From efficiency to optimality in proportional reinsurance under group correlation

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Abstract

Based on our recent discovery of closed form formulae of efficient Mean Variance retentions in variable quota-share proportional reinsurance under group correlation, we analyzed the influence of different combination of correlation and safety loading levels on the efficient frontier, both in a single period stylized problem and in a multiperiod one.

Keywords. Mean-variance efficiency; constrained quadratic optimization; proportional reinsurance; group correlation.

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1 Introduction

It is well known that reinsurance is one of the key strategic variables in risk management of insurance companies, and there is no need after the last global crisis to underline the importance of correlation in problems of financial risk management. Along this line of thought, this paper concerns reward-risk efficiency and optimality in de Finetti's classical variable quota share proportional reinsurance problem under group correlation [1]. Even if de Finetti's paper could be obviously considered an old fashion one, we think that it is the seed of a lot of modern ideas in the theory of financial decisions under uncertainty and still an extraordinary source of inspiration and reflections. Indeed in that paper and with reference to a stylized proportional reinsurance problem, de Finetti was the first to propose an integrated reward-risk approach in finance, well in advance of the celebrated papers by Markowitz ([4], [5], [6]).

By an integrated reward-risk approach we mean a procedure made up by the following steps. The choice of a reward measure and respectively of a risk measure; the introduction of a reward preference and respectively of a risk preference ordering coherent with the chosen measures; the definition of a reward-risk dominance relation based on these preferences; the identification of an efficiency paradigm coherent with such a dominance relation; the selection of an optimality criterion to choose a feasible alternative within the efficient set.

In proportional reinsurance applications, according to the de Finetti's approach, the alternatives are feasible reinsurance (or retention) strategies, the random variables are the related single period net results, the reward measure is the expectation, the risk measure may be either the variance (or its equivalent twin the standard deviation) or the W-ruin probability (W the initial wealth of the company), the optimality criterion should be different depending on the choice of a multiperiod (indeed an asymptotic) strategy or of a (myopic) single period one.

While, as said before, the integrated mean-variance approach is an early version of the celebrated Markowitz approach, the W-ruin probability (widely used at that time in insurance circles) turns out to be an early version of the shortfall measures introduced in financial applications by Roy [14], and widely used nowadays (Ogrycszack-Ruszynsky, [8], [9]).

As regards the efficiency, de Finetti was convinced of the perfect equivalence of the two risk measures or more precisely of the coincidence of the single period efficient retentions sets obtained both in the mean-variance paradigm and in the mean-ruin probability one (for any positive value of W). Moreover he clearly perceived that the mean-variance approach was more manageable for computational purposes and then concentrated on developing a sequential procedure to find the whole efficient retentions set within this approach. In the n dimensional space of retentions this procedure turns out to be the reinsurance counterpart of the critical line algorithm later developed by Markowitz [5], [6].

Implicit in de Finetti's results is that the geometric representation in the mean-variance space of the efficient retentions set is a continuous convex union of parabolae, starting from the origin with zero derivative and differentiable also at the connection points. Obtained at that times without the help of the Karush-Kuhn-Tucker conditions ([2], [3]), this result should be judged as an extraordinary milestone in the history of quantitative finance, even it is not fully accurate. Only recently and following a critical review by Markowitz [7], it has been shown (see Pressacco-Serafini [10]), that there could be a lack of differentiability with an upward jump in the first derivative at some kink connection points: precisely those corresponding to efficient (if any) vertices (i.e. points where each risk is either fully retained or fully reinsured) of the n dimensional cube of retentions.

As we shall see, for our purposes of geometrical interpretations, it may be convenient to switch from the mean-variance plane to the mean-standard deviation one. On one side, in the new reference system of coordinates the properties of the efficient set remain unchanged; precisely, the efficient set is still a continuous convex union of curves, starting from the origin with a straight line, and differentiable also at the connection points, except at those corresponding to the kinks in the mean-variance plane[1] (see Pressacco-Ziani [12]). On the other side, working in the mean-standard deviation plane has the advantage that, under normality, the p iso ruin curves are, for any W, straight lines. For a given value W, initial free capital of the insurance company, a p-iso ruin line is the support of all random variables X sharing the same single period ruin probability (i.e the probability that $X + W \leq 0$). De Finetti suggested to consider as an optimal myopic single period reinsurance decision the one corresponding to the largest expectation compatible with an upper bound of a fixed acceptable ruin probability. In the mean-standard deviation plane, this means the retention corresponding to the intersection point between the p iso-ruin line, corresponding to the upper bound, and the efficient frontier. Obviously if there is no intersection point, because the p iso-ruin line lies completely above the efficient retentions' curve, then a full retention strategy is optimal and the ruin probability at full retention is lower than the upper bound.

The optimality approach previously discussed may be labelled as mean optimality conditional to a ruin probability (a shortfall) constraint. In this optimality approach the risk measure (the W shortfall probability) plays a preliminary role of narrowing the feasible set, then as a second step the mean is the reward preference measure to be maximized over the constrained feasible set. Another more symmetric optimality approach could be the one of maximizing an integrated reward risk measure, to be seen, under normality, as a special case of an utility functional. A trivial example of this integrated measure is

[1] We will check in some stylized examples that the efficient set is almost linear.

the utility $U(m, V) = m - tV$. With this measure, isoutility functions are straight lines of slope $1/t$ in the reference mean-variance plane. In that plane the optimal solution is the tangency point between an isoutility line and the union of parabolae representing the efficient set. It is interesting to note that de Finetti used an approach halfway between the last two solutions in his proposal of mean optimality subject to an upper bound expressed in terms of an asymptotic ruin probability. Despite the great interest of this approach, we do not treat it in this paper.

Going now back to the approach of mean optimality conditional to a ruin probability (shortfall) constraint, it is to be stressed that closed form solutions of the optimal retention vector may be obtained only if the efficient retention set is given in closed form. And contrarily to a widely shared opinion that such a closed form solution exists only under no correlation, this is just what happens in the so called case of group correlation (in this case closed form solution have been given by Pressacco et al [11], see also Pressacco-Ziani [12]). Group correlation means that the portfolio of risks is partitioned into a number of groups and is characterized by two sets of group specific parameters: a set of (group specific) loading coefficients used to charge premiums through a safety loading inspired by the standard deviation principle and thus determining the vector of portfolio expected gains and a set of (group specific) correlation coefficients determining the correlation matrix as a block diagonal one. We think that the feasibility of closed form solutions may greatly enhance the computational speed of multiperiod models based on a sequence of myopic mean optimal decisions conditional to a ruin probability single period upper bound. At the same time closed form solutions make possible a quick synthetic sensitivity analysis of correlation, i.e. to analyze the impact of different correlation levels on the optimal retentions, and hence on the related single period expected gains given the initial free capital endowment of the insurance company and the ruin probability upper bound.

The plan of the paper is as follows: in section 2 there is a short recall of mean-variance efficiency under group correlation *à la* de Finetti, both in the retention space and in the mean-variance one, exploiting our recent fundamental results. Here the efficiency is referred to a single period framework. In Section 3 we introduce an alternative risk measure, that is the single period ruin probability, shortly discussing the idea of iso-ruin lines in the mean-standard deviation reference plane. In section 4 we pass from efficiency to optimality in single period myopic decisions on the basis of our integrated, largest expectation constrained on a ruin probability upper bound, approach and obtained an explicit solution to the problem. Section 5 is dedicated to an interesting property of invariance of the single period profit rate to changes in the free capital under linearity of the efficient frontier, a property which seems to be verified in our simulations. In Section 6, with reference to a stylized portfolio of 5.000 policies, we investigate the consequences on efficient retentions and on the expected profit rate of different combinations of correlation, ruin probability and free capital. Some tables and graphs are also provided. Conclusions follow in section 7.

2 de Finetti's mean-variance approach in proportional reinsurance: basic properties

2.1 Essentials

Let us briefly recall the essentials of a variable quota share proportional reinsurance de Finetti's problem, henceforth de Finetti's problem. An insurance company, with initial free capital W, is faced with n risks (policies). The net profit, that is the difference between net premiums and losses, of these risks is described by a vector of random variables with expected values $\mathbf{m} > 0$, and a non-singular covariance matrix C. We denote the (i, j) entry of the matrix C as σ_{ij}. The diagonal elements are denoted also as $\sigma(i, i) = \sigma_i^2$ and the (i, j) entry may be alternatively denoted as $\sigma(i, j) = \rho_{i,j}\, \sigma_i\, \sigma_j$ with ρ_{ij} the correlation coefficient.

The company has to choose a proportional reinsurance or retention strategy specified by a retention vector $\mathbf{x}$. The retention is feasible if $0 \leq \mathbf{x} \leq 1$. By applying reinsurance on original terms, a retention $\mathbf{x}$ induces a random profit with expected value $E = \mathbf{x}^T\mathbf{m}$ and variance $V = \mathbf{x}^T C\mathbf{x}$. How to choose $\mathbf{x}$? In his seminal paper de Finetti [1] introduced the mean-variance (or equivalently the mean-W ruin probability) paradigm in financial decisions under uncertainty, suggesting that the choice should be restricted to the set of mean-variance (mean-W ruin probability) efficient retentions, that is among those feasible $\mathbf{x}$ such that are no feasible retentions $\mathbf{y}$ with $E(\mathbf{y}) \geq E(\mathbf{x})$, $V(\mathbf{y}) \leq V(\mathbf{x})$ $(p(W, \mathbf{y}) \leq p(W, \mathbf{x}))$ and at least one of the two inequalities holding in a strict sense. By $p(W, \mathbf{x}) = p(W, X)$ we denote, following de Finetti, the single period ruin probability of a retention $\mathbf{x}$, that is the probability that $(W + X) \leq 0$.

Of course, the same paradigm holds if we refer to the set of mean-standard deviation efficient retentions, which are those feasible $\mathbf{x}$ such that are no feasible retentions $\mathbf{y}$ with $E(\mathbf{y}) \geq E(\mathbf{x})$, $\sigma(\mathbf{y}) \leq \sigma(\mathbf{x})$ and at least one of the two inequalities holding in a strict sense.

Making recourse to the Karush-Kuhn-Tucker (KKT) conditions ([2], [3]) in order to solve the constrained quadratic optimization problem of minimizing variance conditional to any fixed value of the mean (between 0 and $m(\mathbf{1})$, that is the largest expectation corresponding to full retention), the following Lagrangian is obtained:

$$L(\mathbf{x}, \lambda, \mathbf{u}, \mathbf{v}) = \frac{1}{2}\,\mathbf{x}^\top C\,\mathbf{x} + \lambda\,(E - \mathbf{m}^T\,\mathbf{x}) + \mathbf{u}\,(\mathbf{x} - \mathbf{1}) - \mathbf{v}\,\mathbf{x} \tag{1}$$

It has been found [10] that the solution of this general problem may be expressed through the so called *advantage functions*[2]:

$$F_i(\mathbf{x}) = \frac{1}{2}\frac{\partial V/\partial x_i}{\partial E/\partial x_i} := \sum_{j=1}^{n} \frac{\sigma(i,j)}{m_i} \cdot x_j \qquad i = 1, \dots, n \tag{2}$$

and that the solutions show the following geometric properties: the graph of the efficient (retention) set, in the mean-standard deviation (mean-variance) space, is a continuous increasing convex curve, connecting the origin (zero retention) with the point of full retention, which is the point of greatest mean. The proof widely exploits the characteristics, in the mean-variance space, of the efficient set sketched by de Finetti under an implicit regularity condition and then recently specified in [10] (sct. 6 and 7) and discussed, in the particular case of group correlation, in [11] (sct. 5).

2.2 Variable quota share proportional reinsurance under group correlation

Shortly de Finetti's problem under group correlation is as follows: the risks of the insurance company are partitioned into a number g of groups $q = 1, \dots, g$. Under the group correlation hypothesis, the elements of $\mathbf{m}$ satisfy $m_{i,q} = \ell_q\, \sigma_{i,q}$, where ℓ_q is a group specific loading coefficient used to charge premiums through a safety loading inspired by the standard deviation principle; C is a block diagonal matrix, $C = \mathrm{diag}(C_1, \dots, C_g)$ i.e. with non null elements only on the main diagonal squared blocks, given for $i_q \neq j_q$ by $\sigma(i_q, j_q) = \rho_q\, \sigma_{i,q}\, \sigma_{j,q}$ with $\rho_q \geq 0$ the group specific correlation coefficient and obviously[3] $\sigma(i_q, i_q) = \sigma^2_{i,q}$. We remark that, under group correlation, the pre-reinsurance random gain of the company is fully described by the couple of $g-$dimensional vectors of correlation and loading coefficients and by the set of g standard deviations' vectors (may be of different dimensions), the latter briefly named *standard deviations structure*.

Under group correlation, the advantage functions (2) become (with $\mathbf{x}_q$ the retention vector of group q):

$$F_{i,q}(\mathbf{x}_q) = \ell_q^{-1}\left(x_{i,q}\sigma_{i,q} + \rho_q \sum_{j \neq i} x_{j,q}\sigma_{j,q}\right) \tag{3}$$

After that, the efficient set is characterized in terms of the advantage functions as follows (for details of the proof, see [11], sect. 3.1):

Optimality conditions under group correlation: $\hat{\mathbf{x}}$ is Mean Variance efficient iff there exists $\lambda \geq 0$ such that, for any $q = 1, \dots, g$:

[2] We recall that such functions have been introduced in [1] as a tool to find, through an intuitive simple procedure, the Mean Variance efficient set at a time where the KKT conditions were not yet available. In a recent paper [10] it has been proposed to call these functions *advantage functions*, as they intuitively capture the advantage coming at a retention point $\mathbf{x}$ from a marginal (additional or initial) reinsurance of the i-th risk. The advantage is measured precisely by the ratio (one half) decrease of variance over decrease of expectation.

[3] Note that $\sigma(i_q, j_q)$ is a covariance symbol, while $\sigma_{i,q}$ is a standard deviation.

I) $F_{i,q}(\hat{\mathbf{x}}) = \ell_q^{-1}\left(\hat{x}_{i,q}\sigma_{i,q} + \rho_q \sum_{j\neq i} \hat{x}_{j,q}\sigma_{j,q}\right) = \lambda \quad \text{if } 0 < \hat{x}_{i,q} < 1$

II) $F_{i,q}(\hat{\mathbf{x}}) = \ell_q^{-1}\,\rho_q \sum_{j\neq i} \hat{x}_{j,q}\sigma_{j,q} \geq \lambda \qquad \text{if } \hat{x}_{i,q} = 0$

III) $F_{i,q}(\hat{\mathbf{x}}) = \ell_q^{-1}\left(\sigma_{i,q} + \rho_q \sum_{j\neq i} \hat{x}_{j,q}\sigma_{j,q}\right) \leq \lambda \quad \text{if } \hat{x}_{i,q} = 1$

To capture the intuitive meaning of the condition, look at the advantage function $F_{i,q}(\mathbf{x})$ as the pseudo marginal utility at $\mathbf{x}$ of buying reinsurance of the i-th risk of the group q and at λ as the shadow price of any (marginal in quota terms) reinsurance. After that, the optimality conditions mean that, given the shadow price, reinsurance of a risk is bought if the marginal utility is larger than the price and up to the point where the (diminishing) marginal utility just matches the price, or obviously if zero retention has been reached this way.

2.3 The efficient set of the group correlation case in the space of retentions

We summarize here the closed form formulae of the efficient retention set recently obtained in [11] as they will appear in [12].

The set of efficient retentions under group correlation is given, for $q = 1, \ldots, g$, by $\hat{\mathbf{x}}(\lambda) = 1$ for $\lambda > \lambda_{1,1}$ and for any $0 \leq \lambda \leq \lambda_{1,1}$ by:

$$\hat{x}_{i,q}(\lambda) = \ell_q\, \sigma_{i,q}^{-1}\, \phi^{-1}\, \lambda - \rho_q \sigma_{i,q}^{-1}\, \phi^{-1} \sum_{j=\chi}^{n_q} \sigma_{j,q} \quad i = 1, \ldots, (\chi - 1) \tag{4a}$$

$$\hat{x}_{i,q}(\lambda) = 1 \quad i = \chi, \ldots, n_q; \tag{4b}$$

where $\chi(q, \lambda)$ is a group specific function of the shadow price to be explained below, while $\phi = \phi(q, \lambda) = [1+\rho_q\cdot(\chi(q, \lambda)-2)]$ is another group specific function of the shadow price. To understand the meaning of (4a) and (4b), keep account that there is a labeling of the risks within each group according to their standard deviation ranking, $\sigma_{1,q} > \sigma_{2,q} > \ldots > \sigma_{n_q,q}$, and a labeling of groups according to their advantage functions ranking at full retention, so as $F_{1,1}(\mathbf{1}_1) > F_{1,2}(\mathbf{1}_2) > \ldots > F_{1,g}(\mathbf{1}_g)$, where coherently with (3):

$$F_{1,q}(\mathbf{1}_q) = \ell_q^{-1}\left(\sigma_{1,q} + \rho_q \sum_{j=2}^{n_q} \sigma_{j,q}\right), \quad q = 1, \ldots, g \tag{5}$$

Now, let us consider the set of "critical" values of the shadow price given by: $\lambda_{i,q} = \ell_q^{-1} \cdot \left(\sigma_{i,q} \cdot (1 + \rho_q \cdot (i-2)) + \rho_q \cdot \sum_{j=i}^{n_q} \sigma_{j,q}\right)$ whose meaning is that of shadow price level at which the risk i_q begins to be reinsured, so as $x_{i,q} = 1$ for $\lambda \geq \lambda_{i,q}$ and $x_{i,q} < 1$ for $\lambda < \lambda_{i,q}$. For any (i, q), it is $\lambda_{i,q} > \lambda_{i+1,q}$, hence, with the dummy positions $\lambda_{0,q} = +\infty$ and $\lambda_{n+1,q} = 0$, $\chi(q, \lambda)$ is the group specific counter of the number of risks already reinsured at λ. In the end, the counter has constant group and interval specific value $\chi(q, \lambda) = h_q$ for $\lambda_{h+1,q} \leq \lambda < \lambda_{h,q}$ and in turn, $\phi(q, \lambda) = \phi_q(h_q) = [1 + \rho_q \cdot (h_q - 2)]$.

In addition, the fact that $\lambda_{1,1}$ is the maximum of $\lambda_{i,q}$ explains why $\hat{\mathbf{x}}(\lambda) = 1$ or $\chi(q,\lambda) = 0$ for any $\lambda > \lambda_{1,1}$.

2.4 The efficient set in the Mean Variance space

Keeping account that in any interval between two consecutive critical values of λ (in general belonging to two different groups), the vector $\mathbf{h} = h(\lambda) = [h_1(\lambda), \ldots, h_q(\lambda), \ldots, h_g(\lambda)]$ does not change with λ (i.e. the vector is interval specific), we may write the following closed form interval specific expressions of expectations and variances of the efficient retentions as functions of λ, both for the groups and for the global portfolio. Denoting in any of such intervals by $E_q(\lambda)$ and respectively by $V_q(\lambda)$ the group expectation and variance, it is:

$$E_q(\lambda) = \lambda\, \alpha(q, h_q) + \beta(q, h_q) \tag{6}$$

$$V_q(\lambda) = \lambda^2\, \alpha(q, h_q) + \gamma(q, h_q) \tag{7}$$

with

$$\alpha(q, h_q) = \ell_q^2\, (h_q - 1)\, [\phi_q(h_q)]^{-1}$$

$$\beta(q, h_q) = \sum_{i=h_q}^{n_q} m_{i,q} - [\phi_q(h_q)]^{-1}\, \ell_q\, \rho_q\, (h_q - 1) \sum_{j=h_q}^{n_q} \sigma_{j,q}$$

$$\gamma(q, h_q) = 2\rho_q \sum_{i=h_q}^{n_q} \sigma_{i,q} \sum_{j=h_q+1}^{n_q} \sigma_{j,q} + \sum_{i=h_q}^{n_q} \sigma_{i,q}^2 - [\phi_q(h_q)]^{-1}\, \rho_q^2\, (h_q - 1) \Big(\sum_{i=h_q}^{n_q} \sigma_{i,q} \Big)^2$$

Note that α, β, γ are functions which (besides the standard deviations of the group) depend on q through the couple (ℓ_q, ρ_q) of group specific parameters as well as on $h_q(\lambda)$ (directly and also indirectly through ϕ), which is both group and interval specific. Then, to obtain the closed form (interval specific) expressions of the global mean and variance of the efficient portfolios as a function of λ, simply add over q the group expectations and respectively (exploiting the zero correlation between different groups) the group variances. After that, the global mean as an interval specific function of λ is given by:

$$E(\lambda) = \lambda\, \alpha(\mathbf{h}) + \beta(\mathbf{h}) \tag{8}$$

and the global variance is the following interval specific function of the global expectation:

$$V(E) = \frac{[E - \beta(\mathbf{h})]^2}{\alpha(\mathbf{h})} + \gamma(\mathbf{h}) \tag{9}$$

where $\alpha(\mathbf{h}) = \sum_q \alpha(q, h_q)$, $\beta(\mathbf{h}) = \sum_q \beta(q, h_q)$ and $\gamma(\mathbf{h}) = \sum_q \gamma(q, h_q)$ are piecewise constant interval specific functions of λ. Hence, the efficient set in the Mean Variance space is a union of parabolas, whose graph turns out to be continuous and differentiable (without kinks) also at the connection points (see [11], p. 13).

As said at the end of 2.1, this implies that in the mean-standard deviation plane the efficient set is a continuous increasing convex curve, connecting the origin (zero retention) with the point of full retention. We stress that the convexity value could be in some examples very small, i.e. the second derivative is positive but near to zero.

3 Iso-ruin lines in the mean-standard deviation plane

As mentioned before, initially de Finetti considered in his single period reinsurance problem the ruin probability as the proper risk measure and tried to study the connection between the efficiency in mean-variance and the one in mean-ruin probability. In this direction he introduced, even if without deepening the related implications, also the explicit equations of iso-ruin parabolae in the mean-variance plane, while he did not focus his attention on a mean-standard deviation framework.

It is advantageous, for computational purposes, to transfer to a mean-standard deviation reference plane where the form of the efficient retention set is given by:

$$\sigma(m) = \left[\frac{(m-\beta)^2}{\alpha} + \gamma\right]^{1/2} \tag{10}$$

where α, β, γ are piecewise constant interval specific parameters of the efficient union of parabolas (according to Formula (9)). It is very important to underline that the mean-standard deviation efficient set is, in our reinsurance problem, coincident with the mean-ruin probability one (see Pressacco-Ziani [13]).

It is also useful to provide the equation of the W iso-ruin lines in our reference plane.

To this aim, let X be the random single period algebraic gain of the insurance company, with mean m and standard deviation σ. If W is the free initial capital or wealth of the company, then its ruin occurs if, at the end of the period, it is $X + W \leq 0$ or $X \leq -W$. So the ruin probability is $p(W, X) = \Phi_X(-W)$ which is the cumulative distribution function of the X variable at $-W$. Under normality, denoting by:

$$-t_X = -\frac{W + m_X}{\sigma_X} \tag{11}$$

it is immediately seen that:

$$\Phi_X(-W) = \Phi_Z(-t_X) \tag{12}$$

with $\Phi_Z(-t)$ the cumulative distribution function of the standard normal variable Z at level t. Such a probability is then a decreasing function of t.

It is relevant to check that, in the mean-standard deviation plane, the straight line whose equation is:

$$\sigma(m) = \frac{m + W}{t_X} \tag{13}$$

that is the ray, going out from $-W_0$ and with slope t^{-1}, is the iso-ruin line which supports random variables sharing, for the given W, the same ruin probability of X.

Remark. Given W, the lower is t the greater are the slope and the ruin probability and viceversa.

4 From efficiency to mean-ruin probability optimality in single period myopic decisions

Passing now from efficiency to single period optimality, that is looking for a unique optimal retention point, we introduce an upper bound of acceptable ruin probability $\overline{p}$ and select as optimal reinsurance decision the one which gives the largest expectation conditional to the upper bound constraint. From a geometric point of view in a mean-standard deviation plane, such a decision is the intersection point of the efficient mean-ruin probability frontier with the iso-ruin line at the given level. The reason is obvious provided that this intersection is not empty. It is clear that if the iso-ruin line lies above the efficient frontier then no intersection occurs, so that the optimal portfolio in this scenario is that with full retention (zero reinsurance), whose ruin probability is lower than the upper bound.

As introduced before, in our specific proportional reinsurance problem of group correlation the expression of the efficient mean-standard deviation frontier is (see Formula (10)):

$$\sigma(m) = \left[\frac{(m-\beta)^2}{\alpha} + \gamma\right]^{1/2}$$

Now solving for m the equation:

$$\left[\frac{(m-\beta)^2}{\alpha} + \gamma\right]^{1/2} = \frac{m+W}{t}$$

after some elementary algebra, we obtain the optimal solution m^* of the intersection problem:

$$m^* = \frac{(\beta t^2 - \alpha W) + \sqrt{\alpha t^2 \cdot [(W+\beta)^2 + \gamma(\alpha - t^2)]}}{t^2 - \alpha} \tag{14}$$

for any given W and for the chosen ruin probability upper bound in the period. Obviously, if an intersection occurs:

$$\sigma^* = \sigma(m^*) = t^{-1} \cdot m^* + t^{-1} \cdot W$$

otherwise

$$\sigma^* = \left[\frac{(m^*-\beta)^2}{\alpha} + \gamma\right]^{1/2}$$

Note that this solution requires the preliminary choice of the specific interval in which the solution is meaningful[4].

The portfolio with coordinates (m^*, σ^*) provides the highest expectation value among those respecting the constraint on the single period ruin probability. To obtain the optimal retentions vector $\mathbf{x}^*$, we recover the shadow price $\lambda(m^*)$ exploiting (8) and then $\mathbf{x}^*$ as a function of $\lambda(m^*)$ according to (4).

We will discuss later an economic rationale for this approach to optimality.

[4]Indeed, we could not accept as a solution an intersection point between the iso-ruin line and an inefficient point of a curve obtained extrapolating the equation of an arc of the efficient frontier.

5 Some considerations on the profit rate

Roughly speaking, the single period expected profit rate is given by the ratio between the expected gain of the optimal decision, m^* and the free capital level, W. An elegant result may be obtained under the simplifying hypothesis that the efficient frontier is a straight line $\sigma = a \cdot m$, which, as we shall see later, is a good proxy of what happens in our simulation[5]. This being the case, given t (or equivalently the ruin probability), the abscissa of the intersection point between the efficient frontier and an iso ruin line of equation $\sigma = t^{-1} \cdot m + t^{-1} \cdot W$, satisfies the equation:

$$m \cdot (a - t^{-1}) = W \cdot t^{-1}$$

Hence the profit rate:

$$\frac{m}{W} = \frac{t^{-1}}{a - t^{-1}}$$

is constant, given t, for any W. Note that a necessary condition to have an intersection is that $a > t^{-1}$, so that the profit rate is surely positive.

This gives an immediate information about the influence on the profit rate of the correlation level (roughly resumed by the coefficient a, slope of the efficient frontier) for the given ruin probability level.

Let us go back now to the economic motivation of choosing the intersection point as the optimal solution. Given a specified ruin probability upper bound and keeping as exogenously fixed (and not a control variable) the free capital level, the optimal choice would be the one maximizing the profit rate (corresponding to the highest expectation). Indeed, choosing another point would imply lowering the expected profit rate.

6 The impact on optimality of correlation, ruin probability and free capital

Elsewhere [12] we deeply investigated the consequences on efficient retentions of different combinations of correlation and loading levels. To that aim we constructed a stylized portfolio of 5.000 policies (1.000 policies for each group) characterized by a *neutral standard deviation structure*[6], i.e. with $\sigma_{i,q} = \sigma_i$ constant, given i, for any q; furthermore, the σ_i are equally spaced with $\sigma_1 = 40$ (the greatest) and $\sigma_{1000} = 0.04$ (the smallest), so as $(\sigma_i - \sigma_{i+1}) = 40/1.000 = (\sigma_1 - \sigma_{1000})/(999)\ \forall q$.

In this paper, we investigate in particular the impact on optimal choices of three key variables: still the level of correlation (given a fixed loading structure); the level of free capital and the level of single period ruin probability.

[5] Quite likely as a consequence of the neutral standard deviation structure and of the evenness of policies between groups in the portfolio.

[6] We are well aware that real life portfolio are very different from this artificial framework.

As regards correlation levels, we consider three different correlation structures: L(ow), M(edium) and H(igh), each one increasing with the group labeling, while keeping constant the ratios $M/L = 2.5$ and $H/L = 4$ across groups (see Table 1, Correlation). Moreover, we choose a loading increasing with labeling on a proper range, with a 10% mean level (see Table 1, Loading).

q	*1*	*2*	*3*	*4*	*5*
			Correlation		
L	2%	4%	6%	8%	10%
M	5%	10%	15%	20%	25%
H	8%	16%	24%	32%	40%
			Loading		
	2%	6%	10%	14%	18%

Table 1: Three correlation structures. One loading structure.

As regards the free capital and the ruin probability, we consider three different levels of both: for the first one $W_1 = 10.000$, $W_2 = 20.000$ and $W_3 = 30.000$; for the second one $p_1 = 0,5\%$, $p_2 = 2,5\%$ and $p_3 = 5\%$. After that, keeping as fixed two of those key variables, we could investigate the impact on the optimal choices of the third one, and in particular the sensitivity of the expected profit rate to the third variable.

To resume results in a couple of expressive figures, we plot the three efficient frontiers (dependent on the correlation structure chosen and, coherently with theoretical results, independent on the level of ruin probability and free capital); additionally we draw a couple of three iso ruin lines. At first, keeping as fixed the ruin probability level (that is the slope of the iso ruin line) and varying the free capital W, three parallel straight lines going out from points with coordinates $(-W; 0)$ are obtained. Then, keeping as fixed the level of W and varying the ruin probability, which means three different straight lines going out from the same point $(-W; 0)$ and with slopes depending on different ruin probability levels.

At the end, in any graph we have six curves (three iso-ruin straight lines and three efficient straight lines) and (at most) nine intersection points (three intersection points for any efficient frontier). See Figures 1 and 2.

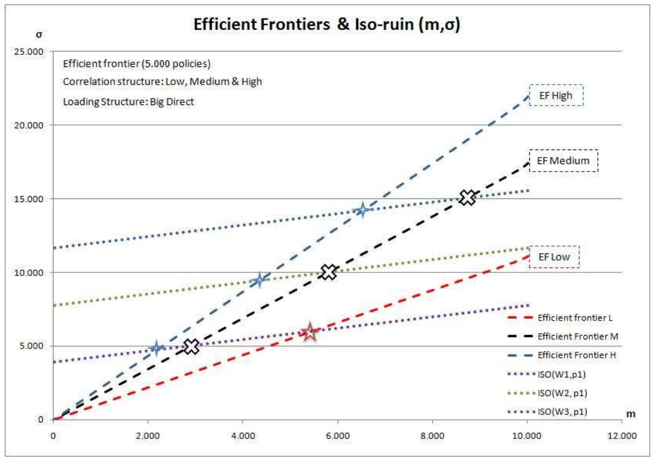

Figure 1: Fixed ruin probability. Different levels of free capital.

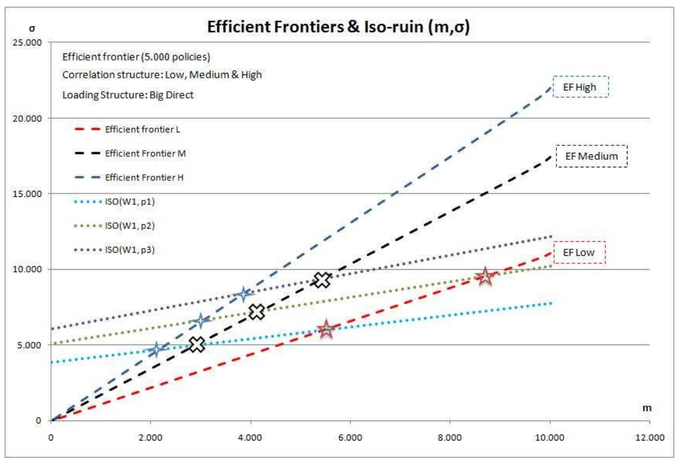

Figure 2: Fixed free capital. Different levels of ruin probability.

At first sight, it is confirmed that, in our example, all efficient sets are almost linear.

Profit Rate			
	L	M	H
W_1, p_3	100,10%	54,47%	38,77%
W_2, p_3	50,05%	50,05%	38,75%
W_3, p_3	33,37%	33,37%	33,37%
W_1, p_2	86,80%	41,99%	30,70%
W_2, p_2	50,05%	42,02%	30,63%
W_3, p_2	33,37%	33,37%	30,59%
W_1, p_1	54,59%	29,06%	21,80%
W_2, p_1	50,05%	29,02%	21,71%
W_3, p_1	33,37%	29,04%	21,71%

Table 2: Profit rates according to the levels of correlation, ruin probability and free capital.

Table 2 show how, for fixed couples of (W, p), correlation levels affect the profit rate.

We underline that the profit rates for High e Medium correlation and probability p_1 and High correlation and probability p_2 are almost constant (at different levels) for any level of free capital. This comes as a consequence of the almost perfect linearity of the efficient frontier and of an existence of an intersection between efficient frontiers and iso-ruin lines.

Another point of view on the problem could take into account that the levels of W may be interpreted as the minimum level of free capital (V@R), requested by the Solvency supervisors to manage the goal optimal portfolio at the desired ruin-probability level. Of course, we could release some capital, given the acceptable ruin-probability, at the cost of lowering the retentions and then the expected profit. Or conversely, increase the free capital in order to increase the retentions and hence the expected gain.

7 Conclusions

In this paper, we keep as a starting point the results concerning closed form formulae of the efficient frontier in a variable quota share proportional reinsurance problem under group correlation.

On this basis, we try to proceed from efficiency to single period optimality (with the perspective of a multiperiod myopic optimality) by connecting three key variables ruin probability, free capital and correlation level.

We develop a procedure able to localize the optimal single period retention set as the one maximizing the expected gain, conditional to a given combination of ruin probability and free capital.

In this framework, we provide a clear theoretical relation, confirmed by simulations on a textbook portfolio, connecting the correlation level and the expected profit rate.

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