in \hat{\Omega}_j$ is defined by

$$\begin{cases} q &= -\phi^{(j)}(x, y) \\ p &= -\psi^{(j)}(x, y) \end{cases} , \quad \forall (x, y) \in \Omega_j \tag{14}$$

and (see Fig. 2).

$$\hat{\Omega}_1 = \{(q, p) \in \mathbb{R}^2 : 0 < p < \frac{m_1}{\rho_1}\} , \quad \hat{\Omega}_2 = \{(q, p) \in \mathbb{R}^2 : -\frac{m_2}{\rho_2} < p < 0\}.$$

The map H transforms the free boundary problem (12) into the following nonlinear problem for the harmonic function $h(q, p) = y$ defined in the domain $\hat{\Omega}_1 \cup \hat{\Omega}_2$

$$\Delta_{qp} h = 0 \quad \text{in} \quad \hat{\Omega}_1 \cup \hat{\Omega}_2 , \tag{15a}$$

$$h = h_1 \quad \text{on} \quad p = \frac{m_1}{\rho_1} , \tag{15b}$$

$$h = -h_2 \quad \text{on} \quad p = -\frac{m_2}{\rho_2}, \tag{15c}$$

$$\frac{\rho_2}{2}\frac{1}{\left(h_p^{(2)}\right)^2 + \left(h_q^{(2)}\right)^2} - \frac{\rho_1}{2}\frac{1}{\left(h_p^{(1)}\right)^2 + \left(h_q^{(1)}\right)^2} + (\rho_2 - \rho_1)gh = E \quad \text{on} \quad p = 0. \tag{15d}$$

Note that, for $(q, p) \in \hat{\Omega}_j$ and $(x, y) \in \Omega_j$ such that $(q, p) = H(x, y)$, $j = 1, 2$, we have the following equalities

$$\begin{aligned} \partial_q &= h_p^{(j)}\,\partial_x + h_q^{(j)}\,\partial_y\,, \\ \partial_p &= -h_q^{(j)}\,\partial_x + h_p^{(j)}\,\partial_y\,, \end{aligned} \tag{16}$$

and

$$\begin{aligned} \partial_y &= -v^{(j)}\,\partial_q - (u^{(j)} - c)\,\partial_p\,, \\ \partial_x &= -(u^{(j)} - c)\,\partial_q + v^{(j)}\,\partial_p\,, \end{aligned} \tag{17}$$

where

$$\begin{aligned} h_q^{(j)} &= -\frac{v^{(j)}}{(u^{(j)} - c)^2 + \left(v^{(j)}\right)^2} = -\frac{\partial x}{\partial p} = \frac{\partial y}{\partial q}\,, \\ h_p^{(j)} &= -\frac{u^{(j)} - c}{\left(u^{(j)} - c\right)^2 + \left(v^{(j)}\right)^2} = \frac{\partial x}{\partial q} = \frac{\partial y}{\partial p}\,. \end{aligned} \tag{18}$$

3. Velocity of propagation of the wave

In 1848, Stokes [1] proposed two definitions for the velocity of propagation of the wave. Firstly, Stokes defined the velocity of propagation to be *the velocity with which the wave form is propagated in space, when the mean horizontal velocity at each point of space occupied by the fluid is zero*. In this paper, since the fluid is layerwise defined, we generalise this definition to

$$-\frac{1}{\lambda(\rho_1 h_1 + \rho_2 h_2)}\int_0^\lambda \left(\rho_1 h_1 (u^{(1)}(x, h_1) - c) + \rho_2 h_2 (u^{(2)}(x, -h_2) - c)\right) \mathrm{d}x\,. \tag{19}$$

Indeed, taking into account that $v^{(j)}$ is periodic with respect to x, and consequently $v^{(j)}(0, y) = v^{(j)}(\lambda, y)$, recalling the irrotationality of the fluid in each layer j, and applying the divergence theorem to the vector field $(v^{(1)}, -u^{(1)})$ in the set $[0, \lambda]\times[y_1, h_1]$, with $y_1 \in \left(\max_{x\in[0,\lambda]} \eta(x), h_1\right)$, and to the vector field $(v^{(2)}, -u^{(2)})$ in the set $[0, \lambda]\times[-h_2, y_2]$, with $y_2 \in \left(-h_2, \min_{x\in[0,\lambda]} \eta(x)\right)$, we arrive at the following equalities

$$\int_0^\lambda u^{(1)}(x, h_1)\mathrm{d}x = \int_0^\lambda u^{(1)}(x, y_1)\mathrm{d}x \quad \text{and} \quad \int_0^\lambda u^{(2)}(x, -h_2)\mathrm{d}x = \int_0^\lambda u^{(2)}(x, y_2)\mathrm{d}x,$$

which means that the quantity defined in (19) is equal to

$$-\frac{1}{\lambda(\rho_1 h_1 + \rho_2 h_2)}\int_0^\lambda \left(\rho_1 h_1 (u^{(1)}(x, y_1) - c) + \rho_2 h_2 (u^{(2)}(x, y_2) - c)\right) \mathrm{d}x\,. \tag{20}$$

with $-h_2 \le y_2 < \eta(x) < y_1 \le h_1$. Note that, in view of Eqs. (8) and (19), the wavespeed c coincides with our generalisation, for a layerwise fluid, of the first definition for the wave velocity proposed by Stokes.

Secondly, Stokes defined the velocity of propagation to be *the velocity of propagation of the wave form in space, when the mean horizontal velocity of the mass of fluid comprised between two very distant planes perpendicular to the x axis is zero*. Here, as above, we generalise this definition for a layerwise fluid to

$$\tilde{c} = -\frac{1}{\lambda(\rho_1 h_1 + \rho_2 h_2)}\int_0^\lambda \left(\rho_2 \int_{-h_2}^{\eta(x)} \left(u^{(2)}(x, y) - c\right) \mathrm{d}y + \rho_1 \int_{\eta(x)}^{h_1} \left(u^{(1)}(x, y) - c\right) \mathrm{d}y\right) \mathrm{d}x,$$

which, in view of Eqs. (11), can be written as

$$\tilde{c} = \frac{m_1 + m_2}{\rho_1 h_1 + \rho_2 h_2}\,. \tag{21}$$

For the case of periodic waves travelling at the free surface in a homogeneous fluid, A. Constantin [10,11], starting from the fact the pressure at the free surface is constant, proved monotonic properties of the horizontal velocity component, and concluded that the *mean horizontal velocity of propagation of the wave* (first Stokes' definition) exceeds the *mean horizontal velocity of the mass of fluid* (second Stokes' definition). In this work, since the pressure along the interface between the two fluid layers is not constant, we have to impose, in each layer, the monotonic conditions (9) to the horizontal velocity field component at the interface. In this way, we are able to prove a similar result for the case of periodic interfacial waves.

Theorem 3.1. *The mean horizontal velocity of propagation of the wave c is greater than the generalised mean horizontal velocity of the mass of fluid $\tilde{c}$.*

Proof. Defining

$$\Omega_{1,per} = \{(x,y) \in \mathbb{R}^2 \,:\, 0 < x < \lambda \;\wedge\; \eta(x) < y < h_1\}\,,$$
$$\Omega_{2,per} = \{(x,y) \in \mathbb{R}^2 \,:\, 0 < x < \lambda \;\wedge\; -h_2 < y < \eta(x)\}\,,$$

and

$$\hat{\Omega}_{1,per} = \{(q,p) \in \mathbb{R}^2 \,:\, 0 < q < c\lambda \;\wedge\; 0 < p < \frac{m_1}{\rho_1}\}\,,$$
$$\hat{\Omega}_{2,per} = \{(q,p) \in \mathbb{R}^2 \,:\, 0 < q < c\lambda \;\wedge\; -\frac{m_2}{\rho_2} < p < 0\}\,,$$

we find, making use of the hodograph transformation defined in (14), that $H(\Omega_{j,per}) = \hat{\Omega}_{j,per}$, for $j = 1, 2$, and

$$\left|\frac{\partial(x,y)}{\partial(q,p)}\right| = \frac{1}{(u^{(j)} - c)^2 + (v^{(j)})^2}.$$

Consequently,

$$\begin{aligned}
\tilde{c} = &- \frac{1}{\lambda(\rho_1 h_1 + \rho_2 h_2)} \int_0^{c\lambda} \left(\rho_2 \int_{-\frac{m_2}{\rho_2}}^{0} (u^{(2)} - c) \left|\frac{\partial(x,y)}{\partial(q,p)}\right| \mathrm{d}p \right) \mathrm{d}q \\
&- \frac{1}{\lambda(\rho_1 h_1 + \rho_2 h_2)} \int_0^{c\lambda} \left(\rho_1 \int_0^{\frac{m_1}{\rho_1}} (u^{(1)} - c) \left|\frac{\partial(x,y)}{\partial(q,p)}\right| \mathrm{d}p \right) \mathrm{d}q \\
= &\frac{1}{\lambda(\rho_1 h_1 + \rho_2 h_2)} \int_0^{c\lambda} \left(\rho_2 \int_{-\frac{m_2}{\rho_2}}^{0} h_p^{(2)}\, \mathrm{d}p + \rho_1 \int_0^{\frac{m_1}{\rho_1}} h_p^{(1)}\, \mathrm{d}p \right) \mathrm{d}q\,.
\end{aligned}$$

Integrating with respect to p and using (17), we arrive at

$$\tilde{c} = c - \frac{1}{\lambda(\rho_1 h_1 + \rho_2 h_2)} \int_0^{\lambda} \eta(x) \left(\rho_2 \left(u^{(2)}(x,\eta(x)) - c\right) - \rho_1 \left(u^{(1)}(x,\eta(x)) - c\right) \right) \mathrm{d}x.$$

The difference between the two generalised Stokes's definitions for the velocity of propagation is therefore

$$c - \tilde{c} = \frac{1}{\lambda(\rho_1 h_1 + \rho_2 h_2)} \int_0^{\lambda} \eta(x) \left(\rho_2\, u^{(2)}(x,\eta(x)) - \rho_1\, u^{(1)}(x,\eta(x)) \right) \mathrm{d}x\,, \tag{22}$$

where we use the fact that the mean value of η is equal to 0. Moreover, since η is even, λ-periodic and strictly decreasing on the interval $(0, \lambda/2)$, it has a (positive) maximum at $x = 0$ and a (negative) minimum at $x = \lambda/2$. Consequently, there is a point $x_0 \in (0, \lambda/2)$ such that $\eta(x_0) = 0$ and therefore

$$\forall x \in (0, x_0) \quad \eta(x) > 0 \qquad \text{and} \qquad \forall x \in (x_0, \lambda/2) \quad \eta(x) < 0\,. \tag{23}$$

Recalling that $u^{(j)}$, $j = 1, 2$, is even in the x variable, we get

$$\begin{aligned}
c - \tilde{c} &= \frac{2}{\lambda(\rho_1 h_1 + \rho_2 h_2)} \int_0^{\lambda/2} \eta(x) \left(\rho_2\, u^{(2)}(x,\eta(x)) - \rho_1\, u^{(1)}(x,\eta(x)) \right) \mathrm{d}x \\
&= \frac{2}{\lambda(\rho_1 h_1 + \rho_2 h_2)} \int_0^{x_0} \eta(x) \left(\rho_2\, u^{(2)}(x,\eta(x)) - \rho_1\, u^{(1)}(x,\eta(x)) \right) \mathrm{d}x \\
&\quad + \frac{2}{\lambda(\rho_1 h_1 + \rho_2 h_2)} \int_{x_0}^{\lambda/2} \eta(x) \left(\rho_2\, u^{(2)}(x,\eta(x)) - \rho_1\, u^{(1)}(x,\eta(x)) \right) \mathrm{d}x\,.
\end{aligned}$$

Taking into account inequalities (23) and (9), we can conclude, recalling again (1), that

$$\begin{aligned}
c - \tilde{c} &> \frac{2}{\lambda(\rho_1 h_1 + \rho_2 h_2)} \int_0^{x_0} \eta(x) \left(\rho_2\, u^{(2)}(x_0,\eta(x_0)) - \rho_1\, u^{(1)}(x_0,\eta(x_0)) \right) \mathrm{d}x \\
&\quad + \frac{2}{\lambda(\rho_1 h_1 + \rho_2 h_2)} \int_{x_0}^{\lambda/2} \eta(x) \left(\rho_2\, u^{(2)}(x_0,\eta(x_0)) - \rho_1 u^{(1)}(x_0,\eta(x_0)) \right) \mathrm{d}x \\
&= \frac{2}{\lambda(\rho_1 h_1 + \rho_2 h_2)} \int_0^{\lambda/2} \eta(x) \left(\rho_2\, u^{(2)}(x_0,\eta(x_0)) - \rho_1 u^{(1)}(x_0,\eta(x_0)) \right) \mathrm{d}x \\
&= \frac{1}{\lambda(\rho_1 h_1 + \rho_2 h_2)} \int_0^{\lambda} \eta(x) \left(\rho_2\, u^{(2)}(x_0,\eta(x_0)) - \rho_1 u^{(1)}(x_0,\eta(x_0)) \right) \mathrm{d}x = 0\,. \quad \square
\end{aligned}$$

4. Wave energy

For the case of periodic waves travelling at the interface between two fluid layers bounded by horizontal planes, the potential energy of the fluid is a consequence of gravity acting at the density jump at the interface and, therefore, we define the excess potential energy of the fluid per horizontal unit area over the value considering the rest position of the interface ($\eta = 0$) by

$$\begin{aligned} E_p &= \frac{g}{\lambda}\int_0^\lambda \left(\rho_2 \int_{-h_2}^{\eta(x)} y\,\mathrm{d}y + \rho_1 \int_{\eta(x)}^{h_1} y\,\mathrm{d}y - \rho_2 \int_{-h_2}^{0} y\,\mathrm{d}y - \rho_1 \int_0^{h_1} y\,\mathrm{d}y\right)\mathrm{d}x \\ &= \frac{g(\rho_2-\rho_1)}{\lambda}\int_0^\lambda \int_0^{\eta(x)} y\,\mathrm{d}y\,\mathrm{d}x = \frac{g(\rho_2-\rho_1)}{2\lambda}\int_0^\lambda \eta^2(x)\,\mathrm{d}x\,. \end{aligned} \tag{24}$$

The kinetic energy of the fluid is due to the motion of the particles in each layer. Thus, in a two-layer fluid setting, we define the excess kinetic energy of the fluid per horizontal unit area over the value for an undisturbed uniform flow, i.e. considering $(u^{(j)}, v^{(j)}) = (0,0)$ and $\eta = 0$, by

$$\begin{aligned} E_k =& \frac{1}{2\lambda}\int_0^\lambda \left(\rho_2 \int_{-h_2}^{\eta(x)} \left((u^{(2)}-c)^2 + (v^{(2)})^2\right)\mathrm{d}y + \rho_1 \int_{\eta(x)}^{h_1} \left((u^{(1)}-c)^2 + (v^{(1)})^2\right)\mathrm{d}y\right)\mathrm{d}x \\ &- \frac{1}{2\lambda}\int_0^\lambda \left(\rho_2 \int_{-h_2}^{0} c^2\mathrm{d}y + \rho_1 \int_0^{h_1} c^2\mathrm{d}y\right)\mathrm{d}x \\ =& \frac{1}{2\lambda}\int_0^\lambda \left(\rho_2 \int_{-h_2}^{\eta(x)} \left((u^{(2)})^2 + (v^{(2)})^2\right)\mathrm{d}y + \rho_1 \int_{\eta(x)}^{h_1} \left((u^{(1)})^2 + (v^{(1)})^2\right)\mathrm{d}y\right)\mathrm{d}x \\ &- \frac{c}{\lambda}\int_0^\lambda \left(\rho_2 \int_{-h_2}^{\eta(x)} u^{(2)}\mathrm{d}y + \rho_1 \int_{\eta(x)}^{h_1} u^{(1)}\mathrm{d}y\right)\mathrm{d}x\,, \end{aligned}$$

where we observed that the mean value of η is equal to 0.

4.1. Linear water waves

Assuming that the wave motion is of small amplitude (wave height is much smaller than the water depth) and of small slope (wave height is much smaller than the wave length), the equations of motion can be linearised (see John [30], Mei et al. [31] and Cal et al. [32]) for a two layer fluid bounded above by a rigid lid as

$$\begin{aligned} &u_x^{(j)} + v_y^{(j)} = 0 \quad \text{in} \quad \Omega_j^L\,, \quad j = 1,2\,, \\ &-c\,u_x^{(j)} = -\frac{P_x^{(j)}}{\rho_j} \quad \text{in} \quad \Omega_j^L\,, \quad j = 1,2\,, \\ &-c\,v_x^{(j)} = -\frac{P_y^{(j)}}{\rho_j} - g \quad \text{in} \quad \Omega_j^L\,, \quad j = 1,2\,, \end{aligned} \tag{25}$$

where

$$\Omega_1^L = \{(x,y) \in \mathbb{R}^2 \,:\, 0 < y < h_1\} \quad \text{and} \quad \Omega_2^L = \{(x,y) \in \mathbb{R}^2 \,:\, -h_2 < y < 0\}.$$

At the rest position of the interface, $y = 0$,

$$\begin{aligned} v^{(j)} &= -c\,\eta_x\,, \quad j = 1,2 \\ P^{(1)} &= P^{(2)}\,, \end{aligned} \tag{26}$$

On the rigid lid and on the flat bottom, we have the Neumann boundary conditions

$$\begin{aligned} &v^{(1)} = 0\,, \quad y = h_1\,, \\ &v^{(2)} = 0\,, \quad y = -h_2\,. \end{aligned} \tag{27}$$

Using the *ansatz*

$$\eta(x) = a\cos(kx)\,, \tag{28}$$

where $k = \frac{2\pi}{\lambda}$ denotes the wave number and $a \in \mathbb{R}$ the amplitude of the interface, we obtain the following solution for the linearised problem (25)–(27) (see e.g. Henry and Villari [33]):

$$\begin{aligned} &u^{(1)}(x,y) = -ack\frac{\cosh(k(y-h_1))}{\sinh kh_1}\cos kx\,, \qquad v^{(1)}(x,y) = -ack\frac{\sinh(k(y-h_1))}{\sinh kh_1}\sin kx\,, \\ &u^{(2)}(x,y) = ack\frac{\cosh(k(y+h_2))}{\sinh kh_2}\cos kx\,, \qquad v^{(2)}(x,y) = ack\frac{\sinh(k(y+h_2))}{\sinh kh_2}\sin kx\,, \end{aligned} \tag{29}$$

satisfying the dispersion relation (cf. Cal et al. [34])

$$c^2 = \frac{g(\rho_2-\rho_1)}{k(\rho_1\coth kh_1 + \rho_2\coth kh_2)}\,. \tag{30}$$

Proposition 4.1. *For internal waves of small amplitude, i.e., solutions of the linearised problem (25)–(27), the excess kinetic and potential energy of the fluid per horizontal unit area have the same magnitude, but different signs.*

Proof. For the case of small amplitude waves, we can use the solution (28)–(29) of the linearised problem (25)–(27) to estimate the excess potential energy

$$E_p^L = \frac{g(\rho_2-\rho_1)}{2\lambda}\int_0^\lambda \eta^2(x)\,\mathrm{d}x = \frac{g(\rho_2-\rho_1)a^2}{4}.$$

In order to estimate the excess kinetic energy, we write

$$\begin{aligned}
E_k^L =& \frac{1}{2\lambda}\int_0^\lambda\left(\rho_2\int_{-h_2}^{\eta(x)}\left((u^{(2)})^2+(v^{(2)})^2\right)\mathrm{d}y+\rho_1\int_{\eta(x)}^{h_1}\left((u^{(1)})^2+(v^{(1)})^2\right)\mathrm{d}y\right)\mathrm{d}x\\
&-\frac{c}{\lambda}\int_0^\lambda\left(\rho_2\int_{-h_2}^{\eta(x)}u^{(2)}\mathrm{d}y+\rho_1\int_{\eta(x)}^{h_1}u^{(1)}\mathrm{d}y\right)\mathrm{d}x\\
=&\frac{1}{2\lambda}\int_0^\lambda\left(\rho_2\int_{-h_2}^{0}\left((u^{(2)})^2+(v^{(2)})^2\right)\mathrm{d}y+\rho_1\int_{0}^{h_1}\left((u^{(1)})^2+(v^{(1)})^2\right)\mathrm{d}y\right)\mathrm{d}x\\
&-\frac{c}{\lambda}\int_0^\lambda\left(\rho_2\int_{-h_2}^{\eta(x)}u^{(2)}\mathrm{d}y+\rho_1\int_{\eta(x)}^{h_1}u^{(1)}\mathrm{d}y\right)\mathrm{d}x+\mathcal{O}(a^3).
\end{aligned}$$

Taking into account that the mean value of $x\mapsto\cos 2kx$ is equal to 0 and considering the dispersion relation (30), the first term of the excess kinetic energy simplifies to

$$\begin{aligned}
&\frac{1}{2\lambda}\int_0^\lambda\left(\rho_2\int_{-h_2}^{0}\left((u^{(2)})^2+(v^{(2)})^2\right)\mathrm{d}y+\rho_1\int_{0}^{h_1}\left((u^{(1)})^2+(v^{(1)})^2\right)\mathrm{d}y\right)\mathrm{d}x\\
&=\frac{a^2c^2k}{8\lambda}\int_0^\lambda\left(\rho_1\frac{2kh_1\cos 2kx+\sinh 2kh_1}{\sinh^2 kh_1}+\rho_2\frac{2kh_2\cos 2kx+\sinh 2kh_2}{\sinh^2 kh_2}\right)\mathrm{d}x\\
&=\frac{a^2c^2k}{4}\left(\rho_1\coth kh_1+\rho_2\coth kh_2\right)\\
&=\frac{a^2\ g(\rho_2-\rho_1)}{4}.
\end{aligned}\tag{31}$$

On the other hand, noting that

$$\begin{aligned}
&\rho_2\int_{-h_2}^{\eta(x)}u^{(2)}(x,y)\,\mathrm{d}y+\rho_1\int_{\eta(x)}^{h_1}u^{(1)}(x,y)\,\mathrm{d}y\\
&=\left(\rho_2\frac{\sinh(k(a\cos kx+h_2))}{\sinh kh_2}+\rho_1\frac{\sinh(k(a\cos kx-h_1))}{\sinh kh_1}\right)ac\cos kx
\end{aligned}$$

and considering the Taylor expansions

$$\begin{aligned}
\sinh(k(a\cos kx+h_2)) &= \sinh kh_2+ak\cosh kh_2\cos kx+\mathcal{O}(a^2),\\
\sinh(k(a\cos kx-h_1)) &= -\sinh kh_1+ak\cosh kh_1\cos kx+\mathcal{O}(a^2),
\end{aligned}$$

we arrive at

$$\begin{aligned}
&\frac{c}{\lambda}\int_0^\lambda\left(\rho_2\int_{-h_2}^{\eta(x)}u^{(2)}\mathrm{d}y+\rho_1\int_{\eta(x)}^{h_1}u^{(1)}\mathrm{d}y\right)\mathrm{d}x\\
&=\frac{a^2c^2}{\lambda}\int_0^\lambda k\left(\rho_2\coth kh_2+\rho_1\coth kh_1\right)\cos^2 kx\,\mathrm{d}x+\mathcal{O}(a^3)\\
&=\frac{a^2c^2}{2}k\left(\rho_2\coth kh_2+\rho_1\coth kh_1\right)+\mathcal{O}(a^3)\\
&=\frac{a^2g(\rho_2-\rho_1)}{2}+\mathcal{O}(a^3),
\end{aligned}\tag{32}$$

where we again used the dispersion relation (30) and the fact that the mean value of $x\mapsto\cos^2 kx$ is equal to $\frac{1}{2}$. Thus, subtracting the quantity obtained in (32) from the quantity obtained in (31), we get the excess kinetic energy for the solution of the linearised problem, up to $\mathcal{O}(a^3)$,

$$E_k^L=\frac{a^2\ g(\rho_2-\rho_1)}{4}-\frac{a^2g(\rho_2-\rho_1)}{2}=-\frac{a^2\ g(\rho_2-\rho_1)}{4}.$$

From this, we conclude that, for the linear water waves case, the total excess energy of the fluid per horizontal unit is, up to $\mathcal{O}(a^3)$,

$$E_t^L=E_p^L+E_k^L=0.\quad\square$$

4.2. Nonlinear water waves

In this section, making use of Theorem 3.1, where we proved that the mean horizontal velocity of propagation of the wave c is greater than generalised mean horizontal velocity of the mass of fluid $\tilde{c}$, we show that for the nonlinear case the excess kinetic energy density is negative. Moreover, we present an expression for the excess total energy per unit horizontal area.

Proposition 4.2. *For all solutions of the nonlinear problem (2)–(6), the excess kinetic energy per unit horizontal area E_k is given by*

$$E_k = -\frac{c}{\lambda}\int_0^{\lambda/2} \eta(x)\left(\rho_2\, u^{(2)}(x,\eta(x)) - \rho_1\, u^{(1)}(x,\eta(x))\right) dx\,. \tag{33}$$

Moreover, $E_k < 0$.

Proof. For the case of nonlinear waves, the excess kinetic energy is given by

$$\begin{aligned}
E_k =& \frac{1}{2\lambda}\int_0^{\lambda}\left(\rho_2\int_{-h_2}^{\eta(x)}\left((u^{(2)}-c)^2+(v^{(2)})^2\right)\mathrm{d}y + \rho_1\int_{\eta(x)}^{h_1}\left((u^{(1)}-c)^2+(v^{(1)})^2\right)\mathrm{d}y\right)\mathrm{d}x\\
&- \frac{1}{2\lambda}\int_0^{\lambda}\left(\rho_2\int_{-h_2}^{0}c^2\mathrm{d}y + \rho_1\int_0^{h_1}c^2\mathrm{d}y\right)\mathrm{d}x\\
=& \frac{1}{2\lambda}\int_0^{\lambda}\left(\rho_2\int_{-h_2}^{\eta(x)}\left((u^{(2)}-c)^2+(v^{(2)})^2\right)\mathrm{d}y + \rho_1\int_{\eta(x)}^{h_1}\left((u^{(1)}-c)^2+(v^{(1)})^2\right)\mathrm{d}y\right)\mathrm{d}x\\
&- \frac{c^2}{2}\left(\rho_1 h_1 + \rho_2 h_2\right)\,.
\end{aligned}$$

Using the hodograph change of variables defined in (14), we get

$$\begin{aligned}
&\frac{1}{2\lambda}\int_0^{c\lambda}\int_{-\frac{m_2}{\rho_2}}^{0}\rho_2\left((u^{(2)}-c)^2+(v^{(2)})^2\right)\left|\frac{\partial(x,y)}{\partial(q,p)}\right|\mathrm{d}p\,\mathrm{d}q\\
&+\frac{1}{2\lambda}\int_0^{c\lambda}\int_0^{\frac{m_1}{\rho_1}}\rho_1\left((u^{(1)}-c)^2+(u^{(1)})^2\right)\left|\frac{\partial(x,y)}{\partial(q,p)}\right|\mathrm{d}p\,\mathrm{d}q\\
=&\frac{1}{2\lambda}\int_0^{c\lambda}\left(\rho_2\int_{-\frac{m_2}{\rho_2}}^{0}\mathrm{d}p + \rho_1\int_0^{\frac{m_1}{\rho_1}}\mathrm{d}p\right)\mathrm{d}q\\
=&\frac{c\,(m_1+m_2)}{2}
\end{aligned}$$

and, consequently, considering Eq. (21), the excess kinetic energy for nonlinear water waves is

$$\begin{aligned}
E_k &= \frac{c}{2}\left(m_1+m_2-c\left(\rho_1 h_1+\rho_2 h_2\right)\right)\\
&= -\frac{c(\rho_1 h_1+\rho_2 h_2)}{2}(c-\tilde{c})\,.
\end{aligned}$$

Finally, in view of Eq. (22) and taking in account Theorem 3.1, we conclude that

$$E_k = -\frac{c}{2\lambda}\int_0^{\lambda}\eta(x)\left(\rho_2\, u^{(2)}(x,\eta(x)) - \rho_1\, u^{(1)}(x,\eta(x))\right)\mathrm{d}x < 0\,. \quad \square$$

Proposition 4.3. *For all solutions of the nonlinear problem (2)–(6), the excess total energy per unit horizontal area E_t is given by*

$$\begin{aligned}
E_t =& -\frac{1}{2\lambda}\int_0^{\lambda/2}\eta(x)\rho_2\left((u^{(2)}(x,\eta(x)))^2+(v^{(2)}(x,\eta(x)))^2\right)dx\\
&-\frac{1}{2\lambda}\int_0^{\lambda/2}\eta(x)\rho_1\left((u^{(1)}(x,\eta(x)))^2+(v^{(1)}(x,\eta(x)))^2\right)dx\,.
\end{aligned}$$

Proof. In view of the equalities (24) and (33), the total excess energy of the fluid per horizontal unit area is

$$\begin{aligned}
E_t =& E_p + E_k\\
=& \frac{g(\rho_2-\rho_1)}{2\lambda}\int_0^{\lambda}\eta^2(x)\,\mathrm{d}x - \frac{c}{2\lambda}\int_0^{\lambda}\eta(x)\left(\rho_2\, u^{(2)}(x,\eta(x)) - \rho_1\, u^{(1)}(x,\eta(x))\right)\mathrm{d}x\\
=& \frac{1}{2\lambda}\int_0^{\lambda}\eta(x)\left(g(\rho_2-\rho_1)\eta(x) - c\left(\rho_2\, u^{(2)}(x,\eta(x)) - \rho_1\, u^{(1)}(x,\eta(x))\right)\right)\mathrm{d}x\\
=& \frac{1}{2\lambda}\int_0^{\lambda}\eta(x)\left(-\frac{\rho_2|\nabla\psi^{(2)}|^2-\rho_1|\nabla\psi^{(1)}|^2}{2} - c\left(\rho_2\, u^{(2)}(x,\eta(x)) - \rho_1\, u^{(1)}(x,\eta(x))\right)\right)\mathrm{d}x
\end{aligned}$$

$$= -\frac{1}{4\lambda}\int_0^\lambda \eta(x)\rho_2\left((u^{(2)}(x,\eta(x)))^2 + (v^{(2)}(x,\eta(x)))^2\right)\mathrm{d}x$$
$$-\frac{1}{4\lambda}\int_0^\lambda \eta(x)\rho_1\left((u^{(1)}(x,\eta(x)))^2 + (v^{(1)}(x,\eta(x)))^2\right)\mathrm{d}x\,,$$

where we used equalities (1) and (12e). □

The total excess energy for interfacial water waves is the average of the contribution of the upper layer and the lower layer to the kinetic energy along the interface, weighed by the wave profile itself. This excess reduces to zero in the case of linear water waves, but can be of any sign in the non-linear regime.

For the case of interfacial periodic water waves there is a dearth of analytical results for the velocity field. The usual kinematical conditions at the interface that the normal velocity is continuous and the parallel component can suffer a jump from one layer to the other does not impede the assumptions made above, that for $x \in (0, \lambda/2)$ we have

$$\partial_x u^{(1)}(x,\eta(x)) > 0 \qquad \text{and} \qquad \partial_x u^{(2)}(x,\eta(x)) < 0.$$

From a physical standpoint, note that a free surface is not separating nothing from a fluid with density, but is a frontier between a less dense fluid and a denser fluid by comparison, allowing for instance the assumption that $P = P_{\mathrm{atm}}$ along the free surface, which cannot be assumed if the densities of the two fluid layers are of the same order of magnitude. Under this assumption, a wave propagating at a free surface is just a particular case of an interfacial wave, if we consider the density of the upper fluid layer equal to zero.

5. Conclusions

In this article we studied the velocity and the energy of non-linear periodic travelling waves propagating at the interface between two layers of homogeneous, irrotational, incompressible and inviscid fluids, in gravitational equilibrium, bounded by horizontal planes. In this analysis it was assumed that the horizontal component of the velocity field is smaller than the wave speed, therefore excluding highest waves (see Hoyler [17]).

Henry [12] showed that the excess potential energy density is positive and the excess kinetic energy density is negative in the general case of non-linear periodic waves travelling at the free surface of a single layer fluid. This recovers the already known same result for linear waves, where, in addition, there is an equipartition in the magnitudes of the excess energy between the energy densities, potential and kinetic, however with opposite signs.

In this paper, we started by introducing new definitions for the velocity of interfacial waves between two homogeneous layers, generalising the Stokes definitions of the velocity of propagation of surface waves in a homogeneous fluid. Under the monotonic conditions (9) for the horizontal component $u^{(j)}$ of the velocity field at the interface, and imposing condition (7), according to which the horizontal fluid velocity $u^{(j)}$ throughout the fluid never reaches the wave speed c, we were able to show that the mean horizontal velocity of propagation of the wave c is greater than the generalised mean horizontal velocity of the mass of the fluid $\tilde{c}$.

Based on this result, we were able to prove that Henry's result for surface waves in a homogeneous fluid can be extended to the case of interfacial waves: in general, in the non-linear setting, the excess potential energy density is positive and the excess kinetic energy density is negative, having the same magnitude, but different signs, in the case of linear interfacial waves.

Further studies are now required to understand what happens in the case of a multi-layer fluid, with and without the presence of a free surface.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Filipe S. Cal reports financial support was provided by Fundação para a Ciência e a Tecnologia (FCT).

Data availability

No data was used for the research described in the article.

Acknowledgements

F. S. Cal was partially supported by FCT Fundação para a Ciência e a Tecnologia, Portugal, through the project UIDB/04674/2020 (CIMA). G. A. S. Dias received no funding for this work. The authors have no conflict of interest in this work.

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