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New symmetries of the two-Higgs-doublet model

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Abstract The Two Higgs Doublet Model invariant under the gauge group $SU(2) \times U(1)$ is known to have six additional global discrete or continuous symmetries of its scalar sector. We have discovered regions of parameter space of the model which are basis and renormalization group invariant to all orders of perturbation theory in the scalar and gauge sectors, but correspond to none of the hitherto considered symmetries. We therefore identify seven new symmetries of the model and discuss their phenomenology. Soft symmetry breaking is required for some of these models so that electroweak symmetry breaking can occur. We show that, at least at the two-loop level, it is possible to extend some of these symmetries to include fermions.

1 Introduction

The Two-Higgs-Doublet Model (2HDM) is one of the more popular extensions of the Standard Model (SM) of particle physics. It was introduced by Lee in 1973 [1] to provide a way of having spontaneous CP violation. In its simplest form, the model has the same gauge symmetries as the SM, same fermionic content – but instead of a single $SU(2)$ spin-0 doublet, the 2HDM has two, Φ_1 and Φ_2. The model has a rich phenomenology with a scalar spectrum comprising three neutral and one charged elementary spin-0 states. Different versions of the 2HDM allow for the possibility of spontaneous CP-violation; provide dark matter candidates whose stability is guaranteed by a discrete symmetry; may have tree-level flavour changing neutral currents (FCNCs) mediated by neutral scalars; may have sizeable contributions to flavour physics. For a review, see for instance [2].

The scalar potential of the SM is characterized by 2 real, independent parameters, out of which one obtains the value of the Higgs field vacuum expectation value (vev), $v = 246$ GeV, and the Higgs mass, $m_h \simeq 125$ GeV. For the 2HDM, however, the scalar potential is much more complex: the most general 2HDM has a potential with 11 independent real parameters [3]. Simultaneously, that model has scalar-mediated FCNC, which experimentally are known to be very constrained – this arises because, in the most general 2HDM, both doublets couple to all fermions. For that reason, in 1976 a discrete Z_2 symmetry was proposed to eliminate those FCNCs, so that right-handed fermions of the same hypercharge (charged leptons, up-like and down-like quarks) are made to couple to a single Higgs doublet [4,5]. Along the way the number of free scalar parameters is reduced to 7, and thus the predictivity of the model is increased. This Z_2 symmetry required invariance of the lagrangian under a transformation for which one of the doublets changes sign while the other remains unchanged, for instance $\Phi_2 \to -\Phi_2$. In another example, Peccei and Quinn [6] observed that a 2HDM endowed with a continuous global $U(1)$ symmetry was a possible solution to the strong CP problem – and in that model the number of free parameters of the scalar potential is 6. The Peccei-Quinn symmetry may be obtained by requiring invariance under a transformation like $\Phi_2 \to e^{i\alpha}\Phi_2$, for an arbitrary real phase α. These are examples of unitary symmetry transformations between the doublets, sometimes called *Higgs family symmetries*. Another possibility is to transform doublets into a linear combination of their complex conjugates, or more precisely their CP conjugates, and these are called *generalized CP symmetries*. These two types of field transformations leave invariant the doublets' kinetic terms, and it has been shown [7,8] that, for the $SU(2)\times U(1)$ invari-

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ant scalar potential, there only *six* possible symmetries. Since in the 2HDM both doublets have the same quantum numbers, any linear combination thereof which preserves the kinetic terms is equally acceptable. This freedom to choose a basis of scalar fields may mask the form of the symmetries, so that it may seem there are more than six of them. In fact a basis-independent analysis shows that indeed, only six different symmetries – and therefore six different versions of the 2HDM, with different numbers of free parameters and possible phenomenology – are allowed, when one considers all possible doublet transformations which preserve the kinetic terms and gauge symmetries.

A fingerprint of continuous symmetries, from Noether's theorem [9], is the existence of some quantities (charges) which are conserved during the evolution of the system under its equations of motion. Indeed, for each of the six symmetries mentioned above (and explained in greater detail in Sect. 2.3) certain relations between parameters of the 2HDM scalar potential are found to be preserved under renormalization. Symmetry-constrained relations between the dimensionless couplings of the model will even remain invariant to all orders of perturbation after spontaneous symmetry breaking of that symmetry has occurred.[1]

In this paper we will investigate a curious situation in which we have been able to identify a region of 2HDM parameter space characterized by specific relations between couplings which are not only basis invariant but also left invariant under the renormalization group (RG) – and which do not correspond to any of the six aforementioned symmetries. In terms of the most usual notation used to write the 2HDM scalar potential, these conditions are

$$m_{11}^2 + m_{22}^2 = 0, \quad \lambda_1 = \lambda_2, \quad \lambda_7 = -\lambda_6. \tag{1.1}$$

Using arguments of basis invariance, we will show how this specific region of the 2HDM parameter space remains invariant under renormalization to all orders of perturbation theory, not considering fermions. We were unable to extend the all-order argument to the Yukawa sector, but will show, via an explicit calculation, that the relations between parameters we have found remain invariant at least to two loops when fermions are taken into account. We therefore conclude that, at least to two-loop order, the specific relations between couplings which we found are invariant under renormalization when the whole lagrangian is taken into account. Indeed, it could be that invariance at one-loop would be the consequence of some unphysical fine-tuning, but to see that those relations between couplings remain valid even when two-loop contributions are taken into account suggests that invariance to all orders is a strong possibility. To put things into perspective, consider that multi Higgs doublets are many times studied under the so-called "custodial symmetry", which is an approximate symmetry of the lagrangian. The scalar potential can be made invariant implying a specific mass spectrum for scalars. Invariance of the kinetic terms under custodial symmetry would imply equal masses for the W and Z bosons, and is therefore broken by the presence of the gauge coupling constant corresponding to the hypercharge gauge group. It is also broken by Yukawa interactions, namely by the fact that up-type and down-type quarks have different masses. Therefore, custodial symmetry relations are not preserved under radiative corrections even at the one-loop level.

However, we cannot find what specific field transformation yields these RG-invariant conditions. They cannot come from a Higgs family or a generalized CP symmetry. We have identified a transformation on scalar bilinears – quadratic combinations of scalar doublets which are gauge invariant – which seemingly produces exactly the region of parameter space we are interested in, but not only such a transformation is impossible to reproduce on the basis of operations upon doublets, it does not seem possible to extend it to the gauge sector, let alone the Yukawa one. Nonetheless, though ignorant of the transformation on fields which produces this RG invariant region, we will nevertheless refer to it as being produced by a *symmetry*, which we call the r_0 *symmetry*. It is possible to combine the r_0 symmetry with the other six known 2HDM symmetries and find new models, which boast (new) combinations of parameters which are RG-invariant to all orders, and quite interesting phenomenologies, including, for specific models: existence of explicit CP violation; impossibility of spontaneous breaking of a Z_2 symmetry or CP violation; mass degeneracy of neutral scalars; and no decoupling limit possible when the r_0 symmetry holds. While extending the r_0 symmetry to the fermion sector we will prove that the Yukawa matrices found obeying previously known 2HDM symmetries (to wit, the symmetries called CP2 and CP3) also preserve the new conditions among parameters characteristic of the new symmetry to at least two-loop order.

This paper is structured as follows: in Sect. 2 we review the 2HDM, with an emphasis on basis transformations, the bilinear formalism, the known symmetries of the model and the model's one-loop renormalization group equations, which will be the stepping stone for our reasoning. In Sect. 3 we will show how the set of relations among 2HDM parameters shown above is preserved under renormalization at the one-loop level. We will then demonstrate how, considering only the scalar and gauge sectors of the model to begin with, that invariance is indeed an all-order result, using arguments of basis invariance and dimensional analysis to perform an analysis of the model's β-functions at an arbitrary number of loops. We then provide a heuristic interpretation of this symmetry using the bilinear formalism, which shows how the desired conditions upon the parameters may be obtained via a sign change in one of the bilinears in a formal manner, which

[1] Finite contributions to those couplings from radiative corrections may spoil those relations, however.

justifies the name r_0 symmetry we chose. We then combine the r_0 symmetry conditions with those of the known 6 2HDM symmetries and list 7 new possible symmetries of the model. Some of those lead to vanishing quadratic terms and must be softly broken. In Sect. 4 we analyse the phenomenology of the scalar sector of each of the symmetries considered, including soft breaking terms when necessary or interesting. Section 5 sees us tackling the fermion sector and arguing that the CP2 or CP3 Yukawa textures would adequately preserve the r_0 symmetry relations between quartic couplings to all orders, and showing by means of an explicit β-function calculation, that those same Yukawa structures also preserve, at least to two-loop order, the relation $m_{11}^2 + m_{22}^2 = 0$. An overview of our results and conclusions are drawn in Sect. 6.

2 The two-Higgs doublet model

The 2HDM is one of the simplest extensions of the SM, wherein one considers two $SU(2)$ doublets with hypercharge one instead of just one doublet. In the following we will briefly review the basic aspects of a useful formalism to understand the structure of the scalar sector of the model, and the global symmetries one can impose on it.

2.1 The scalar potential

The most general scalar potential involving two hypercharge $Y = 1$ scalar doublets invariant under the electroweak gauge group $SU(2)_L \times U(1)_Y$ is given by

$$\begin{aligned} V =\ & m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger \Phi_2 - [m_{12}^2 \Phi_1^\dagger \Phi_2 + \text{h.c.}] \\ & + \tfrac{1}{2}\lambda_1 (\Phi_1^\dagger \Phi_1)^2 + \tfrac{1}{2}\lambda_2 (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) \\ & + \lambda_4 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) + \Big\{ \tfrac{1}{2}\lambda_5 (\Phi_1^\dagger \Phi_2)^2 \\ & + \big[\lambda_6 (\Phi_1^\dagger \Phi_1) + \lambda_7 (\Phi_2^\dagger \Phi_2)\big] \Phi_1^\dagger \Phi_2 + \text{h.c.} \Big\}, \end{aligned} \tag{2.1}$$

where, other than m_{12}^2 and $\lambda_{5,6,7}$, all parameters are real. An alternative notation uses four gauge-invariant bilinears constructed from the doublets [7,8,10–18],

$$\begin{aligned} r_0 &= \frac{1}{2}\left(\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2\right), \\ r_1 &= \frac{1}{2}\left(\Phi_1^\dagger \Phi_2 + \Phi_2^\dagger \Phi_1\right) = \text{Re}\left(\Phi_1^\dagger \Phi_2\right), \\ r_2 &= -\frac{i}{2}\left(\Phi_1^\dagger \Phi_2 - \Phi_2^\dagger \Phi_1\right) = \text{Im}\left(\Phi_1^\dagger \Phi_2\right), \\ r_3 &= \frac{1}{2}\left(\Phi_1^\dagger \Phi_1 - \Phi_2^\dagger \Phi_2\right). \end{aligned} \tag{2.2}$$

In terms of these quantities, then, the potential of Eq. (2.1) may be written as

$$V = M_\mu r^\mu + \Lambda_{\mu\nu} r^\mu r^\nu, \tag{2.3}$$

where we use a Minkowski-like formalism to define the 4-vectors

$$\begin{aligned} r^\mu &= (r_0,\ r_1,\ r_2,\ r_3) = (r_0,\ \vec{r}), \\ M^\mu &= \left(m_{11}^2 + m_{22}^2,\ 2\text{Re}(m_{12}^2),\ -2\text{Im}(m_{12}^2),\ m_{22}^2 - m_{11}^2\right) \\ &= (M_0,\ \vec{M}), \end{aligned} \tag{2.4}$$

as well as the tensor

$$\Lambda^{\mu\nu} = \begin{pmatrix} \Lambda_{00} & \vec{\Lambda} \\ \vec{\Lambda}^T & \Lambda \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\lambda_1+\lambda_2)+\lambda_3 & -\text{Re}\,(\lambda_6+\lambda_7) & \text{Im}\,(\lambda_6+\lambda_7) & \frac{1}{2}(\lambda_2-\lambda_1) \\ -\text{Re}\,(\lambda_6+\lambda_7) & \lambda_4+\text{Re}\,(\lambda_5) & -\text{Im}\,(\lambda_5) & \text{Re}\,(\lambda_6-\lambda_7) \\ \text{Im}\,(\lambda_6+\lambda_7) & -\text{Im}\,(\lambda_5) & \lambda_4-\text{Re}\,(\lambda_5) & -\text{Im}\,(\lambda_6-\lambda_7) \\ \frac{1}{2}(\lambda_2-\lambda_1) & \text{Re}\,(\lambda_6-\lambda_7) & -\text{Im}\,(\lambda_6-\lambda_7) & \frac{1}{2}(\lambda_1+\lambda_2)-\lambda_3 \end{pmatrix}. \tag{2.5}$$

For future convenience, we defined the singlet Λ_{00} and the vector $\vec{\Lambda}$ as

$$\begin{aligned} \Lambda_{00} &= \frac{1}{2}(\lambda_1+\lambda_2)+\lambda_3, \\ \vec{\Lambda} &= \left(-\text{Re}\,(\lambda_6+\lambda_7),\ \text{Im}\,(\lambda_6+\lambda_7),\ \frac{1}{2}(\lambda_2-\lambda_1)\right), \end{aligned} \tag{2.6}$$

and therefore the matrix Λ from Eq. (2.5) is the right-bottom 3×3 block within the $\Lambda^{\mu\nu}$ tensor above. The bilinear notation can be very useful for studies of 2HDM symmetries or vacuum structure. Recently, bilinears were used to analyse the 1-loop 2HDM effective potential and analyse its global and CP symmetries [19,20].

2.2 Basis transformations

Since both doublets have exactly the same quantum numbers, there is nothing a priori that distinguishes one from the other – thus any linear combination of the two that preserves the kinetic terms of the theory should yield the same physics. Specifically, if one considers a new set of doublets $\{\Phi_1',\ \Phi_2'\}$ related to the first by $\Phi_a' = U_{ab}\,\Phi_b$, for any unitary 2×2 matrix U, the model, and all physics thereof originating, is left invariant. These are called *basis transformations*, and the parameters of the potential will in general change from basis to basis. If we parameterize the matrix U as

$$U = \begin{pmatrix} e^{i\chi} c_\psi & e^{i(\chi-\xi)} s_\psi \\ -e^{i(\xi-\chi)} s_\psi & e^{-i\chi} c_\psi \end{pmatrix}, \tag{2.7}$$

where we have defined $c_x = \cos x$ and $s_x = \sin x$, we obtain relations between the parameters of the potential in the new basis as a function of those in the original one and the angles and phases which form U [2,21]:

$$m_{11}^{2}{}' = m_{11}^2 c_\psi^2 + m_{22}^2 s_\psi^2 - \mathrm{Re}\left(m_{12}^2 e^{i\xi}\right) s_{2\psi}, \tag{2.8a}$$

$$m_{22}^{2}{}' = m_{11}^2 s_\psi^2 + m_{22}^2 c_\psi^2 + \mathrm{Re}\left(m_{12}^2 e^{i\xi}\right) s_{2\psi}, \tag{2.8b}$$

$$m_{12}^{2}{}' = e^{i(2\chi-\xi)} \left[\frac{1}{2}\left(m_{11}^2 - m_{22}^2\right) s_{2\psi} + \mathrm{Re}\left(m_{12}^2 e^{i\xi}\right) c_{2\psi} + i\,\mathrm{Im}\left(m_{12}^2 e^{i\xi}\right)\right], \tag{2.8c}$$

$$\lambda_1' = \lambda_1 c_\psi^4 + \lambda_2 s_\psi^4 + \frac{1}{2}\lambda_{345} s_{2\psi}^2 + 2 s_{2\psi}\left[c_\psi^2 \mathrm{Re}\left(\lambda_6 e^{i\xi}\right) + s_\psi^2 \mathrm{Re}\left(\lambda_7 e^{i\xi}\right)\right], \tag{2.8d}$$

$$\lambda_2' = \lambda_1 s_\psi^4 + \lambda_2 c_\psi^4 + \frac{1}{2}\lambda_{345} s_{2\psi}^2 - 2 s_{2\psi}\left[s_\psi^2 \mathrm{Re}\left(\lambda_6 e^{i\xi}\right) + c_\psi^2 \mathrm{Re}\left(\lambda_7 e^{i\xi}\right)\right], \tag{2.8e}$$

$$\lambda_3' = \lambda_3 + \tfrac{1}{4} s_{2\psi}^2 \left(\lambda_1 + \lambda_2 - 2\lambda_{345}\right) - s_{2\psi} c_{2\psi} \mathrm{Re}\left[\left(\lambda_6 - \lambda_7\right) e^{i\xi}\right], \tag{2.8f}$$

$$\lambda_4' = \lambda_4 + \frac{1}{4} s_{2\psi}^2 \left(\lambda_1 + \lambda_2 - 2\lambda_{345}\right) - s_{2\psi} c_{2\psi} \mathrm{Re}\left[\left(\lambda_6 - \lambda_7\right) e^{i\xi}\right], \tag{2.8g}$$

$$\lambda_5' = e^{2i(2\chi-\xi)} \left\{\frac{1}{4} s_{2\psi}^2 \left(\lambda_1 + \lambda_2 - 2\lambda_{345}\right) + \mathrm{Re}\left(\lambda_5 e^{2i\xi}\right) + i c_{2\psi} \mathrm{Im}\left(\lambda_5 e^{2i\xi}\right) - s_{2\psi} c_{2\psi} \mathrm{Re}\left[\left(\lambda_6 - \lambda_7\right) e^{i\xi}\right] - i s_{2\psi} \mathrm{Im}\left[\left(\lambda_6 - \lambda_7\right) e^{i\xi}\right]\right\}, \tag{2.8h}$$

$$\lambda_6' = e^{i(2\chi-\xi)} \left\{-\frac{1}{2} s_{2\psi}\left[\lambda_1 c_\psi^2 - \lambda_2 s_\psi^2 - \lambda_{345} c_{2\psi} - i\,\mathrm{Im}\left(\lambda_5 e^{2i\xi}\right)\right] + c_\psi c_{3\psi} \mathrm{Re}\left(\lambda_6 e^{i\xi}\right) + s_\psi s_{3\psi} \mathrm{Re}\left(\lambda_7 e^{i\xi}\right) + i c_\psi^2 \mathrm{Im}\left(\lambda_6 e^{i\xi}\right) + i s_\psi^2 \mathrm{Im}\left(\lambda_7 e^{i\xi}\right)\right\}, \tag{2.8i}$$

$$\lambda_7' = e^{i(2\chi-\xi)} \left\{-\frac{1}{2} s_{2\psi}\left[\lambda_1 s_\psi^2 - \lambda_2 c_\psi^2 + \lambda_{345} c_{2\psi} + i\,\mathrm{Im}\left(\lambda_5 e^{2i\xi}\right)\right] + s_\psi s_{3\psi} \mathrm{Re}\left(\lambda_6 e^{i\xi}\right) + c_\psi c_{3\psi} \mathrm{Re}\left(\lambda_7 e^{i\xi}\right) + i s_\psi^2 \mathrm{Im}\left(\lambda_6 e^{i\xi}\right) + i c_\psi^2 \mathrm{Im}\left(\lambda_7 e^{i\xi}\right)\right\}, \tag{2.8j}$$

where for convenience we write

$$\lambda_{345} = \lambda_3 + \lambda_4 + \mathrm{Re}\left(\lambda_5 e^{2i\xi}\right). \tag{2.9}$$

Basis transformations are exceedingly simple to write in the bilinear formalism. Defining the 3×3 matrix $O(3)$ rotation matrix $R_{ij}(U) = \mathrm{Tr}\left(U^\dagger \sigma_i U \sigma_j\right)/2$, where σ_i ($i = 1, 2, 3$) are the Pauli matrices, we find that $\vec{r}$, $\vec{M}$ and $\vec{\Lambda}$ transform as vectors for these basis changes, i.e.

$$\vec{r}\,' = R\,\vec{r}, \quad \vec{M}' = R\,\vec{M}, \quad \vec{\Lambda}' = R\,\vec{\Lambda}, \tag{2.10}$$

whereas r_0, M_0 and Λ_{00} do not change under basis transformations – they are *basis invariants* – and Λ transforms as a 3×3 matrix would under rotations, $\Lambda' = R\,\Lambda\,R^T$.

The most general potential of Eq. (2.1) has seemingly 14 independent real parameters, but in fact, once basis freedom is taken into account (which allows one to choose a basis so as to eliminate several parameters), the real number of independent real parameters is 11 [3]. This may be seen in several ways, but perhaps the simplest of those is using the bilinear formalism described above: since basis transformations correspond, in this formalism, to $O(3)$ rotations, the matrix R is characterized by 3 independent angles, which can be used to "rotate away" three of the 14 parameters of the potential. For instance, one can chose R so as to diagonalize the symmetric 3×3 Λ matrix, thus eliminating three out of its six parameters.

It is also possible to express the kinetic terms in terms of bilinears, though the limitations of this formalism start to appear. As explained in [12], the scalar kinetic terms T (excluding gauge interactions) may be written as

$$T = K^\mu \left(\partial_\alpha \Phi_i\right)^\dagger \left(\sigma_\mu\right)_{ij} \left(\partial^\alpha \Phi_j\right), \tag{2.11}$$

where a sum on $i, j = 1, 2$ is assumed, and the 4-vectors in this expression are $\sigma^\mu = (\mathbb{1}, \sigma_i)$, with σ_i the Pauli matrices, and $K^\mu = (1, 0, 0, 0)$. Though we write the bilinears and the potential in a Minkowski-like formalism, we should not consider boost transformations of the 4-vectors or tensors considered: in fact, such transformations would change K^μ in such a way that Eq. (2.11) would no longer yield the correct kinetic terms for the scalar doublets.

2.3 Global symmetries of the 2HDM

One can impose global symmetries on the 2HDM potential of Eq. (2.1) to obtain models with different and interesting phenomenology. Following the usual procedure, one takes scalar field transformations which preserve their kinetic terms, and there are two possibilities for that to occur: one may consider *Higgs-family symmetries*, where unitary transformations mix both doublets,

$$\Phi_i \;\rightarrow\; \Phi'_i = \sum_{j=1}^{2} U_{ij}\,\Phi_j \tag{2.12}$$

where U is a generic 2×2 unitary matrix; the other possibility are transformations with transformed doublets being linear combinations of their complex conjugates,

$$\Phi_i \;\rightarrow\; \Phi'_i = \sum_{j=1}^{2} X_{ij}\,\Phi^*_j \tag{2.13}$$

where once again $X \in U(2)$ is a generic matrix but now the transformed fields are combinations of the complex conjugates of the original doublets. These are called *generalized CP (GCP) symmetries*. The simplest example of a transformation like those of Eq. (2.12) is a simple Z_2 one, with one of the doublets changing sign, while the other remains the same,

$$\Phi_1 \;\rightarrow\; \Phi_1, \quad \Phi_2 \;\rightarrow\; -\Phi_2. \tag{2.14}$$

This symmetry, when extended to the Yukawa sector, prevents the occurrence of tree-level flavour-changing neutral currents (FCNC) [4,5], and eliminates the m^2_{12}, λ_6 and λ_7 couplings. And the simplest example of a symmetry like those of Eq. (2.13) is the "standard" CP transformation, i.e. requiring invariance of the potential under the field transformation

$$\Phi_i \;\rightarrow\; \Phi^*_i. \tag{2.15}$$

This symmetry, sometimes called CP1, yields a potential such that all parameters are real, and the possibility of CP-conserving vacua exists, as well as vacua with spontaneous CP violation – unlike the most general potential of Eq. (2.1), where CP breaking is explicit.

In the bilinear formalism, both Higgs-family and GCP field transformations are represented as rotations in the 3-dimensional space defined by the vector $\vec{r}$, namely

$$\vec{r} \;\rightarrow\; \vec{r}\,' = S\,\vec{r}, \tag{2.16}$$

where $S \in O(3)$ defines a rotation of $\vec{r}$. When such rotations are proper (i.e., $\det(S) = 1$) we have a Higgs-family symmetry. Improper rotations ($\det(S) = -1$) yield GCP symmetries. Both types of symmetries/rotations leave the value of r_0 invariant.[2] The two examples of symmetries described above correspond to S matrices given by

$$S_{Z_2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_{CP1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.17}$$

[2] Indeed, there is a well-defined procedure to obtain the matrix S from the U and X matrices defined in Eqs. (2.12) and (2.13), see [2,7,8,15,18] for details.

Given the freedom to change basis that the 2HDM scalar potential possesses, the same symmetry may look differently in different bases, but its physical implications remain the same. For instance, on a different basis, the Z_2 transformation actually looks like a permutation transformation S_2, where the field transformation corresponds to an exchange between the doublet fields, $\Phi_1 \leftrightarrow \Phi_2$. The resulting potential looks different from the one mentioned above (now we would have $m^2_{11} = m^2_{22}$, $\lambda_1 = \lambda_2$ and $\lambda_6 = \lambda_7$), but it is simply a basis change from the basis wherein the Z_2 field transformation is given by Eq. (2.14). Indeed, the matrix (2.16) for the S_2 transformation is simply given by $S_{S_2} = \mathrm{diag}(1,\,-1,\,-1)$, which is clearly obtained from S_{Z_2} by a permutation of axis. Such permutations correspond, in the bilinear formalism, to basis changes. In fact, it may be shown [7,8] that the Z_2 transformation corresponds, in an arbitrary basis, to a parity transformation (i.e. a sign flip) on two of the three axes of the $\vec{r}$ vector. Likewise, the CP1 transformation will always be given by a parity transformation on a single of the three axes of this space, and there is no physical distinction between a parity transformation on the first, second or third axis (these would correspond to transformations of $\vec{r}$ such that $r_1 \rightarrow -r_1$, or $r_2 \rightarrow -r_2$ or $r_3 \rightarrow -r_3$, respectively). This is why, in the bilinear formalism, the Z_2 and CP1 transformations yield the symmetry groups of the potential $Z_2 \times Z_2$ and Z_2, respectively.

With arbitrary unitary 2×2 matrices U and X for Higgs-family and GCP field transformations, it would appear that the number of these symmetries one might impose on the 2HDM potential would be difficult to establish, but using the bilinear formalism it is simple to see that the maximum number of different such symmetries is *six* [7,8]. In fact, since in the bilinear formalism symmetry transformations translate as $O(3)$ rotations imposed on the $\vec{r}$ vector, and any rotation in 3-dimensional space can be decomposed on parity transformations about the axes, or simple proper rotations about one or more axes, the total number of different possibilities is:

- A parity transformation about a single axis. This corresponds to CP1, and the bilinear symmetry group is Z_2.
- A parity transformation about two axes. This is the Z_2 transformation, and the bilinear symmetry group is $Z_2 \times Z_2$.
- A parity transformation about the three axes. This is the CP2 symmetry, and in terms of doublet transformations, it corresponds to $\Phi_1 \rightarrow \Phi^*_2$, $\Phi_2 \rightarrow -\Phi^*_1$, but in the bilinear formalism the corresponding transformation matrix is quite simple:

$$S_{CP2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.18}$$

The potential has a bilinear symmetry group $Z_2 \times Z_2 \times Z_2$.[3]

- A rotation about one of the axes. This corresponds to a $U(1)$ Peccei-Quinn symmetry [6]. It is obtained requiring invariance under the doublet transformation, $\Phi_1 \rightarrow \Phi_1$, $\Phi_2 \rightarrow e^{i\alpha}\Phi_2$ (with α an arbitrary real number), which corresponds to an S matrix in bilinear space given by

$$S_{U(1)} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.19}$$

and we recognise a rotation around the third axis, in the plane defined by the first two. Again, this field/bilinear transformation is expressed in a specific basis, but the potential one would obtain would be physically equivalent if one were to consider a rotation around any of the other two axes. The symmetry group in the bilinear formalism is $O(2)$.

- A rotation about one of the axes along with a parity transformation on the same axis. This is another GCP symmetry, dubbed CP3, and is obtained via the doublet transformation

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \Phi_1^* \\ \Phi_2^* \end{pmatrix}, \tag{2.20}$$

where, without loss of generality, $0 < \alpha < \pi/2$. This corresponds to an improper rotation around the direction of the second axis of $\vec{r}$,

$$S_{CP3} = \begin{pmatrix} \cos 2\alpha & 0 & \sin 2\alpha \\ 0 & -1 & 0 \\ -\sin 2\alpha & 0 & \cos 2\alpha \end{pmatrix}, \tag{2.21}$$

corresponding to a $Z_2 \times O(2)$ symmetry group in the bilinear formalism.

- A generic rotation in the 3-dimensional space of the vector $\vec{r}$, corresponding to the most general matrix $U \in U(2)$ in Eq. (2.12). This is commonly referred to as the $SO(3)$-symmetric potential, and the rotation matrix in the bilinear formalism is the most generic $SO(3)$ matrix possible. The bilinear formalism symmetry is therefore $SO(3)$.

These are the six symmetries of the $SU(2) \times U(1)$ invariant[4] 2HDM scalar potential that emerge from the transformations that generate Higgs-family symmetries or generalized CP transformations. In Table 1 we summarise the impact each symmetry has on the parameters of the scalar potential. This table considers that each symmetry was imposed in the basis for which the field transformations are as shown above.[5]

Having reviewed the way the 2HDM global symmetries are obtained, we will argue, in Sect. 3 that there are indeed other symmetries not considered in the classification shown above.

2.4 Renormalization group equations

The one-loop renormalization group (RG) equations yield the model's β-functions. They are given, for the most general 2HDM of Eq. (2.1), by [2,24,25]

$$\begin{aligned}
\beta_{m_{11}^2} &= 3\lambda_1 m_{11}^2 + (2\lambda_3 + \lambda_4)\, m_{22}^2 - 3\left(\lambda_6^* m_{12}^2 + \text{h.c.}\right) \\
&\quad -\frac{1}{4}\left(9g^2 + 3g'^2\right) m_{11}^2 + \beta^F_{m_{11}^2}, \\
\beta_{m_{22}^2} &= (2\lambda_3 + \lambda_4)\, m_{11}^2 + 3\lambda_2\, m_{22}^2 - 3\left(\lambda_7^* m_{12}^2 + \text{h.c.}\right) \\
&\quad -\frac{1}{4}\left(9g^2 + 3g'^2\right) m_{22}^2 + \beta^F_{m_{22}^2}, \\
\beta_{m_{12}^2} &= -3\left(\lambda_6\, m_{11}^2 + \lambda_7\, m_{22}^2\right) + (\lambda_3 + 2\lambda_4)\, m_{12}^2 \\
&\quad +3\lambda_5\, m_{12}^{2\,*} - \frac{1}{4}\left(9g^2 + 3g'^2\right) m_{12}^2 + \beta^F_{m_{12}^2},
\end{aligned} \tag{2.22}$$

for the quadratic couplings, and for the quartic ones,

$$\begin{aligned}
\beta_{\lambda_1} &= 6\lambda_1^2 + 2\lambda_3^2 + 2\lambda_3\lambda_4 + \lambda_4^2 + |\lambda_5|^2 + 12\,|\lambda_6|^2 \\
&\quad + \frac{3}{8}(3g^4 + g'^4 + 2g^2 g'^2) - \frac{3}{2}\lambda_1(3g^2 + g'^2) + \beta^F_{\lambda_1},
\end{aligned} \tag{2.23a}$$

$$\begin{aligned}
\beta_{\lambda_2} &= 6\lambda_2^2 + 2\lambda_3^2 + 2\lambda_3\lambda_4 + \lambda_4^2 + |\lambda_5|^2 + 12\,|\lambda_7|^2 \\
&\quad + \frac{3}{8}(3g^4 + g'^4 + 2g^2 g'^2) - \frac{3}{2}\lambda_2(3g^2 + g'^2) + \beta^F_{\lambda_2},
\end{aligned} \tag{2.23b}$$

$$\begin{aligned}
\beta_{\lambda_3} &= (\lambda_1 + \lambda_2)(3\lambda_3 + \lambda_4) + 2\lambda_3^2 + \lambda_4^2 + |\lambda_5|^2 \\
&\quad + 2\left(|\lambda_6|^2 + |\lambda_7|^2\right) + 8\,\text{Re}\left(\lambda_6\lambda_7^*\right) \\
&\quad + \frac{3}{8}(3g^4 + g'^4 - 2g^2 g'^2) - \frac{3}{2}\lambda_3(3g^2 + g'^2) + \beta^F_{\lambda_3},
\end{aligned} \tag{2.23c}$$

$$\begin{aligned}
\beta_{\lambda_4} &= (\lambda_1 + \lambda_2)\lambda_4 + 4\lambda_3\lambda_4 + 2\lambda_4^2 + 4\,|\lambda_5|^2 \\
&\quad + 5\left(|\lambda_6|^2 + |\lambda_7|^2\right) + 2\,\text{Re}\left(\lambda_6\lambda_7^*\right) \\
&\quad + \frac{3}{2}g^2 g'^2 - \frac{3}{2}\lambda_4(3g^2 + g'^2) + \beta^F_{\lambda_4},
\end{aligned} \tag{2.23d}$$

[3] Note that although CP2 corresponds to a single transformation of the doublets/bilinears, it yields a symmetry group $Z_2 \times Z_2 \times Z_2$, which has three generators. This is a particular feature of the 2HDM and is not reproduced beyond two doublets.

[4] If one disregards hypercharge, the number of symmetries obtained is larger, including for instance the custodial symmetry group [22,23].

[5] The counting of parameters may seem odd for the CP2 case in the chosen basis. In a simpler basis, proposed in [3], the conditions on the model's parameters make λ_5 real and $\lambda_6 = \lambda_7 = 0$, yielding 5 independent parameters. Likewise, for the Z_2 symmetry, notice that the λ_5 coupling can always be made real through a basis redefinition, which eliminates one of the parameters.

Table 1 Relations between 2HDM scalar potential parameters for each of the six symmetries discussed, and the number N of independent real parameters for each symmetry-constrained scalar potential

Symmetry	m_{11}^2	m_{22}^2	m_{12}^2	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	N
CP1			Real					Real	Real	Real	9
Z_2			0						0	0	7
U(1)			0					0	0	0	6
CP2		m_{11}^2	0		λ_1					$-\lambda_6$	5
CP3		m_{11}^2	0		λ_1			$\lambda_1 - \lambda_3 - \lambda_4$	0	0	4
$SO(3)$		m_{11}^2	0		λ_1		$\lambda_1 - \lambda_3$	0	0	0	3

$$\beta_{\lambda_5} = (\lambda_1 + \lambda_2 + 4\lambda_3 + 6\lambda_4)\,\lambda_5 + 5\left(\lambda_6^2 + \lambda_7^2\right) + 2\lambda_6\lambda_7 - \frac{3}{2}\lambda_5(3g^2 + g'^2) + \beta^F_{\lambda_5}, \tag{2.23e}$$

$$\beta_{\lambda_6} = (6\lambda_1 + 3\lambda_3 + 4\lambda_4)\,\lambda_6 + (3\lambda_3 + 2\lambda_4)\,\lambda_7 + 5\lambda_5\lambda_6^* + \lambda_5\lambda_7^* - \frac{3}{2}\lambda_6(3g^2 + g'^2) + \beta^F_{\lambda_6}, \tag{2.23f}$$

$$\beta_{\lambda_7} = (6\lambda_2 + 3\lambda_3 + 4\lambda_4)\,\lambda_7 + (3\lambda_3 + 2\lambda_4)\,\lambda_6 + 5\lambda_5\lambda_7^* + \lambda_5\lambda_6^* - \frac{3}{2}\lambda_7(3g^2 + g'^2) + \beta^F_{\lambda_7}, \tag{2.23g}$$

where the β^F_x terms contain all contributions coming from fermions, which we will disregard for the moment,[6] and return to in Sect. 5. For simplicity, we have absorbed factors of $16\pi^2$ within the definition of the β-functions. g and g', obviously, represent the SU(2) and U(1) gauge couplings. The results for the 2HDM two-loop-β functions for the quartic couplings may be found, for instance, in the package SARAH [26–30]. The 2HDM three-loop β-functions have been obtained by Bednyakov [25]. The above β-functions allow us to verify that the relations obtained in the previous section among parameters are RG-invariant to one-loop order. For instance, we observe that if all of the quartic couplings are real, as consequence of a CP1 symmetry, no imaginary components for the λ_i are generated at one-loop. Likewise, we see that if one imposes a Z_2 symmetry so that $\lambda_6 = \lambda_7 = 0$ one immediately obtains $\beta_{\lambda_6} = \beta_{\lambda_7} = 0$, confirming that the symmetry-obtained condition on the λ's is preserved under radiative corrections at the one-loop order. Indeed, we may expect that condition to hold to all orders of perturbation theory, precisely because it is obtained via a symmetry. Another interesting perspective is obtained looking at the λ_5 β-function for the Z_2 model,

$$\beta_{\lambda_5} = \left[\lambda_1 + \lambda_2 + 4\lambda_3 + 6\lambda_4 - \frac{3}{2}\left(3g^2 + g'^2\right)\right]\lambda_5, \tag{2.24}$$

wherein one identifies a *fixed point* of this RG equation – if at any scale one should have $\lambda_5 = 0$, that coupling will remain equal to zero for all renormalization scales. Such fixed points of RG equations are usually fingerprints of hidden symmetries, and indeed that is the case here: if $\lambda_6 = \lambda_7 = 0$, the extra constraint $\lambda_5 = 0$ takes us from a Z_2-symmetric model to a $U(1)$-symmetric one, as can be seen from Table 1.

At this point, and as it will be crucial for the discussion in the next section, let us observe that the set of conditions

$$\left\{m_{11}^2 + m_{22}^2 = 0, \quad \lambda_1 = \lambda_2, \quad \lambda_6 = -\lambda_7\right\} \tag{2.25}$$

constitutes a fixed point of the one-loop RG equations. In fact, by manipulating the above β-functions, we obtain

$$\beta_{m_{11}^2 + m_{22}^2} = 3(\lambda_1 m_{11}^2 + \lambda_2 m_{22}^2) + (2\lambda_3 + \lambda_4)\,(m_{11}^2 + m_{22}^2) - 3\left[(\lambda_6^* + \lambda_7^*)m_{12}^2 + \text{h.c.}\right] - \frac{1}{4}\,(9g^2 + 3g'^2)(m_{11}^2 + m_{22}^2), \tag{2.26}$$

$$\beta_{\lambda_1 - \lambda_2} = 6\left(\lambda_1^2 - \lambda_2^2\right) + 12\left(|\lambda_6|^2 - |\lambda_7|^2\right) - \frac{3}{2}(\lambda_1 - \lambda_2)\,(3g^2 + g'^2), \tag{2.27}$$

$$\beta_{\lambda_6 + \lambda_7} = 6\,(\lambda_1\lambda_6 + \lambda_2\lambda_7) + (3\lambda_3 + 2\lambda_4)\,(\lambda_6 + \lambda_7) + 6\lambda_5\left(\lambda_6^* + \lambda_7^*\right) - \frac{3}{2}(\lambda_6 + \lambda_7)\,(3g^2 + g'^2), \tag{2.28}$$

and we see that the conditions listed in Eq. (2.25) do constitute a fixed point of these RG equations. Of course, it must be said that just because the one-loop β-functions have a fixed point that is not guaranteed to indicate a symmetry – it may be, unlike the $U(1)$ example discussed above, simply a one-loop accident that such a fixed point occurs. As we will argue in the next section, though, that is not the case, and the conditions of Eq. (2.25) are indeed invariant for all orders of perturbation theory.

We also take the opportunity to point out that the parameter conditions of Eq. (2.25) are *basis invariant*. This can be shown explicitly by using the general basis transformations presented in Eqs. (2.8a)–(2.8j).

Finally, we remark that the two relations between quartic couplings in Eq. (2.25) may look familiar: they are exactly

[6] Or we can disregard them altogether and think of the symmetries existing in a theory without fermions.

the ones we obtain from the application of the CP2 symmetry (check Table 1). Notice, however, that the conditions on the quadratic parameters arising from CP2 are *not* the same as those in Eq. (2.25). We will return to this subject shortly.

3 New 2HDM symmetries

In this section we will argue that new symmetries of the 2HDM $SU(2) \times U(1)$ scalar potential of Eq. (2.1) exist, other than those discussed in Sect. 2.3. We will arrive at this conclusion by identifying all-order fixed points in the 2HDM β-functions, and to reach that argument we will use a curious interplay between basis invariance, dimensional analysis and RG equations.

3.1 All-orders fixed points in 2HDM RG equations

As explained in Sect. 2.2, basis transformations are extremely simple to represent in the bilinear formalism. A generic basis transformation corresponds, in bilinear space, to a generic $O(3)$ rotation matrix R, and as such $\vec{r}$, $\vec{M}$ and $\vec{\Lambda}$ transform as 3-vectors under these rotations; the 3×3 matrix Λ is also transformed as under a rotation in this space; and the quantities r_0, M_0 and Λ_{00} are basis invariants. It is then possible to write the most generic set of basis invariant quantities one can form with the quartic parameters of the potential [7,25]. These are

$$I_{1,1} = \Lambda_{00}, \quad I_{1,2} = \mathrm{Tr}\,\Lambda$$
$$I_{2,1} = \vec{\Lambda} \cdot \vec{\Lambda}, \quad I_{2,2} = \mathrm{Tr}\,\Lambda^2$$
$$I_{3,1} = \vec{\Lambda} \cdot \Lambda\vec{\Lambda}, \quad I_{3,2} = \mathrm{Tr}\,\Lambda^3$$
$$I_{4,1} = \vec{\Lambda} \cdot \Lambda^2\vec{\Lambda}. \qquad (3.1)$$

One might think that higher powers of the Λ matrix could be used to build further invariants, but that is not the case. In fact, this matrix satisfies [25]

$$\Lambda^3 = (\mathrm{Tr}\,\Lambda)\Lambda^2 - \frac{1}{2}\left[(\mathrm{Tr}\,\Lambda)^2 - \mathrm{Tr}\,\Lambda^2\right]\Lambda + \frac{1}{6}\left[(\mathrm{Tr}\,\Lambda)^3 - 3\mathrm{Tr}\,\Lambda\,\mathrm{Tr}\,\Lambda^2 + 2\mathrm{Tr}\,\Lambda^3\right] \mathbb{1}_{3\times 3}. \qquad (3.2)$$

This relation, obtained via the Cayley–Hamilton theorem, shows that powers of Λ higher than 2 can always be expressed as a sum of powers of up to 2 of that matrix.[7]

[7] This is also the reason why we do not need to consider the basis-invariant determinant of Λ in this discussion, since the determinant of a 3×3 matrix can be expressed as a linear combination of the traces of its powers up to 3.

As explained in [25], then, the β function of the vector $\vec{\Lambda}$ is given, to all orders of perturbation theory, by

$$\beta_{\vec{\Lambda}} = a_0\,\vec{\Lambda} + a_1\,\Lambda\,\vec{\Lambda} + a_2\,\Lambda^2\,\vec{\Lambda} \qquad (3.3)$$

where the a_i are polynomial expressions involving the invariants of Eq. (3.1). If one computes this β-function at an arbitrary number of loops in perturbation theory, basis invariance will always require that it is given by the structure shown above. Indeed, Eq. (3.3) expresses a very elegant interplay between basis invariance and RG equations: since $\vec{\Lambda}$ transforms as a vector under basis transformations, its β-function must transform in the same manner; therefore, the right-hand side of (3.3) must be composed of terms proportional to vector-like combinations of couplings, and the only three that can be used are $\vec{\Lambda}$, $\Lambda\,\vec{\Lambda}$ and $\Lambda^2\,\vec{\Lambda}$ – higher powers of Λ, as explained above, are superfluous. There is another vector for basis transformation involving couplings of the potential – $\vec{M}$ – but due to its dimensions of mass, it cannot enter in (3.3). This argument can easily be extended to accommodate the contributions from the gauge couplings – as the gauge sector is left unchanged under basis transformations, terms involving the couplings g and g' will simply contribute to the coefficients a_i in (3.3).

With basis transformation properties dictating that the structure of $\beta_{\vec{\Lambda}}$ is, to all orders, a series of terms all linear in $\vec{\Lambda}$, we reach a straightforward conclusion:

- $\vec{\Lambda} = \vec{0}$ *is a fixed point of the RG equation for this quantity, to all orders of perturbation theory.*

Now, $\vec{\Lambda} = \vec{0}$ implies, in terms of the notation of Eq. (2.1), that $\lambda_1 = \lambda_2$ and $\lambda_6 = -\lambda_7$, which are the conditions on quartic couplings we discussed in Eq. (2.25). They are also, as we already mentioned, the conditions one obtains for the quartic couplings from the CP2 symmetry. So this β-function argument seems to have led us to re-discover the CP2 symmetry, but as we will shortly see that is not necessarily so.

Continuing to follow the reasoning of [25], the β-function for the quadratic parameter singlet $M_0 = m_{11}^2 + m_{22}^2$ defined in Eq. (2.4) must obey two constraints: it must have dimensions of (mass)^2 and it must be a singlet under basis transformations. Given the property of the Λ matrix shown in Eq. (3.2), it is easy to conclude that β_{M_0} is a linear combination, via basis invariant dimensionless coefficients b_i, of four different quantities,

$$\beta_{M_0} = b_0\,M_0 + b_1\,\vec{\Lambda} \cdot \vec{M} + b_2\,\vec{\Lambda} \cdot \left(\Lambda\vec{M}\right) + b_3\,\vec{\Lambda} \cdot \left(\Lambda^2\vec{M}\right). \qquad (3.4)$$

It is easy to understand the structure of this equation – since all terms must have the same mass dimension they are either built with M_0 or the vector $\vec{M}$; and any term involving $\vec{M}$

must involve an internal product with a dimensionless vector to form a basis transformation singlet, and the only such vector available is $\vec{\Lambda}$. And as before, this structure is easily generalizable to include gauge couplings – since there are no other terms in the 2HDM lagrangian with the appropriate dimensions, all gauge contributions will simply be contained in the b_i coefficients of Eq. (3.4). The structure of this equation also allows us to reach a simple conclusion:

- *If* $\vec{\Lambda} = \vec{0}$, *then* $M_0 = 0$ *is a fixed point of the RG equation for this quantity, to all orders of perturbation theory.*

Following the same line of reasoning, the β-function for the vector $\vec{M}$ of Eq. (2.4) should be given by a linear combination of terms with dimension (mass)^2 which behave as vectors under basis transformations. This leads us to

$$\beta_{\vec{M}} = c_0\, \vec{M} + c_1\, \Lambda\, \vec{M} + c_2\, \Lambda^2\, \vec{M} + c_3\, I_M\, \vec{\Lambda} + c_4\, I_M\, \Lambda\, \vec{\Lambda} + c_5\, I_M\, \Lambda^2\, \vec{\Lambda} \quad (3.5)$$

where I_M stands for some linear combination of the four basis-invariant quantities (with the same dimension as $\vec{M}$) used in Eq. (3.4). And once again, we see that this RG equation possesses a fixed point:

- *If* $\vec{\Lambda} = \vec{0}$, *then* $\vec{M} = \vec{0}$ *is a fixed point of the RG equation for this quantity, to all orders of perturbation theory.*

Notice how the existence of this fixed point is completely independent of the previous one. We have therefore identified two all-orders fixed points of the 2HDM RG equations:

- $\{\vec{M} = \vec{0},\ \vec{\Lambda} = \vec{0}\}$. *This is equivalent, in the notation of* (2.1), *to*

  $$m_{11}^2 = m_{22}^2, \quad m_{12}^2 = 0, \quad \lambda_1 = \lambda_2, \quad \lambda_6 = -\lambda_7. \quad (3.6)$$

  These are exactly the CP2 symmetry conditions.
- $\{M_0 = 0,\ \vec{\Lambda} = \vec{0}\}$. *This is equivalent, in the notation of* (2.1), *to*

  $$m_{11}^2 = -m_{22}^2, \quad \lambda_1 = \lambda_2, \quad \lambda_6 = -\lambda_7. \quad (3.7)$$

  These are the conditions mentioned before in Eq. (2.25) and they coincide with the CP2 symmetry conditions for the quartic couplings, but have different conditions for the quadratic ones. As we have already discussed these conditions are basis invariant, so they are *not* a basis change of the previous ones.

We have already shown explicitly that the conditions $\{M_0 = 0,\ \vec{\Lambda} = \vec{0}\}$ (i.e. Eq. (2.25)) are RG-invariant at the one-loop level. The reader is encouraged to verify, as we have done, that this statement holds at least to two and three-loop level, using the explicit results for the β-functions of [25,26].

It may be tempting to think of the above second set of conditions on the parameters of the potential as a special soft breaking version of the CP2 model. In fact, it is not unheard of that some soft breaking conditions are RG invariant. We can imagine one such scenario for the CP2 model – according to Table 1, the exact CP2 symmetry implies $m_{11}^2 = m_{22}^2$ and $m_{12}^2 = 0$. If one now considers a softly broken potential with $m_{12}^2 \neq 0$, the condition $m_{11}^2 = m_{22}^2$ will be RG-preserved to all orders, since this potential has a residual S_2 permutation symmetry ($\Phi_1 \leftrightarrow \Phi_2$). If instead one were to consider a softly broken potential with $m_{11}^2 \neq m_{22}^2$ one would still have $m_{12}^2 = 0$ at all orders of perturbation theory, since this model has a residual Z_2 symmetry.

However, notice that if the set of constraints $\{M_0 = 0,\ \vec{\Lambda} = \vec{0}\}$ is satisfied, that imposes conditions on the quadratic part of the potential which are (a) invariant to all orders of perturbation theory and (b) different from any conditions any of the six symmetries listed in Table 1 manages to impose on those parameters. In fact, the most that Higgs-family or GCP symmetries manage to do about the quadratic parameters is to impose the equality of m_{11}^2 and m_{22}^2, m_{12}^2 being real or its vanishing – *never* such a distinct relation as $m_{11}^2 = -m_{22}^2$. Indeed, this all-order constraint imposed on the quadratic parameters cannot be obtained via the two types of symmetries we have been discussing – how then can we obtain it? In the following section we will provide a simple interpretation, in the bilinear formalism, of how $\{M_0 = 0,\ \vec{\Lambda} = \vec{0}\}$ may arise, and argue it constitutes a new type of 2HDM symmetry.

3.2 The r_0 symmetry – bilinear interpretation

Let us begin by remarking that another useful way of writing the scalar potential of Eq. (2.1) is by making obvious the dependence on the basis invariant and vector-like objects. This is very easily expressed in terms of the bilinear formalism and the quantities defined in Eqs. (2.4)–(2.6), so that

$$V = M_0\, r_0 + \Lambda_{00}\, r_0^2 - \vec{M} . \vec{r} - 2 \left(\vec{\Lambda} . \vec{r}\right) r_0 + \vec{r}\, . \, (\Lambda\, \vec{r})\, . \quad (3.8)$$

As defined in Sect. 2.3, the CP2 symmetry corresponds, in the bilinear formalism, to a parity transformation about the three axes of the vector $\vec{r}$, such that $\vec{r} \to -\vec{r}$. Applied to the potential written in the bilinear notation above, it is immediate to see what the result of the application of CP2 is: the potential can only remain invariant under the symmetry if $\vec{\Lambda} = \vec{0}$ and $\vec{M} = \vec{0}$.

The bilinear writing of the potential makes it also clear that there is seemingly *another* way to obtain $\vec{\Lambda} = \vec{0}$. To wit, consider what happens to the potential if one changes the sign of r_0:

$$r_0 \to -r_0 \implies \{M_0 = 0,\ \vec{\Lambda} = \vec{0}\}. \tag{3.9}$$

These are exactly the conditions we obtained from the second all-orders fixed point identified above, that lead to the parameter relations shown in Eq. (2.25). Let us call this the *r_0 symmetry*.

The seminal work of [8,12] did not consider any transformations of the type $r_0 \to -r_0$, for two very good reasons: first, the way r_0 is defined (check Eq. (2.2)), this quantity is always positive; second, r_0 is left invariant under any Higgs-family or GCP symmetry. The first of these objections can be remedied in the sense that Eq. (2.2) can be trivially changed, so that r_0 is defined as having both signs:

$$r_0 = \pm \frac{1}{2}\left(\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2\right). \tag{3.10}$$

Having expanded the range of variation of r_0, the conclusions derived in [8,12] for the positivity conditions of the potential, or the number and types of minima possible, would remain valid under the assumption that one would study only the "forward light cone" of the Minkowski-like space of the r bilinears.

As for the second consideration, it is part of the reason why we argue that the conditions of Eq. (2.25) constitute a new type of 2HDM symmetry – we have shown that they are preserved under renormalization to all orders of perturbation theory, which is the hallmark of the presence of a symmetry. We have shown that they can be obtained, at least formally, via a parity transformation on the "time" axis of the r_μ bilinear vector. There is neither a Higgs-family nor a GCP symmetry that can yield $r_0 \to -r_0$, nor can such symmetries yield a parameter condition like $M_0 = 0 \iff m_{11}^2 = -m_{22}^2$. Nonetheless, that condition was found to be both basis invariant *and* RG invariant to all orders. The six symmetries described in Sect. 2.3 can be described via transformations on the doublets, which have a counterpart as transformations on the bilinears – for the r_0 symmetry, we can obtain RG-invariant conditions on the potential via a bilinear transformation, which seemingly has no equivalent on transformations expressed in terms of the doublets themselves. In this regard, it is almost as if the bilinear formalism is more "fundamental" in what concerns the scalar sector of the 2HDM, as aspects of the model can be understood in terms of the r_μ but not in terms of the Φ_i.

We must worry about the kinetic terms too, and in particular the gauge interactions of the doublets. Here, again, the limitations of the bilinear formalism make themselves manifest. In Eq. (2.11) the 2HDM kinetic terms were written using the same Minkowski formalism used for the bilinears and the potential, but not considering the gauge interactions.

The doublets' covariant derivatives are defined as

$$D_\mu = \partial_\mu + i\frac{g'}{2} Y B_\mu + i\frac{g}{2}\sigma^a W_\mu^a, \tag{3.11}$$

where Y is the hypercharge of the fields the derivative operates upon, an implicit sum on $a = 1, 2, 3$ is assumed and W_μ^a and B_μ are the $SU(2)_L$ and $U(1)_Y$ gauge fields respectively. The kinetic terms are therefore given by (using the fact that both scalar doublets have hypercharge $Y = 1$)

$$\begin{aligned} T &= \left(D_\mu \Phi_i\right)^\dagger D^\mu \Phi_i \\ &= \partial_\mu \Phi_i^\dagger \partial^\mu \Phi_i + \frac{ig'}{2}\left[\left(\partial_\mu \Phi_i^\dagger\right) \Phi_i - \Phi_i^\dagger \left(\partial_\mu \Phi_i\right)\right] B^\mu \\ &\quad + \frac{ig}{2}\left[\left(\partial_\mu \Phi_i^\dagger\right) \sigma^a \Phi_i - \Phi_i^\dagger \sigma^a \left(\partial^\mu \Phi_i\right)\right] W^{a\mu} \\ &\quad + \frac{1}{2} g g' \left(\Phi_1^\dagger \sigma^a \Phi_1 + \Phi_2^\dagger \sigma^a \Phi_2\right) W_\mu^a B^\mu \\ &\quad + \frac{1}{4}\left(g'^2 B_\mu B^\mu + g^2 W_\mu^a W^{a\mu}\right)\left(|\Phi_1|^2 + |\Phi_2|^2\right), \end{aligned} \tag{3.12}$$

where again an implicit sum on the indices $i = 1, 2$ and $a = 1, 2, 3$ is assumed. Hence, we can rewrite this equation as

$$\begin{aligned} T = K_1^\mu \Big\{ &\left(\partial_\alpha \Phi_i^\dagger\right) (\sigma_\mu)_{ij} \left(\partial^\alpha \Phi_j\right) \\ &+ \frac{ig'}{2}\left[\left(\partial_\alpha \Phi_i^\dagger\right) (\sigma_\mu)_{ij}\, \Phi_j - \Phi_i^\dagger (\sigma_\mu)_{ij} \left(\partial_\alpha \Phi_j\right)\right] B^\alpha \\ &+ \frac{ig}{2}\left[\left(\partial_\alpha \Phi_i^\dagger\right) (\sigma_\mu)_{ij}\, \sigma^a \Phi_j - \Phi_i^\dagger (\sigma_\mu)_{ij}\, \sigma^a \left(\partial^\alpha \Phi_j\right)\right] W^{a\alpha} \\ &+ \frac{g g'}{2} \Phi_i^\dagger (\sigma_\mu)_{ij}\, \sigma^a\, \Phi_j\, W_\alpha^a B^\alpha \Big\} \\ &+ \frac{1}{2} K_2^\mu \left(g'^2 B_\alpha B^\alpha + g^2 W_\alpha^a W^{a\alpha}\right) r_\mu, \end{aligned} \tag{3.13}$$

with $K_1^\mu = K_2^\mu = (1, 0, 0, 0)$, and care must be taken to not confuse the 4-vector σ_μ (defined in Eq. (2.11)) and the three Pauli matrices σ^a. The last term can be made to remain invariant under the transformation $r_0 \to -r_0$ if one assumes $K_2 \to -K_2$, as well. However, that does not explain how the remaining terms, involving derivatives and gauge fields, could remain invariant. This once more emphasizes that we do not know what the expression of the r_0 symmetry in terms of doublet fields (and their derivatives) ought to be. However, in Appendix A we show how a peculiar transformation of fields and spacetime coordinates could reproduce the r_0 symmetry, at least formally.

However, though we may be unable to write the kinetic terms in a satisfactory way as a function of bilinears, this does not invalidate the fact that the region of parameter space we identify with the r_0 symmetry is RG invariant to all orders, and we must remember that reasoning included the contributions of gauge interactions as well.

We therefore argue that the conditions of Eq. (2.25), which are basis and RG invariant, are obtained from the imposition on the potential of a new type of symmetry, which we have dubbed the r_0 symmetry. We have provided a bilinear transformation which, applied to the potential, yields these conditions on the parameters of the potential. Though the conditions on the quartic couplings can be obtained via a GCP symmetry (CP2), no unitary transformations on the doublets or their complex conjugates can reproduce the all-orders RG-invariant conditions on the quadratic parameters of Eq. (2.25). Of course, there are plenty of examples of symmetries in particle physics models which do not involve this type of transformations, such as supersymmetry, for instance.

3.3 List of new symmetries

The r_0 symmetry yields CP2-like quartic couplings and $m_{11}^2 = -m_{22}^2$. When combined with the bilinear transformations which yield the six symmetries listed on Table 1, we can obtain a total of seven new symmetry classes. We will designate the new symmetries with the prefix "0" – so for instance, "0CP1" will refer to the application of the r_0 and CP1 symmetries, as "0Z_2" refers to the application of r_0 and Z_2. We therefore obtain the constraints on the parameters of the potential shown in Table 2.

The last three symmetries listed in Table 2 have the odd property of not having any quadratic parameters – the combination of the r_0 symmetry with others eliminating all of those coefficients. We reached the parameter relation $m_{22}^2 = -m_{11}^2$ through an analysis of all-orders RG invariance, and of course that, due to dimensional analysis, for any potential with all quadratic couplings vanishing they will remain zero at all orders of perturbation theory. Such models, however, are clearly not interesting, since electroweak symmetry breaking is not possible with vanishing quadratic couplings.[8] However, soft breaking versions of such models, in particular soft breakings which include the condition $m_{22}^2 = -m_{11}^2$, may be of interest, and we will consider several such cases in Sect. 5.

The parameter relations presented in Table 2 are not in the simplest form that the 2HDM potential can have under each of those symmetries, since basis freedom can still be used to eliminate some spurious parameters. In particular, we can use the result of Refs. [3,21], in which it was shown that if $\lambda_1 = \lambda_2$ and $\lambda_7 = -\lambda_6$, then a basis exists for which all λ_i are real and $\lambda_6 = \lambda_7 = 0$, without any loss of generality. Proceeding to this basis, we obtain the most simple form of the potential for each symmetry, and can establish the number of independent parameters for each case. We list the relations between couplings in this new basis, and the number N of free parameters, in Table 3. Some of the symmetries shown in Table 2 already had $\lambda_6 = \lambda_7 = 0$, so for those there is no change.

Again, any soft breaking of the above potentials preserves the renormalizability of the model, in particular the relations between the quartic couplings.

4 Scalar phenomenology of the new symmetric models

We have shown how the condition $m_{22}^2 = -m_{11}^2$, coupled with $\lambda_1 = \lambda_2$ and $\lambda_7 = -\lambda_6$, constitutes an all-orders RG invariant region of parameter space, which can seemingly be obtained in the bilinear formalism via the transformation $r_0 \to -r_0$. We now wish to investigate the consequences that this condition in particular can have on the phenomenology of the 2HDM scalars. To do this we must investigate how electroweak symmetry breaking occurs. For this purpose, we start by writing out our potential in a basis in which the r_0 symmetry is manifest. It is given by

$$\begin{aligned} V =& m_{11}^2\left[\Phi_1^\dagger\Phi_1 - \Phi_2^\dagger\Phi_2\right] - \left[m_{12}^2\Phi_1^\dagger\Phi_2 + \text{h.c.}\right] \\ &+\tfrac{1}{2}\lambda_1\left[(\Phi_1^\dagger\Phi_1)^2 + (\Phi_2^\dagger\Phi_2)^2\right] + \lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) \\ &+\lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1) + \Big\{\tfrac{1}{2}\lambda_5(\Phi_1^\dagger\Phi_2)^2 \\ &+\lambda_6\left[(\Phi_1^\dagger\Phi_1) - (\Phi_2^\dagger\Phi_2)\right]\Phi_1^\dagger\Phi_2 \\ &+\text{h.c.}\Big\}, \end{aligned} \tag{4.1}$$

where all parameters are real, except for m_{12}^2, λ_5 and λ_6 which may be complex. Without loss of generality one can rotate into a simpler basis in which $\lambda_6 = \lambda_7 = 0$ and λ_5 is real to get

$$\begin{aligned} V =& m_{11}^2\left[\Phi_1^\dagger\Phi_1 - \Phi_2^\dagger\Phi_2\right] - \left[m_{12}^2\Phi_1^\dagger\Phi_2 + \text{h.c.}\right] \\ &+\tfrac{1}{2}\lambda_1\left[(\Phi_1^\dagger\Phi_1)^2 + (\Phi_2^\dagger\Phi_2)^2\right] + \lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) \\ &+\lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1) + \frac{\lambda_5}{2}\left[(\Phi_1^\dagger\Phi_2)^2 + (\Phi_2^\dagger\Phi_1)^2\right]. \end{aligned} \tag{4.2}$$

The Higgs doublets can be parameterized as

$$\Phi_j = e^{i\xi_j}\begin{pmatrix} \varphi_j^+ \\ (v_j + \eta_j + i\chi_j)/\sqrt{2} \end{pmatrix}, \quad j = 1, 2. \tag{4.3}$$

Here v_j are real numbers, so that $v_1^2 + v_2^2 = v^2$. The fields η_j and χ_j are real, whereas φ_j^+ are complex fields. Then the most general form of the vacuum will have the form

$$\langle\Phi_j\rangle = \frac{e^{i\xi_j}}{\sqrt{2}}\begin{pmatrix} 0 \\ v_j \end{pmatrix}, \quad j = 1, 2, \tag{4.4}$$

where we may without loss of generality choose $\xi_1 = 0$ and put $\xi_2 \equiv \xi$. We may also assume that both $v_i \geq 0$. Massless

[8] Though it might occur when radiative corrections are taken into account, as in the Coleman–Weinberg mechanism [31].

Table 2 Relations between 2HDM scalar potential parameters for each of the new seven symmetries discussed

Symmetry	m_{11}^2	m_{22}^2	m_{12}^2	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
r_0		$-m_{11}^2$			λ_1					$-\lambda_6$
0CP1		$-m_{11}^2$	Real		λ_1			Real	Real	$-\lambda_6$
$0Z_2$		$-m_{11}^2$	0		λ_1				0	0
0U(1)		$-m_{11}^2$	0		λ_1			0	0	0
0CP2	0	0	0		λ_1					$-\lambda_6$
0CP3	0	0	0		λ_1			$\lambda_1 - \lambda_3 - \lambda_4$	0	0
$0SO(3)$	0	0	0		λ_1		$\lambda_1 - \lambda_3$	0	0	0

Table 3 Relations between 2HDM scalar potential parameters for each of the new symmetries in a special basis, and the number N of independent real parameters for each symmetry-constrained scalar potential

Symmetry	m_{11}^2	m_{22}^2	m_{12}^2	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	N
r_0		$-m_{11}^2$			λ_1			Real	0	0	7
0CP1		$-m_{11}^2$	Real		λ_1			Real	0	0	6
$0Z_2$		$-m_{11}^2$	0		λ_1			Real	0	0	5
0U(1)		$-m_{11}^2$	0		λ_1			0	0	0	4
0CP2	0	0	0		λ_1			Real	0	0	4
0CP3	0	0	0		λ_1			$\lambda_1 - \lambda_3 - \lambda_4$	0	0	3
$0SO(3)$	0	0	0		λ_1		$\lambda_1 - \lambda_3$	0	0	0	2

Goldstone states are extracted by defining orthogonal states

$$\begin{pmatrix} G_0 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} v_1/v & v_2/v \\ -v_2/v & v_1/v \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix},$$
$$\begin{pmatrix} G^\pm \\ H^\pm \end{pmatrix} = \begin{pmatrix} v_1/v & v_2/v \\ -v_2/v & v_1/v \end{pmatrix} \begin{pmatrix} \varphi_1^\pm \\ \varphi_2^\pm \end{pmatrix}. \tag{4.5}$$

Then G_0 and $G^\pm$ become the massless Goldstone fields, and $H^\pm$ are the charged scalars. The model also contains three neutral scalars, which are linear compositions of the η_i,

$$\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = R \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}, \tag{4.6}$$

with the 3×3 orthogonal rotation matrix R satisfying

$$R\mathcal{M}^2 R^{\mathrm{T}} = \mathcal{M}^2_{\text{diag}} = \text{diag}(M_1^2, M_2^2, M_3^2). \tag{4.7}$$

4.1 Relations among physical parameters

The most general 2HDM has 11 independent real parameters. Clearly, it would be desirable to express such parameters in terms of physical quantities, which can be measured experimentally and are, by definition, basis-invariant. Recently [32–34] a set of 11 independent physical parameters was proposed, described by

$$\mathcal{P} \equiv \{M^2_{H^\pm}, M_1^2, M_2^2, M_3^2, e_1, e_2, e_3, q_1, q_2, q_3, q\}. \tag{4.8}$$

In this set, $M_{H^\pm}$ is the mass of the charged scalars, and $M_{1,2,3}$ are the masses of the three neutral scalars. These are, in the Higgs basis, the eigenvalues of the 3×3 mass matrix of the neutral sector, diagonalized by an orthogonal matrix R according to (4.7). As for the e_i, they are obtained from the interactions of the neutral scalars with gauge bosons, which arise from the doublets' kinetic terms:

$$\mathcal{L}_k = (D_\mu \Phi_1)^\dagger (D^\mu \Phi_1) + (D_\mu \Phi_2)^\dagger (D^\mu \Phi_2). \tag{4.9}$$

From these terms, and with the usual definitions for covariant derivatives, we identify trilinear gauge-scalar interaction terms,

$$\begin{aligned} \text{Coefficient}\left(\mathcal{L}_k, Z^\mu \left[H_j \overleftrightarrow{\partial_\mu} H_i\right]\right) &= \frac{g}{2v\cos\theta_{\mathrm{W}}} \epsilon_{ijk} e_k, \\ \text{Coefficient}\left(\mathcal{L}_k, H_i Z^\mu Z^\nu\right) &= \frac{g^2}{4\cos^2\theta_{\mathrm{W}}} e_i\, g_{\mu\nu}, \\ \text{Coefficient}\left(\mathcal{L}_k, H_i W^{+\mu} W^{-\nu}\right) &= \frac{g^2}{2} e_i\, g_{\mu\nu}. \end{aligned} \tag{4.10}$$

All interactions between the H_i and the electroweak gauge bosons involve the quantities e_i – for instance, e_1 is related to the coupling modifier κ_V used by the LHC experimental collaborations by $\kappa_V = e_1/v$. In a general basis,[9] the e_i are given by

$$e_i \equiv v_1 R_{i1} + v_2 R_{i2}, \tag{4.11}$$

[9] In the Higgs basis, the expressions simplify to $e_i = v R_{i1}$.

where R is the diagonalization matrix of the neutral scalars mentioned above (see [32–34] for details). Interestingly, the e_i coefficients obey a "sum rule"

$$e_1^2 + e_2^2 + e_3^3 = v^2. \tag{4.12}$$

The three trilinear $H_i H^+ H^-$ couplings and the quadrilinear $H^+ H^+ H^- H^-$ coupling complete the physical parameter set. These couplings, respectively denoted by q_i and q, are quite complicated in a general basis, but in the Higgs basis they simplify to

$$\begin{aligned} q_i &\equiv \text{Coefficient}(V, H_i H^+ H^-) \\ &= v(R_{i1}\lambda_3 + R_{i2}\text{Re}\,\lambda_7 - R_{i3}\text{Im}\,\lambda_7), \end{aligned} \tag{4.13}$$

$$q \equiv \text{Coefficient}(V, H^+ H^+ H^- H^-) = \frac{1}{2}\lambda_2, \tag{4.14}$$

where again the R_{ij} are elements of the rotation matrix R mentioned above.

The elements of $\mathcal{P}$ give therefore expressions in terms of tree-level masses and couplings, and all physical observables of the scalar sector are expressible in terms of these 11 parameters. When symmetries are imposed on the 2HDM the number of free parameters is reduced and relations among some of them arise. This was studied, for the six familiar symmetries of the 2HDM, in [35]. The analysis was extended to softly broken symmetries in [36].

4.2 The r_0 model

Out of all the possibilities described in Table 3, the r_0 model is the only one for which explicit CP violation occurs. Since m_{12}^2 is complex, it is easy to see that its phase cannot be absorbed by a basis transformation without it rendering parameters in the quartic part of the potential complex. Explicit CP violation can also be established using the four basis invariants whose vanishing heralds explicit CP conservation for a given 2HDM [21]. To be more precise, we will use the equivalent formulation of those four invariants in the bilinear formalism [15], given by

$$\begin{aligned} I_1 &= \left(\vec{M} \times \vec{\Lambda}\right) \cdot \left(\Lambda \vec{M}\right) \\ I_2 &= \left(\vec{M} \times \vec{\Lambda}\right) \cdot \left(\Lambda \vec{\Lambda}\right) \\ I_3 &= \left[\vec{M} \times \left(\Lambda \vec{M}\right)\right] \cdot \left(\Lambda^2 \vec{M}\right) \\ I_4 &= \left[\vec{\Lambda} \times \left(\Lambda \vec{\Lambda}\right)\right] \cdot \left(\Lambda^2 \vec{\Lambda}\right). \end{aligned} \tag{4.15}$$

Since the r_0 symmetry implies $\vec{\Lambda} = \vec{0}$ the invariants $I_{1,2,4}$ are automatically zero. This leaves I_3, a simple calculation shows that

$$I_3 = -16\lambda_5\, m_{11}^2\, \text{Im}(m_{12}^2)\, \text{Re}(m_{12}^2) \left[(\lambda_1 - \lambda_3 - \lambda_4)^2 - \lambda_5^2\right], \tag{4.16}$$

so in general we will have $I_3 \neq 0$ for the r_0 model – and therefore there is explicit CP violation in this model. The CP violation is not hard, but soft since the CP violating phase resides in m_{12}^2.

Working out the stationary-point equations for the general r_0 model, we find that they are solved by

$$\begin{aligned} m_{11}^2 &= \frac{1}{2}\lambda_1\left(v_2^2 - v_1^2\right), \\ \text{Re}\, m_{12}^2 &= \frac{1}{2} v_1 v_2 \cos\xi\, (\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5), \\ \text{Im}\, m_{12}^2 &= -\frac{1}{2} v_1 v_2 \sin\xi\, (\lambda_1 + \lambda_3 + \lambda_4 - \lambda_5). \end{aligned} \tag{4.17}$$

The elements of the neutral sector mass matrix become

$$\begin{aligned} \left(\mathcal{M}^2\right)_{11} &= \frac{1}{2}\left(2\lambda_1 v_1^2 + (\lambda_1 + \lambda_3 + \lambda_4 + \cos 2\xi\, \lambda_5)\, v_2^2\right), \\ \left(\mathcal{M}^2\right)_{22} &= \frac{1}{2}\left((\lambda_1 + \lambda_3 + \lambda_4 + \cos 2\xi\, \lambda_5)\, v_1^2 + 2\lambda_1 v_2^2\right), \\ \left(\mathcal{M}^2\right)_{33} &= \frac{1}{2} v^2\, (\lambda_1 + \lambda_3 + \lambda_4 - \cos 2\xi\, \lambda_5), \\ \left(\mathcal{M}^2\right)_{12} &= -\frac{1}{2} v_1 v_2\, (\lambda_1 - \lambda_3 - \lambda_4 - \cos 2\xi\, \lambda_5), \\ \left(\mathcal{M}^2\right)_{13} &= -\frac{1}{2} v_2 v \sin 2\xi\, \lambda_5, \\ \left(\mathcal{M}^2\right)_{23} &= -\frac{1}{2} v_1 v \sin 2\xi\, \lambda_5. \end{aligned} \tag{4.18}$$

The neutral sector rotation matrix is then given by

$$R = \begin{pmatrix} \frac{v_2 \cos\xi}{v} & \frac{v_1 \cos\xi}{v} & -\sin\xi \\ -\frac{v_1}{v} & \frac{v_2}{v} & 0 \\ \frac{v_2 \sin\xi}{v} & \frac{v_1 \sin\xi}{v} & \cos\xi \end{pmatrix}, \tag{4.19}$$

yielding masses

$$\begin{aligned} M_1^2 &= \frac{1}{2} v^2 (\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5), \quad M_2^2 = \lambda_1 v^2, \\ M_3^2 &= \frac{1}{2} v^2 (\lambda_1 + \lambda_3 + \lambda_4 - \lambda_5), \quad M_{H^\pm}^2 = \frac{1}{2}(\lambda_1 + \lambda_3)\, v^2. \end{aligned} \tag{4.20}$$

The r_0 symmetry conditions, coupled with the minimisation equations, eliminates the dependence on the quadratic couplings. All squared masses therefore become products of quartic couplings with v^2. A *decoupling limit* is not possible in this model – since all quartic couplings are constrained by perturbativity, the values of the scalar masses cannot be too large.

With m_{SM} the SM Higgs mass (125 GeV), and from perturbativity alone, $|\lambda_i| < 4\pi$, it is easy to see from (4.20) that we obtain an upper bound on the masses,

$$\max\{M_i\} = \sqrt{\tfrac{1}{2} m_{\text{SM}}^2 + 6\pi\, v^2} \simeq 1132 \text{ GeV}. \tag{4.21}$$

Unitarity constraints [37] on the 2HDM will however restrict the size of several combinations of quartic couplings, so we can obtain more restrictive bounds on the scalars' masses. A scan over parameters, imposing unitarity and also boundedness-from-below constraints [8,12], shows that indeed it is not possible to obtain scalar masses arbitrarily large, due to a combination of symmetry and unitarity conditions. Assuming that M_2 is the SM-like Higgs boson, we obtain

$$\begin{aligned} M_{H^\pm} &\leq 711 \text{ GeV}, \\ M_3 &\leq 712 \text{ GeV}, \\ M_1 &\leq 711 \text{ GeV}, \end{aligned} \tag{4.22}$$

and $M_1 + M_3 \leq 1400$ GeV.

Working out the three gauge couplings and the four scalar couplings contained in the physical parameter set $\mathcal{P}$ described in Sect. 4.1, we get

$$\begin{aligned} e_1 &= \frac{2v_1 v_2 \cos\xi}{v}, \quad e_2 = \frac{v_2^2 - v_1^2}{v}, \quad e_3 = \frac{2v_1 v_2 \sin\xi}{v}, \\ q_1 &= \frac{v_1 v_2 \cos\xi\,(\lambda_1 + \lambda_3 - \lambda_4 - \lambda_5)}{v}, \\ q_2 &= -\frac{\lambda_3\left(v_1^2 - v_2^2\right)}{v}, \\ q_3 &= \frac{v_1 v_2 \sin\xi\,(\lambda_1 + \lambda_3 - \lambda_4 + \lambda_5)}{v}, \\ q &= \frac{\lambda_1\left(v_1^4 + v_2^4\right) + 2\left(\lambda_3 + \lambda_4\right) v_1^2 v_2^2 + +2 v_1^2 v_2^2 \cos 2\xi\,\lambda_5}{2v^4}. \end{aligned} \tag{4.23}$$

The r_0 conditions are easily translated into constraints among the parameters of $\mathcal{P}$. Using the techniques laid out [35,38], the basis-invariant constraint $m_{11}^2 + m_{22}^2 = 0$ translates into

$$\begin{aligned} M_{H^\pm}^2 = &\frac{1}{2}(e_1 q_1 + e_2 q_2 + e_3 q_3) \\ &+ \frac{1}{2v^2}(e_1^2 M_1^2 + e_2^2 M_2^2 + e_3^2 M_3^2). \end{aligned} \tag{4.24}$$

The combined constraints $\lambda_2 = \lambda_1$ and $\lambda_6 + \lambda_7 = 0$ are also basis-invariant. These conditions are those of a CP2 invariant V_4, and were already translated into constraints among the parameters of $\mathcal{P}$ in [36], dubbed Case SOFT-CP2. Combining the constraints of Case SOFT-CP2 with (4.24), we arrive at

$$\begin{aligned} \textbf{Case } r_0: \quad & v^2(e_1 q_2 - e_2 q_1) + e_1 e_2 (M_2^2 - M_1^2) = 0, \\ & v^2(e_1 q_3 - e_3 q_1) + e_1 e_3 (M_3^2 - M_1^2) = 0, \\ & v^2(e_2 q_3 - e_3 q_2) + e_2 e_3 (M_3^2 - M_2^2) = 0, \\ & q = \frac{1}{2v^4}(e_1^2 M_1^2 + e_2^2 M_2^2 + e_3^2 M_3^2), \\ M_{H^\pm}^2 = & \frac{1}{2}(e_1 q_1 + e_2 q_2 + e_3 q_3) \\ & + \frac{1}{2v^2}(e_1^2 M_1^2 + e_2^2 M_2^2 + e_3^2 M_3^2), \end{aligned}$$

which fully describes the physical consequences of the r_0 symmetry when imposed upon the 2HDM potential. Superficially, this looks like five constraints, but it is in fact only four since the first three are not independent. Thus, the most general potential invariant under r_0 has $11 - 4 = 7$ free parameters. It is now easy to check that the masses and couplings we worked out for this model satisfy the constraints of Case r_0.

4.2.1 Soft breaking of r_0

If we try to softly break r_0 by relaxing the condition $m_{11}^2 + m_{22}^2 = 0$, we just go back to the softly broken CP2-model described by Case SOFT-CP2 in [36], except for the situation where $m_{22}^2 = m_{11}^2$, and the whole potential is CP2 invariant. Such cases were described in [35], and there only one case, namely Case CCD was found to be RG-stable.

4.3 The 0CP1 model

In the 0CP1 model, the r_0 symmetry is imposed on the potential alongside the CP1 symmetry, yielding a potential which, in its symmetry basis, has parameters such as are described in Table 2, but without loss of generality we can go to a simpler basis as indicated in Table 3 to get

$$\begin{aligned} V = &m_{11}^2 \left[\Phi_1^\dagger\Phi_1 - \Phi_2^\dagger\Phi_2\right] - m_{12}^2\left[\Phi_1^\dagger\Phi_2 + \Phi_2^\dagger\Phi_1\right] \\ &+ \tfrac{1}{2}\lambda_1\left[(\Phi_1^\dagger\Phi_1)^2 + (\Phi_2^\dagger\Phi_2)^2\right] \\ &+ \lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1) \\ &+ \frac{\lambda_5}{2}\left[(\Phi_1^\dagger\Phi_2)^2 + (\Phi_2^\dagger\Phi_1)^2\right], \end{aligned} \tag{4.25}$$

where now all parameters are real. There are different ways to solve the resulting stationary point equations. There are solutions with $v_1 = 0$ or $v_2 = 0$. Such solutions imply $m_{12}^2 = 0$, and are situations where the potential is Z_2 invariant. They will therefore not be discussed in this section. We may thus safely assume $v_1 v_2 \neq 0$ in the following. Next, there are solutions requiring $\sin\xi = 0$, thus describing a model which preserves CP, and there are also solutions where $\sin\xi \neq 0$ leaving open the possibility for spontaneous CP violation.

4.4 CP conserving 0CP1

We consider only a model with $\xi = 0$ (letting $\xi = \pi$ yields similar results). Now, the stationary point equations are solved by

$$\begin{aligned} m_{11}^2 &= \frac{1}{2}\lambda_1\left(v_2^2 - v_1^2\right), \\ m_{12}^2 &= \frac{1}{2}v_1 v_2\left(\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5\right). \end{aligned} \tag{4.26}$$

The elements of the neutral-sector mass-squared matrix are

$$\left(\mathcal{M}^2\right)_{11} = \frac{1}{2}\left(2v_1^2\lambda_1 + v_2^2\left(\lambda_1+\lambda_3+\lambda_4+\lambda_5\right)\right),$$
$$\left(\mathcal{M}^2\right)_{22} = \frac{1}{2}\left(v_1^2\left(\lambda_1+\lambda_3+\lambda_4+\lambda_5\right) + 2v_2^2\lambda_1\right),$$
$$\left(\mathcal{M}^2\right)_{33} = \frac{1}{2}v^2\left(\lambda_1+\lambda_3+\lambda_4-\lambda_5\right),$$
$$\left(\mathcal{M}^2\right)_{12} = -\frac{1}{2}v_1v_2\left(\lambda_1-\lambda_3-\lambda_4-\lambda_5\right),$$
$$\left(\mathcal{M}^2\right)_{13} = \left(\mathcal{M}^2\right)_{23} = 0. \tag{4.27}$$

The rotation matrix is given by

$$R = \begin{pmatrix} \frac{v_2}{v} & \frac{v_1}{v} & 0 \\ -\frac{v_1}{v} & \frac{v_2}{v} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.28}$$

yielding masses

$$M_1^2 = \frac{1}{2}v^2\left(\lambda_1+\lambda_3+\lambda_4+\lambda_5\right), \quad M_2^2 = \lambda_1 v^2,$$
$$M_3^2 = \frac{1}{2}v^2\left(\lambda_1+\lambda_3+\lambda_4-\lambda_5\right), \quad M_{H^\pm}^2 = \frac{1}{2}\left(\lambda_1+\lambda_3\right)v^2. \tag{4.29}$$

The neutral-sector mass-squared matrix is broken into two blocks – a 2 × 2 block, indicating the presence of two CP-even states H_1 and H_2 (this latter being the SM Higgs), and an isolated diagonal entry indicating the mass of a pseudoscalar H_3. We find that

$$M_{H^\pm} \le 711 \text{ GeV},$$
$$M_3 \le 708 \text{ GeV},$$
$$M_1 \le 961 \text{ GeV}, \tag{4.30}$$

where we have assumed $M_2 = 125$ GeV. Once again, we see how a decoupling limit is not achievable, as there are upper bounds on the extra scalar masses.

Working out the three gauge couplings and the four scalar couplings contained in the physical parameter set $\mathcal{P}$ described in Sect. 4.1, we get

$$e_1 = \frac{2v_1v_2}{v}, \quad e_2 = \frac{v_2^2 - v_1^2}{v}, \quad e_3 = 0,$$
$$q_1 = \frac{v_1v_2\left(\lambda_1+\lambda_3-\lambda_4-\lambda_5\right)}{v},$$
$$q_2 = \frac{\lambda_3\left(v_2^2-v_1^2\right)}{v}, \quad q_3 = 0,$$
$$q = \frac{\lambda_1\left(v_1^4+v_2^4\right) + 2\left(\lambda_3+\lambda_4+\lambda_5\right)v_1^2v_2^2}{2v^4}. \tag{4.31}$$

We see from these constraints that the model is CP conserving since $e_3 = q_3 = 0$. This corresponds to Case C of CP conservation in [35]. Thus, combining the constraints of Case r_0 with the constraints of Case C, we arrive at

$$\textbf{Case 0CP1-C}: \quad \begin{aligned} & e_k = q_k = 0, \\ & v^2(e_iq_j - e_jq_i) + e_ie_j(M_j^2 - M_i^2) = 0, \\ & q = \frac{1}{2v^4}(e_1^2M_1^2 + e_2^2M_2^2), \\ & M_{H^\pm}^2 = \frac{1}{2}(e_1q_1 + e_2q_2) \\ & \quad + \frac{1}{2v^2}(e_1^2M_1^2 + e_2^2M_2^2) \end{aligned}$$

which fully describes the physical consequences of the CP conserving 0CP1 model. There are five constraints, implying that this model has $11 - 5 = 6$ free parameters. It is now easy to check that the masses and couplings we worked out for this model satisfy the constraints of Case 0CP1-C for $k = 3$.

4.4.1 Spontaneous CP violation in a 0CP1 model

For the regular CP1-conserving potential, we know that, for certain regions of parameter space, spontaneous CP violation (SCPV) can occur, so let us investigate whether the same can happen for the 0CP1 model. Solving the stationary point equations assuming $\sin\xi \neq 0$ yields

$$m_{11}^2 = \frac{1}{2}\lambda_1\left(v_2^2 - v_1^2\right), \tag{4.32}$$
$$m_{12}^2 = \left(\lambda_1+\lambda_3+\lambda_4\right)v_1v_2\cos\xi, \tag{4.33}$$
$$\lambda_5 = \lambda_1+\lambda_3+\lambda_4. \tag{4.34}$$

The last of these equations is a condition among quartic couplings only, not enforced by the model's symmetries, and which therefore would be a tree-level fine-tuning, unstable under radiative corrections. Since we are assuming that $v_1v_2 \neq 0$, as otherwise that would imply a Z_2 invariant vacuum, the only way to avoid RG-instability is to assume $\sin\xi = 0$, as we did in Eq. (4.26). Notice, also, that in the situation encountered, the minimization conditions do not allow for a full determination of the parameters v_1, v_2 and ξ in terms of potential parameters. This is presumably a situation where the tree-level minimisation conditions are not sufficient to determine whether SCPV can occur, and one would need to perform a one-loop analysis to settle the issue. We will meet this issue again, for the $0Z_2$ and 0U(1) models. The only vacua in the model for which we can rely on the tree-level solutions are therefore those which preserve CP, i.e., with $\sin\xi = 0$. We will therefore not investigate this model further in the present work, since a full one-loop analysis is needed to settle the issue of SCPV.

4.4.2 *Soft breaking of CP1*

Let us consider the possibility of keeping the r_0 symmetry intact, but softly breaking CP1. That would imply that we allow for complex m_{12}^2. From Table 3 we see that this simply takes us back to the general r_0 model and yields nothing new.

4.5 The $0Z_2$ model

As seen from Table 3, the $0Z_2$ model is characterized (in the reduced basis) by, on top of the relations between parameters from the 0CP1 model, also having $m_{12}^2 = 0$. The potential then reads

$$V = m_{11}^2\left[\Phi_1^\dagger\Phi_1 - \Phi_2^\dagger\Phi_2\right] + \tfrac{1}{2}\lambda_1\left[(\Phi_1^\dagger\Phi_1)^2 + (\Phi_2^\dagger\Phi_2)^2\right] + \lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1) + \frac{\lambda_5}{2}\left[(\Phi_1^\dagger\Phi_2)^2 + (\Phi_2^\dagger\Phi_1)^2\right]. \tag{4.35}$$

There are different ways to solve the stationary point equations. There are solutions with $v_1 = 0$ or $v_2 = 0$. Such solutions imply that the model is Z_2 invariant (inert). Next, there are solutions requiring $\sin 2\xi = 0$. They represent models where Z_2 may be spontaneously broken. There are also solutions where $v_1 v_2 \neq 0$ and $\sin 2\xi \neq 0$. Such solutions will imply $\lambda_5 = 0$, and this yields a U(1) invariant potential. Such solutions will therefore not be discussed in this section.

4.5.1 *Z_2 conserving vacuum in $0Z_2$*

We consider only a model with $v_2 = 0$ (letting $v_1 = 0$ yields similar results). We may then without loss of generality rotate to a basis where $\xi = 0$. Now, the stationary point equations are solved by

$$m_{11}^2 = -\frac{1}{2}\lambda_1 v^2. \tag{4.36}$$

The neutral sector mass matrix is diagonal and without mass degeneracy, so the rotation matrix is simply $R = I_3$, and masses are given by

$$M_1^2 = \lambda_1 v^2, \quad M_2^2 = \frac{1}{2}v^2(\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5), \quad M_3^2 = \frac{1}{2}v^2(\lambda_1 + \lambda_3 + \lambda_4 - \lambda_5), \quad M_{H^\pm}^2 = \frac{1}{2}(\lambda_1 + \lambda_3)v^2. \tag{4.37}$$

The vacuum with $v_2 = 0$ we just came across is clearly the realization, within the $0Z_2$ model, of the Inert 2HDM [39–42]. In such a model, the observed Higgs boson ($h \equiv H_1$) stems from the real, neutral component of Φ_1 and has (tree-level) interactions with gauge bosons (and fermions) identical to those of the SM, whereas the extra scalars arising from Φ_2 – which would be $H \equiv H_2$, $A \equiv H_3$ and $H^\pm$ – have no triple vertex interactions with gauge bosons (and fermions). The lightest of those states is therefore stable and the resulting IDM has been studied extensively as a possible model to provide dark matter.[10]

A quick scan demanding unitarity and boundedness from below for the quartic couplings yields an upper bound of roughly 710 GeV for all extra scalar (non-SM Higgs states, therefore) masses. This is in stark contrast with the usual IDM, for which there is no upper bound for the inert scalar masses, since the m_{22}^2 parameter is free in that model. But not here – the r_0 symmetry forces $m_{22}^2 = -m_{11}^2 = M_1^2/2$, and thus upper bounds on the H, A and $H^\pm$ arise.

Working out the three gauge couplings and the four scalar couplings contained in the physical parameter set $\mathcal{P}$ described in Sect. 4.1, we get

$$e_1 = v, \quad e_2 = e_3 = 0, \quad q_1 = \lambda_3 v, \quad q_2 = q_3 = 0, \quad q = \frac{\lambda_1}{2}. \tag{4.38}$$

We see that the model is Z_2 invariant since $e_2 = e_3 = q_2 = q_3 = 0$. This corresponds to Case CC of Z_2 conservation in [35]. Thus, combining the constraints of Case r_0 with the constraints of Case CC, we arrive at

$$\textbf{Case } 0Z_2\textbf{-CC}: \quad e_j = q_j = e_k = q_k = 0, \quad q = \frac{M_i^2}{2v^2}, \quad M_{H^\pm}^2 = \frac{e_i q_i}{2} + \frac{M_i^2}{2},$$

which fully describes the physical consequences of the Z_2 invariant $0Z_2$ model. There are six constraints, implying that this model has $11 - 6 = 5$ free parameters. It is now easy to check that the masses and couplings we worked out for this model satisfy the constraints of Case $0Z_2$-CC for $i = 1$, $j = 2$ and $k = 3$.

4.5.2 *Spontaneous Z_2 violation in a $0Z_2$ model*

We solve the stationary point equations for $\sin 2\xi = 0$, assuming $v_1 v_2 \neq 0$. We restrict ourselves to $\xi = 0$ ($\xi = \pm\pi/2, \pi$ yields similar results). We get

$$m_{11}^2 = \frac{1}{2}\lambda_1\left(v_2^2 - v_1^2\right), \quad \lambda_5 = -\lambda_1 - \lambda_3 - \lambda_4. \tag{4.39}$$

Again, the last of these equations is a condition among quartic couplings only, not enforced by the model's symmetries, and which therefore would be a tree-level fine-tuning, unstable under radiative corrections. See the discussion following (4.34). The tree-level minimization conditions are not letting us determine both vevs in terms of the potential parameters,

[10] Obviously, we could also have a vacuum with $v_1 = 0$, but in what concerns the scalar sector that solution is equivalent, via a basis change, to the $v_2 = 0$ solution. When fermions are taken into account they are not, however, equivalent.

only $v_1^2 - v_2^2$ could be found. Note also that the condition $\lambda_5 = -\lambda_1 - \lambda_3 - \lambda_4$ is not preserved by the RGE, i.e. the corresponding β-function is not vanishing. So once again, a one-loop minimization is necessary to investigate the possibility of $v_2 \neq 0$ – we can expect that higher orders of perturbation expansion are necessary to generate a non-zero true vev somewhere along the tree-level one in accordance with the Georgi–Pais theorem [43]. Since again a full one-loop analysis is necessary, we will not investigate this model further here.

4.5.3 *Soft breaking of Z_2*

Let us also consider the possibility of keeping the r_0 symmetry intact, but softly breaking Z_2. That would imply that we allow for a nonzero m_{12}^2. From Table 3 we see that this simply takes us back to the 0CP1 model in the case of a real m_{12}^2 and to the general r_0 model in the case of a complex m_{12}^2, hence this yields nothing new.

4.6 The 0U(1) model

As can be appreciated from Table 3, the 0U(1) has in the reduced basis all the parameter constraints of the $0Z_2$ one, plus the condition $\lambda_5 = 0$. The potential reads

$$V = m_{11}^2 \left[\Phi_1^\dagger\Phi_1 - \Phi_2^\dagger\Phi_2\right] + \tfrac{1}{2}\lambda_1 \left[(\Phi_1^\dagger\Phi_1)^2 + (\Phi_2^\dagger\Phi_2)^2\right] + \lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1), \quad (4.40)$$

and we may without loss of generality rotate into a basis where $\xi = 0$. There are different ways to solve the stationary point equations. There are solutions with $v_1 = 0$ or $v_2 = 0$. Such solutions imply that the whole model is U(1) invariant. There are also solutions where $v_1 v_2 \neq 0$. In such models, U(1) is spontaneously broken.

4.6.1 *U(1) invariant vacuum in 0U(1)*

We consider only a model with $v_2 = 0$ (letting $v_1 = 0$ yields similar results). Now, the stationary point equations are solved by

$$m_{11}^2 = -\frac{1}{2}\lambda_1 v^2. \quad (4.41)$$

The neutral sector mass matrix is diagonal with masses given by[11]

$$M_1^2 = \lambda_1 v^2, \quad M_2^2 = M_3^2 = \frac{1}{2}(\lambda_1 + \lambda_3 + \lambda_4)\, v^2,$$
$$M_{H^\pm}^2 = \frac{1}{2}(\lambda_1 + \lambda_3)\, v^2. \quad (4.42)$$

We conclude that if the U(1) symmetry is preserved by the vacuum (when only one of the doublets acquires a vev) we are left with a version of the IDM, where the two neutral inert scalars are degenerate in mass. Working out the three gauge couplings and the four scalar couplings contained in the physical parameter set $\mathcal{P}$ described in Sect. 4.1, we get

$$e_1 = v, \quad e_2 = e_3 = 0, \quad q_1 = \lambda_3 v,$$
$$q_2 = q_3 = 0, \quad q = \frac{\lambda_1}{2}. \quad (4.43)$$

We see that the model is U(1) conserving since all the constraints defining Case BCC of U(1) conservation in [35] are satisfied. Thus, combining the constraints of Case r_0 with the constraints of Case BCC, we arrive at

Case 0U(1)-BCC : $M_j = M_k,$

$$e_j = q_j = e_k = q_k = 0, \quad q = \frac{M_i^2}{2v^2},$$
$$M_{H^\pm}^2 = \frac{e_i q_i}{2} + \frac{M_i^2}{2},$$

which fully describes the physical consequences of the U(1) conserving 0U(1) model. There are seven constraints, implying that this model has $11 - 7 = 4$ free parameters. It is now easy to check that the masses and couplings we worked out for this model satisfy the constraints of Case 0U(1)-BCC for $i = 1,\ j = 2$ and $k = 3$.

4.6.2 *Spontaneous U(1) violation in a 0U(1) model*

We solve the stationary point equations for $v_1 v_2 \neq 0$,

$$m_{11}^2 = \frac{1}{2}\lambda_1\left(v_2^2 - v_1^2\right), \quad \lambda_4 = -\lambda_1 - \lambda_3. \quad (4.44)$$

Again, $\lambda_4 = -\lambda_1 - \lambda_3$ is an RGE unstable condition. Furthermore, this vacuum would leave undetermined the values of the vevs v_1 and v_2. As in the previous RGE-unstable cases encountered, a one-loop calculation would be necessary to investigate the possibility of spontaneous 0U(1) breaking. We will not pursue this model further in the present work, a full

[11] Since there is mass degeneracy one can imagine mixing the two mass degenerate states H_2 and H_3 using a rotation matrix $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$, where α is completely arbitrary. Note that none of the masses or couplings depend on α, so simply putting $\alpha = 0$ yields the exact same result.

one-loop analysis is needed to settle the issue of spontaneous breaking of U(1).

4.6.3 Soft breaking of 0U(1)

Let us consider the possibility of keeping the r_0 symmetry intact, but softly break U(1). That would imply that we allow for a nonzero m_{12}^2. The potential for an r_0 invariant potential with a softly broken U(1) is

$$\begin{aligned} V &= m_{11}^2\left[\Phi_1^\dagger\Phi_1 - \Phi_2^\dagger\Phi_2\right] - \left[m_{12}^2\Phi_1^\dagger\Phi_2 + \text{h.c.}\right] \\ &\quad + \tfrac{1}{2}\lambda_1\left[(\Phi_1^\dagger\Phi_1)^2 + (\Phi_2^\dagger\Phi_2)^2\right] \\ &\quad + \lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1). \end{aligned} \tag{4.45}$$

Without loss of generality we may rotate into a basis in which m_{12}^2 is real to get

$$\begin{aligned} V &= m_{11}^2\left[\Phi_1^\dagger\Phi_1 - \Phi_2^\dagger\Phi_2\right] - m_{12}^2\left[\Phi_1^\dagger\Phi_2 + \Phi_2^\dagger\Phi_1\right] \\ &\quad + \tfrac{1}{2}\lambda_1\left[(\Phi_1^\dagger\Phi_1)^2 + (\Phi_2^\dagger\Phi_2)^2\right] \\ &\quad + \lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1). \end{aligned} \tag{4.46}$$

The only viable vacua require $\sin\xi = 0$. We choose to analyze the situation where $\xi = 0$ (if $\xi = \pi$ we get similar results). The stationary point equations are then solved by

$$\begin{aligned} m_{11}^2 &= \frac{1}{2}\lambda_1\left(v_2^2 - v_1^2\right), \\ \text{Re}\, m_{12}^2 &= \frac{1}{2}\left(\lambda_1 + \lambda_3 + \lambda_4\right) v_1 v_2. \end{aligned} \tag{4.47}$$

The neutral sector mass matrix is given by

$$\frac{1}{2}\begin{pmatrix} 2\lambda_1 v_1^2 + (\lambda_1+\lambda_3+\lambda_4)\, v_2^2 & (-\lambda_1+\lambda_3+\lambda_4)\, v_1 v_2 & 0 \\ (-\lambda_1+\lambda_3+\lambda_4)\, v_1 v_2 & (\lambda_1+\lambda_3+\lambda_4)\, v_1^2 + 2\lambda_1 v_2^2 & 0 \\ 0 & 0 & (\lambda_1+\lambda_3+\lambda_4)\, v^2 \end{pmatrix}, \tag{4.48}$$

and since there is mass degeneracy, the most general rotation matrix is given by

$$R = \begin{pmatrix} \frac{v_2\cos\alpha}{v} & \frac{v_1\cos\alpha}{v} & \sin\alpha \\ -\frac{v_1}{v} & \frac{v_2}{v} & 0 \\ -\frac{v_2\sin\alpha}{v} & -\frac{v_1\sin\alpha}{v} & \cos\alpha \end{pmatrix}, \tag{4.49}$$

where α is arbitrary (and simply mixes the two mass degenerate fields H_1 and H_3). Masses are given as

$$\begin{aligned} M_1^2 &= M_3^2 = \frac{1}{2}\left(\lambda_1+\lambda_3+\lambda_4\right) v^2, \quad M_2^2 = \lambda_1 v^2, \\ M_{H^\pm}^2 &= \frac{1}{2}(\lambda_1+\lambda_3) v^2, \end{aligned} \tag{4.50}$$

and the couplings are

$$\begin{aligned} e_1 &= \frac{2v_1v_2\cos\alpha}{v}, \quad e_2 = \frac{v_2^2 - v_1^2}{v}, \quad e_3 = -\frac{2v_1v_2\sin\alpha}{v}, \\ q_1 &= \frac{(\lambda_1+\lambda_3-\lambda_4)\, v_1 v_2\cos\alpha}{v}, \quad q_2 = \frac{\lambda_3\left(v_2^2 - v_1^2\right)}{v}, \\ q_3 &= -\frac{(\lambda_1+\lambda_3-\lambda_4)\, v_1 v_2 \sin\alpha}{v}, \\ q &= \frac{\lambda_1\left(v_1^4+v_2^4\right) + 2\left(\lambda_3+\lambda_4\right) v_1^2 v_2^2}{2v^4}. \end{aligned} \tag{4.51}$$

Now it is easy to check that the physical constraints are satisfied for this model with $i = 1$, $j = 3$, $k = 2$:

Case SOFT-0U1-B :

$$\begin{aligned} & M_i = M_j, \quad e_i q_j - e_j q_i = 0, \\ & v^2(e_i q_k - e_k q_i) + e_i e_k (M_k^2 - M_i^2) = 0, \\ & v^2(e_j q_k - e_k q_j) + e_j e_k (M_k^2 - M_i^2) = 0, \\ & q = \frac{1}{2v^4}([e_i^2 + e_j^2] M_i^2 + e_k^2 M_k^2), \\ & M_{H^\pm}^2 = \frac{1}{2}(e_i q_i + e_j q_j + e_k q_k) \\ & \quad + \frac{1}{2v^2}([e_i^2 + e_j^2] M_i^2 + e_k^2 M_k^2), \end{aligned}$$

and the presence of Case B [35] of CP conservation tells us that CP violation is not possible.

Notice that, interestingly, one obtains a degeneracy (at tree-level) between two of the neutral states, both in the case of the inert 0U(1) model and the softly broken version of 0U(1). Indeed, both models have analogous expressions for the masses, but there is a crucial distinction between them: in the inert 0U(1) model, only one of those scalars (denoted H_1) will have tree-level couplings to W and Z pairs, whereas the others (H_2 and H_3) are indeed inert states – thus, neither H_2 nor H_3 couple to electroweak gauge bosons at tree level. In the softly broken 0U(1) model, for which both doublets have vevs, the CP-even mass matrix is not diagonal in the symmetry basis, indicating that mixing occurs between the CP-even parts of the two doublets. Also, we see that some couplings depend on the arbitrary angle α. In [35], we argued that in such situations, what we can observe in experiments cannot depend on the unphysical α, only combinations independent of α may appear.[12] Nevertheless, we can pick a particular value of α and perform our analysis and calculations of observables with the chosen value of α. Picking $\alpha = 0$ leads to $e_3 = q_3 = 0$ and identifies H_3 as a pseudoscalar that does not couple to CP-even pairs of gauge bosons (ZZ, W^+W^+) or charged scalars (H^+H^-). Therefore, though degenerate

[12] For instance combinations $e_1^2 + e_3^2$, $q_1^2 + q_3^2$, $e_1q_1 + e_3q_3$ are independent of α.

at tree-level, H_1 and H_3 have different interactions, which indicates that their mass degeneracy will be lifted by radiative corrections. This argument cannot hold for one value of α only, but holds irrespective of the value of α one chooses.

There is also a sub-case of SOFT-0U1-B that we get if we put $m_{11}^2 = 0$. The only viable vacuum then is whenever $\sin\xi = 0$ and $v_1 = v_2 = v/\sqrt{2}$. The analysis is identical to the steps above, leading to

$$\begin{aligned}
&e_1 = v\cos\alpha, \quad e_2 = 0, \quad e_3 = -v\sin\alpha,\\
&q_1 = \tfrac{1}{2}(\lambda_1 + \lambda_3 - \lambda_4)\, v\cos\alpha, \quad q_2 = 0,\\
&q_3 = -\tfrac{1}{2}(\lambda_1 + \lambda_3 - \lambda_4)\, v\sin\alpha,\\
&q = \frac{1}{4}(\lambda_1 + \lambda_3 + \lambda_4)\,. \qquad (4.52)
\end{aligned}$$

Now it is easy to check that the physical constraints are satisfied for this model with $i = 1$, $j = 3$, $k = 2$:

Case SOFT-0U1-BC : $$\begin{aligned}
&M_i = M_j,\, e_i q_j - e_j q_i = 0,\\
&e_k = q_k = 0\\
&q = \frac{M_i^2}{2v^2},\\
&M_{H^\pm}^2 = \frac{1}{2}(e_i q_i + e_j q_j) + \frac{M_i^2}{2}.
\end{aligned}$$

Note that this is the same model that one gets if one in the softly broken 0CP3 model of (4.54) considers the sub-case where m_{12}^2 is real. As is shown later, the softly broken 0U(1) models and the softly broken 0CP3 models are simply related via a change of basis.

4.7 The 0CP2 model

In the 0CP2 model there are no quadratic terms, therefore no spontaneous electroweak breaking may occur for those cases (at tree-level, at least). Adding soft terms that break CP2 while keeping the r_0 symmetry intact simply takes us back to the r_0, 0CP1 or 0Z_2 model (depending on whether m_{12}^2 is complex, real or vanishing) as can be seen from Table 3. Hence, there are no new realistic models to be found by studying 0CP2 models.

4.8 The 0CP3 model (softly broken)

In the 0CP3 model there are no quadratic terms as well, therefore no spontaneous electroweak breaking may occur for those cases (at tree-level, at least). Adding soft terms that break CP3 while keeping the r_0 symmetry intact yields the following potential

$$\begin{aligned}
V =\;& m_{11}^2\left[\Phi_1^\dagger\Phi_1 - \Phi_2^\dagger\Phi_2\right] - \left[m_{12}^2\Phi_1^\dagger\Phi_2 + \text{h.c.}\right]\\
&+\tfrac{1}{2}\lambda_1\left[(\Phi_1^\dagger\Phi_1)^2 + (\Phi_2^\dagger\Phi_2)^2\right]\\
&+\lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1)\\
&+\frac{\lambda_1 - \lambda_3 - \lambda_4}{2}\left[(\Phi_1^\dagger\Phi_2)^2 + (\Phi_2^\dagger\Phi_1)^2\right]. \qquad (4.53)
\end{aligned}$$

Without loss of generality we can employ a change of basis with an orthogonal rotation among the two doublets, with a choice of either making $m_{11}^2 = 0$ or making m_{12}^2 purely imaginary to further simplify the potential (m_{12}^2 cannot be made real using an orthogonal change of basis). We choose to simplify the potential further by making $m_{11}^2 = 0$ to get

$$\begin{aligned}
V =\;& -\left[m_{12}^2\Phi_1^\dagger\Phi_2 + \text{h.c.}\right] + \tfrac{1}{2}\lambda_1\left[(\Phi_1^\dagger\Phi_1)^2 + (\Phi_2^\dagger\Phi_2)^2\right]\\
&+\lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1)\\
&+\frac{\lambda_1 - \lambda_3 - \lambda_4}{2}\left[(\Phi_1^\dagger\Phi_2)^2 + (\Phi_2^\dagger\Phi_1)^2\right]. \qquad (4.54)
\end{aligned}$$

This would seem a completely new possibility, but indeed it is not – it is in fact the same potential of the softly-broken 0U(1) model, Eq. (4.46), but expressed in a different basis. To see this, start from that equation and use the expressions for basis changes shown in Sect. 2.2, for the following basis transformation:

$$\begin{pmatrix}\Phi_1'\\ \Phi_2'\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}. \qquad (4.55)$$

In the new basis, the potential will have the exact form of Eq. (4.54). This case therefore yields nothing new.

4.9 The 0SO(3) model (softly broken)

In the 0SO(3) model there are again no quadratic terms, therefore no spontaneous electroweak breaking may occur for those cases (at tree-level, at least). Adding soft terms that break SO(3) while keeping the r_0 symmetry intact yield the following potential

$$\begin{aligned}
V =\;& m_{11}^2\left[\Phi_1^\dagger\Phi_1 - \Phi_2^\dagger\Phi_2\right] - \left[m_{12}^2\Phi_1^\dagger\Phi_2 + \text{h.c.}\right]\\
&+\tfrac{1}{2}\lambda_1\left[(\Phi_1^\dagger\Phi_1)^2 + (\Phi_2^\dagger\Phi_2)^2\right]\\
&+\lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + (\lambda_1 - \lambda_3)(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1). \qquad (4.56)
\end{aligned}$$

Since the quartic part of the SO(3) potential is insensitive to basis changes, we may without loss of generality rotate into a basis where $m_{12}^2 = 0$ to get

$$\begin{aligned}
V =\;& m_{11}^2\left[\Phi_1^\dagger\Phi_1 - \Phi_2^\dagger\Phi_2\right] + \tfrac{1}{2}\lambda_1\left[(\Phi_1^\dagger\Phi_1)^2 + (\Phi_2^\dagger\Phi_2)^2\right]\\
&+\lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + (\lambda_1 - \lambda_3)(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1). \qquad (4.57)
\end{aligned}$$

We may also without loss of generality assume $\xi = 0$. The only viable solution[13] of the stationary point equations is found for $v_2 = 0$ ($v_1 = 0$ yields similar results),

$$m_{11}^2 = -\frac{1}{2}\lambda_1 v^2, \quad v_2 = 0. \tag{4.58}$$

The mass matrix is diagonal with full mass degeneracy.

$$M_1^2 = M_2^2 = M_3^2 = \lambda_1 v^2, \quad M_{H^\pm}^2 = \frac{1}{2}(\lambda_1 + \lambda_3)v^2. \tag{4.59}$$

The most general rotation matrix is therefore given by

$$R = \begin{pmatrix} c_1\, c_2 & s_1\, c_2 & s_2 \\ -(c_1\, s_2\, s_3 + s_1\, c_3) & c_1\, c_3 - s_1\, s_2\, s_3 & c_2\, s_3 \\ -c_1\, s_2\, c_3 + s_1\, s_3 & -(c_1\, s_3 + s_1\, s_2\, c_3) & c_2\, c_3 \end{pmatrix}, \tag{4.60}$$

where $c_i = \cos\alpha_i$, $s_i = \sin\alpha_i$ and all α_i are completely arbitrary due to the full mass degeneracy.

The couplings are

$$\begin{aligned} &e_1 = v c_1\, c_2, \quad e_2 = -v(c_1\, s_2\, s_3 + s_1\, c_3), \\ &e_3 = v(-c_1\, s_2\, c_3 + s_1\, s_3), \\ &q_1 = v\lambda_3 c_1\, c_2, \quad q_2 = -v\lambda_3(c_1\, s_2\, s_3 + s_1\, c_3), \\ &q_3 = v\lambda_3(-c_1\, s_2\, c_3 + s_1\, s_3), \quad q = \frac{\lambda_1}{2}. \end{aligned} \tag{4.61}$$

This model was discussed in [36] where we dubbed it Case SOFT-SO3-ABBB. Here, we add a zero to the name since it is invariant under r_0. In terms of masses and couplings, this model is then described by

Case SOFT-0SO3-ABBB : $M_1 = M_2 = M_3$, $e_1 q_2 - e_2 q_1 = 0$, $e_1 q_3 - e_3 q_1 = 0$, $e_2 q_3 - e_3 q_2 = 0$, $2M_{H^\pm}^2 = M_1^2 + e_1 q_1 + e_2 q_2 + e_3 q_3$, $2v^2 q = M_1^2$.

We may also here pick specific values of the arbitrary rotation angles. Picking all $\alpha_i = 0$, yields $e_2 = e_3 = q_2 = q_3 = 0$, thereby identifying H_2 and H_3 as the inert scalars that do not couple to CP-even pairs of gauge bosons (ZZ, W^+W^+) or charged scalars (H^+H^-). Since then H_1 couples differently to the gauge bosons than the inert fields H_2 and H_3 do, we expect the full mass degeneracy to be lifted at one-loop level. A partial mass degeneracy between the two inert fields H_2 and H_3 may very well be preserved at one loop level.

[13] Another solution with $m_{11}^2 = 0$, $\lambda_1 = 0$ also exist, but then we are back to the situation where we have no quadratic terms, so electroweak symmetry breaking does not occur.

5 The fermion sector

We have established in previous sections that the conditions described by Eq. (2.25) are RG invariant to all orders. Our demonstrations, however, involved only the scalar and gauge sectors. That by itself is interesting, as we may consider conceptual models without fermions, but as we will now show, the conditions behind the r_0 symmetry (as well as several other of the new symmetries studied above) can be satisfied to at least two-loop order, even if one includes the Yukawa sector. This is more than can be said, for instance, for the "custodial symmetry", which not only is broken by the $U(1)_Y$ gauge group, but also by the different masses for up and down quarks. In this section we do not wish to exhaust all possibilities, but simply show that it is possible to find Yukawa textures which are invariant, up to two-loop order, under some of the symmetries discussed earlier.

Concerning the new symmetries proposed in the current work, whose effects on the parameters of the potential are summarised in Table 2, we observe that they all have a scalar quartic sector with couplings which obey, *at least*, the CP2 symmetry relations. For models with symmetries such as r_0, 0CP1 and 0CP2, indeed, the quartic sector obeys exactly the same relations as the CP2 case. This means that, if we can find Yukawa matrices with textures which comply with the CP2 symmetry, we automatically will have ensured that:

- Those textures will be preserved under radiative corrections, since they are the result of a symmetry (CP2) which extends to all dimensionless couplings of the model.
- The relations between quartic scalar couplings in (at least) models r_0, 0CP1 and 0CP2 (softly broken or not) will be preserved to all orders in perturbation theory.
- The theory will be renormalizable regardless of the quadratic parameters of the scalar potential, but it may be possible that the r_0 relation $m_{22}^2 = -m_{11}^2$ is RG-preserved even when considering Yukawa interactions.

In other words, in what concerns the dimensionless couplings of the model (scalar quartic, gauge or Yukawa), choosing CP2 Yukawa textures is consistent from the renormalization point of view: CP2 Yukawas will not spoil the 0CP2 scalar quartic relations because they are identical to the CP2 ones, and vice-versa. It remains to be seen whether CP2 Yukawas respect the full 0CP2 symmetry-imposed relations, *i.e.,* the relation $m_{22}^2 = -m_{11}^2$. We will show that this is what happens, at least up to two-loop order.

The same arguments are valid if we consider the 0CP3 model (softly broken or not) – since that model has quartic coupling relations which are identical to the CP3 case, if one considers a CP3-symmetric Yukawa sector all relations between dimensionless couplings are preserved under renormalization. Again, it remains also a possibility that the

$m_{22}^2 = -m_{11}^2$ relation is itself found to be preserved under radiative corrections. It is this aspect which we will now investigate, since this relation between quadratic parameters is what distinguishes the new symmetries we are proposing from those already known.

Let us recall that the most generic 2HDM Yukawa sector may be written as[14]

$$-\mathcal{L}_Y = \bar{q}_L(\Gamma_1\Phi_1 + \Gamma_2\Phi_2)n_R + \bar{q}_L(\Delta_1\tilde{\Phi}_1 + \Delta_2\tilde{\Phi}_2)p_R + \bar{l}_L(\Pi_1\Phi_1 + \Pi_2\Phi_2)l_R + \text{H.c.} \quad (5.1)$$

In this equation, $\tilde{\Phi}_i = i\sigma_2\Phi_i^*$ are the doublets' charge conjugates; q_L and l_L are 3-vectors in flavour space containing the quark and lepton left doublets; likewise, n_R, p_R and l_R are 3-vectors in flavour space, containing, respectively, the righthanded down, up and charged lepton fields. The Γ_i, Δ_i and Π_i are 3×3 complex matrices containing Yukawa couplings. The fermionic fields in this equation do not correspond to the quark and lepton mass states. The physical fields (corresponding to quark and lepton mass eigenstates) are related to these via unitary transformations in flavour space which involve 3×3 U(3) matrices in flavour space. For the quarks, for instance, we would have

$$p_L = U_{uL}\,u_L,\quad p_R = U_{uR}\,u_R,\quad n_L = U_{dL}\,d_L,\quad n_R = U_{dR}\,d_R \quad (5.2)$$

so that the down and up quark mass matrices, given by

$$M_d = \frac{1}{\sqrt{2}}(\Gamma_1 v_1 + \Gamma_2 v_2),\quad M_u = \frac{1}{\sqrt{2}}\left(\Delta_1 v_1^* + \Delta_2 v_2^*\right) \quad (5.3)$$

are bi-diagonalised so that one obtains the physical quark masses,

$$\text{diag}(m_d, m_s, m_b) = U_{dL}^\dagger\, M_d\, U_{dR},\quad \text{diag}(m_u, m_c, m_t) = U_{uL}^\dagger\, M_u\, U_{uR}. \quad (5.4)$$

Similar relations hold for the leptons as well. The transformations (5.2) mean that there is additional basis freedom in the Yukawa sector of the 2HDM, by redefining the fermion fields alongside the scalar ones. Concerning the CP symmetries – CP1, CP2 and CP3 – of the 2HDM, as was explained in Ref. [44], they may be extended to the Yukawa sector. The Yukawa matrices Γ_i, then, must obey the following relations,

$$X_\alpha\Gamma_1^* - (\cos\theta\Gamma_1 - \sin\theta\Gamma_2)X_\beta = 0,\quad X_\alpha\Gamma_2^* - (\sin\theta\Gamma_1 + \cos\theta\Gamma_2)X_\beta = 0 \quad (5.5)$$

where the matrices X_x are given by

$$X_x = \begin{pmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (5.6)$$

and the angle θ (like the angles α and β) can be taken, without loss of generality, to be between 0 and $\pi/2$ and describes each possible CP symmetry: $\theta = 0$ corresponds to CP1; $\theta = \pi/2$ to CP2; and any arbitrary angle $0 < \theta < \pi/2$ yields CP3. An analogous equation to (5.5) (with different, independent angles γ replacing β) is valid for the up quark Yukawa matrices Δ_i. Solving (5.5) for CP2 and CP3 one then finds:

- For the CP2 symmetry, Eq. (5.5) is satisfied (for $\theta = \pi/2$ and $\alpha = \beta = \pi/4$) by Γ matrices of the form [45,46] [44]

$$\Gamma_1 = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{12} & -a_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad \Gamma_2 = \begin{pmatrix} -a_{12}^* & a_{11}^* & 0 \\ a_{11}^* & a_{12}^* & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (5.7)$$

  Analogous expressions are then found for the Δ and Π matrices, with different coefficients b_{ij} and c_{ij} instead of a_{ij}. As is plain to see, these matrices imply that one of the up and down quarks and a charged lepton will be massless.
- For the CP3 symmetry, Eq. (5.5) is satisfied (for $\theta = \alpha = \beta = \pi/3$) by Γ matrices of the form [44]

$$\Gamma_1 = \begin{pmatrix} i\,a_{11} & i\,a_{12} & a_{13} \\ i\,a_{12} & -i\,a_{11} & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix},\quad \Gamma_2 = \begin{pmatrix} i\,a_{12} & -i\,a_{11} & -a_{23} \\ -i\,a_{11} & -i\,a_{12} & a_{13} \\ -a_{32} & a_{31} & 0 \end{pmatrix}, \quad (5.8)$$

  with the a_{ij} real. Analogous matrices are then found for the Δ and Π matrices, with different coefficients b_{ij} and c_{ij} instead of a_{ij}. These matrices yield three generations of massive charged fermions, and it was possible to perform a numerical fit reproducing the known quark and lepton masses; however, that fit could not reproduce the value of the Jarlskog invariant.[15]

With Yukawa matrices that comply with symmetries CP2 and CP3 – and therefore, as has been explained, fermionic contributions to RG running will respect the relations between scalar quartic couplings for those models – we

[14] We will neglect neutrinos in this study; pure Dirac mass terms for neutrinos could be trivially added to this lagrangian, of course.

[15] Notice, however, that the phenomenological problems with the CP2 and CP3 Yukawa sector may be solved by adding vector-like fermions to the model [47,48].

can verify whether the unusual $m_{11}^2 + m_{22}^2 = 0$ relation is also preserved when one includes the fermion sector in the model. Let us show how this works explicitly at one-loop – the fermionic contributions to the β-functions of m_{11}^2 and m_{22}^2 in Eq. (2.22) are given, for the most general 2HDM, by (see, for instance, [26–30,49]):

$$\beta^{F,1L}_{m_{11}^2} = \left[3\,\mathrm{Tr}(\Delta_1\Delta_1^\dagger) + 3\,\mathrm{Tr}(\Gamma_1\Gamma_1^\dagger) + \mathrm{Tr}(\Pi_1\Pi_1^\dagger)\right] m_{11}^2 - \left\{\left[3\,\mathrm{Tr}(\Delta_1^\dagger\Delta_2) + 3\,\mathrm{Tr}(\Gamma_1^\dagger\Gamma_2) + \mathrm{Tr}(\Pi_1^\dagger\Pi_2)\right] \times\, m_{12}^2 + \mathrm{h.c.}\right\},$$
$$\beta^{F,1L}_{m_{22}^2} = \left[3\mathrm{Tr}(\Delta_2\Delta_2^\dagger) + 3\mathrm{Tr}(\Gamma_2\Gamma_2^\dagger) + \mathrm{Tr}(\Pi_2\Pi_2^\dagger)\right] m_{22}^2 - \left\{\left[3\,\mathrm{Tr}(\Delta_1^\dagger\Delta_2) + 3\,\mathrm{Tr}(\Gamma_1^\dagger\Gamma_2) + \mathrm{Tr}(\Pi_1^\dagger\Pi_2)\right] \times\, m_{12}^2 + \mathrm{h.c.}\right\}. \tag{5.9}$$

We then see something remarkable – for both the CP2 or CP3 Yukawa textures (Eqs. (5.7) and (5.8) respectively), one obtains

$$\mathrm{Tr}(\Delta_1\Delta_1^\dagger) = \mathrm{Tr}(\Delta_2\Delta_2^\dagger), \quad \mathrm{Tr}(\Gamma_1\Gamma_1^\dagger) = \mathrm{Tr}(\Gamma_2\Gamma_2^\dagger), \quad \mathrm{Tr}(\Pi_1\Pi_1^\dagger) = \mathrm{Tr}(\Pi_2\Pi_2^\dagger), \tag{5.10}$$

as well as

$$\mathrm{Tr}(\Delta_1\Delta_2^\dagger) = \mathrm{Tr}(\Gamma_1\Gamma_2^\dagger) = \mathrm{Tr}(\Pi_1\Pi_2^\dagger) = 0. \tag{5.11}$$

Hence,

$$\beta^{F,1L}_{m_{11}^2+m_{22}^2} = \left[3\,\mathrm{Tr}(\Delta_1\Delta_1^\dagger) + 3\,\mathrm{Tr}(\Gamma_1\Gamma_1^\dagger) + \mathrm{Tr}(\Pi_1\Pi_1^\dagger)\right] \times \left(m_{11}^2 + m_{22}^2\right) \tag{5.12}$$

and therefore it has been shown that the $m_{11}^2 + m_{22}^2 = 0$ condition is preserved under RGE running for CP2 and CP3 invariant theories at one-loop, even including the fermionic sector.

An all-order result is beyond our skills, but we can at least extend this demonstration to two loops. We can use the SARAH [26–30] package and adapt its results for a 2HDM Type-III model[16] for the specific Yukawa matrices of Eqs. (5.7) and (5.8). Doing so, we find that when the Yukawa matrices have the CP2/CP3 structures and the potential obeys the 0CP2/0CP3 symmetry, the two-loop beta functions for the quadratic scalar couplings, including fermions, satisfy

$$\beta^{2L}_{m_{11}^2+m_{22}^2} = X\,(m_{11}^2 + m_{22}^2), \tag{5.13}$$

where the quantity "X" contains contributions from all dimensionless couplings of the model (scalar, gauge, Yukawa). And therefore, as in the one-loop case, we verify that $m_{11}^2 + m_{22}^2 = 0$ is preserved by RG running up to two loops at least. This may also be independently verified using the PyR@TE package [49]. Wishing to go beyond the "black box" of these remarkable packages, we performed a simplified verification of these results to understand the cancellations between different terms necessary for them to occur, and show it, as a curiosity, in Appendix B.

It seems therefore likely that invariance under the r_0 symmetry, at least for the 0CP2/0CP3 versions, can be extended to three-loop order in the Yukawa sector – or indeed to all orders, as we argued was the case for the scalar and gauge contributions.

6 Conclusions

We found a set of constraints on 2HDM scalar parameters which is RG invariant to all orders when one considers only scalar and gauge interactions – and which can be invariant to at least two loops if fermions are also included. To do so we analysed the beta-functions of the parameters of the model and discovered fixed points – valid to all orders in scalar and gauge couplings – which correspond to relations between 2HDM parameters which do not coincide with any of the known six symmetries of the $SU(2) \times U(1)$ scalar potential. Those relations, given by what we called the r_0-symmetry,[17] are

$$m_{11}^2 + m_{22}^2 = 0, \quad \lambda_1 = \lambda_2, \quad \lambda_6 = -\lambda_7, \tag{6.1}$$

which have also been shown to be basis-invariant.

It is well known that invariance of a system under a symmetry imposes certain relations among its parameters; and those relations will be preserved under renormalization to all orders, constituting fixed points of its RG equations. What we have found here for the 2HDM is in some sense the opposite situation: we found fixed points of the RG equations, valid to all orders of perturbation theory, but do not know what symmetry operation upon the model's fields may cause them to appear. We showed that one way of understanding the relations obtained for the parameters of the scalar potential is to consider an "r_0 sign change" in the gauge invariant bilinear r_0 – but though that is helpful as a formal way of understanding our results, it raises serious questions, as the scalar kinetic terms are not invariant under the transformation $r_0 \to -r_0$. Further, this transformation is impossible to obtain via unitary transformations of the doublets or their

[16] The reader should be aware that, up to version 4.15.1 of SARAH, there is a bug in the code concerning non-supersymmetric beta-functions for the squared mass couplings in model III. This arises when Yukawa couplings induce mixing between the doublets. The issue has been identified and a patch is available from SARAH's keepers. Many thanks to Mark Goodsell for his help in this matter.

[17] Considering the names of the authors, the only other reasonable nomenclature would be the GOOF symmetry, and we do not think such possibility would be well-met in the community.

complex conjugates. We propose a (very strange) set of transformations on the scalar and gauge fields in Appendix A, but it is unclear whether or not it constitutes a mathematical trick only.

Therefore, strictly speaking, we may not have identified "symmetries" of the 2HDM. If the reader wishes, call them instead "*relations between 2HDM parameters which yield fixed points of the RG equations to all orders of perturbation theory and are therefore preserved under renormalization*". But given that the several models we discuss here will benefit from all features of symmetries when these all-order invariant relations are considered, we believe that calling these "symmetries" is justified, and challenge our colleagues to find the field transformations which yield them.

Combining the r_0-symmetry with the other known six symmetries yields seven new symmetry classes. We briefly investigated the phenomenological aspects of each of those symmetries, considering possible soft breaking terms. The impact of the new symmetries on physical parameters – masses, couplings of scalar-gauge boson and scalar self interactions – was shown, with simple relations between those observables obtained. We also concluded that the r_0-symmetry has measurable impacts on the 2HDM, namely it prevents the existence of a *decoupling limit* – the r_0-symmetry, coupled with minimization conditions, eliminates all dependence on squared mass parameters in the scalars' physical masses, and therefore, in models invariant under the r_0-symmetry, the non-SM-like scalars cannot be arbitrarily heavy. We found bounds of a little above 700 GeV for both charged and neutral scalars. Therefore these models can easily be disproven experimentally, if bounds on extra scalar masses are found to be well above $\sim$ 700 GeV when new LHC data is analysed.

Another possibly interesting phenomenological consequence of the r_0-symmetry occurs for the softly broken 0U(1) model, where the extra CP-even scalar and the CP-odd one were found to be mass degenerate at tree level. However, since the CP-even particle has different interactions than the CP-odd one (it couples to W and Z pairs, for instance, as well as to charged scalar pairs, H^+H^-), this mass degeneracy will be lifted via radiative corrections. A full one-loop calculation is necessary to determine the mass splitting between these two scalars, but one might expect that it will not be sizeable. Hence, if a CP-even scalar and a CP-odd one were discovered at the LHC with a small mass difference between them, the r_0-symmetry coupled with a Peccei-Quinn U(1) symmetry may therefore provide a simple, natural way to explain it.

We have found several instances where the r_0-symmetry prevents tree-level spontaneous symmetry breaking – in the 0CP1 model, spontaneous CP violation is found to require, at tree-level, an RG-unstable relation among quartic couplings; likewise, in the 0Z2 model, a vacuum where both doublets acquire vevs and spontaneous breaking of the Z_2 symmetry would occur, is found to also require an RG-unstable relation among quartic couplings (albeit a different one); and the same occurs for spontaneous U(1) breaking in the 0U(1) model, with yet another RG-unstable condition on the quartic couplings necessary for the tree-level minimization equations to have a solution. Further, in these cases, it was found that the tree-level minimization equations did not allow for the unequivocal determination of the doublets' vevs. These are situations where a one-loop minimization is necessary, to verify whether radiative corrections allow the spontaneous breaking of these symmetries, as in the Georgi–Pais mechanism [43].

We showed that, at least for some of the models proposed, it is possible to extend the r_0-symmetry to the full lagrangian, including fermions. We did not obtain an all-order result, but were capable of showing that, at least up to two loops, the 0CP2 symmetry, including CP-symmetric Yukawa matrices, was a symmetry of the full lagrangian. Likewise, the 0CP3 model, with CP3-symmetric Yukawa matrices, is fully consistent up to two loops in the fermionic sector, at least. This strongly suggests that these parameter relations may indeed be preserved under renormalization to all orders of perturbation theory, even including Yukawa interactions. The CP2 and CP3 Yukawas considered were just a "case study" to prove extension to fermions of the r_0-symmetry was possible, but they are not necessarily the only ones – others may be found. The CP2 an CP3 Yukawa textures have phenomenological problems associated with them (massless fermions in the former case; wrong values for the Jarlskog invariant for the latter), which may be solved by enlarging the particle content of the 2HDM via the introduction of vector-like fermions [47,48]. It would be interesting to verify whether, with the extra fermion content, it would still be possible for the 0CP2 and 0CP3 models to be RG-invariant (at least to two loops) when including the Yukawa sector.

In conclusion, we have shown that the 2HDM includes regions of parameter space where relations between scalar couplings are RG-invariant to all orders – and for at least two cases, at least to two-loop order when one includes Yukawa interactions. The models boasting the new r_0-symmetry have interesting phenomenology and leave plenty of questions for future avenues of research, the more pressing one of which may well be whether there are transformations on the fields of the model which reproduce the r_0-symmetry. Appendix A has one such proposal which works mathematically, but whose physical meaning is unclear.

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Appendix A: Imaginary spacetime – a proposed transformation originating the r_0 symmetry

In Sect. 3.2 we showed that the r_0 symmetry could be interpreted as a change in sign in the r_0 bilinear. However, though this formally worked for the scalar potential, the transformation $r_0 \to -r_0$ could not be extended to the theory's kinetic terms in any obvious manner. In this appendix we will show a curiosity: it is possible to obtain a transformation of fields and spacetime coordinates which leave the lagrangian invariant under the r_0 symmetry, but such a transformation involves a complex spacetime and gauge-breaking relations between the fields, though the final theory is gauge-invariant. We stress that we do not pretend that what follows is a fully-fledged transformation that explains the origin of the r_0-symmetry. Instead, it may well be just a mathematical curiosity.

Let us for this purpose parameterize the doublets as

$$\Phi_1 = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} \phi_5 + i\phi_6 \\ \phi_7 + i\phi_8 \end{pmatrix}, \tag{A.1}$$

with all fields ϕ_i Hermitian. We find that the bilinears can be expressed as

$$\begin{aligned} r_0 &= \frac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 + \phi_5^2 + \phi_6^2 + \phi_7^2 + \phi_8^2), \\ r_1 &= \phi_1\phi_5 + \phi_2\phi_6 + \phi_3\phi_7 + \phi_4\phi_8, \\ r_2 &= -\phi_2\phi_5 + \phi_1\phi_6 - \phi_4\phi_7 + \phi_3\phi_8, \\ r_3 &= \frac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 - \phi_5^2 - \phi_6^2 - \phi_7^2 - \phi_8^2). \end{aligned} \tag{A.2}$$

We are looking for a transformation that makes r_0 change sign, while r_1, r_2 and r_3 are unchanged – this, indeed, is the interpretation we made of the r_0 symmetry in Sect. 3.2. The transformation

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \end{pmatrix} \tag{A.3}$$

accomplishes this. This transformation implies

$$\begin{aligned} &\Phi_1 \to -\Phi_2^* \quad \Phi_1^\dagger \to \Phi_2^T, \\ &\Phi_2 \to \Phi_1^*, \quad \Phi_2^\dagger \to -\Phi_1^T. \end{aligned} \tag{A.4}$$

Notice that this transformation, applied to the real component fields of the doublets, forces each doublet and their hermitian conjugates to transform differently than they should. Indeed, the transformation of $\Phi_1^\dagger$ above is *not* the hermitian conjugate of the transformation of Φ_1, and the same holds for the second doublet. This suggests that behind the r_0 symmetry is a type of formalism in which Φ_i and $\Phi_i^\dagger$ should be treated as independent objects. Notice, too, that the transformation of Eq. (A.3) is akin to a Z_4 symmetry in the sense that, to recover the original doublets, one needs to apply it *four* times. This is more easily seen from Eq. (A.4).

Let us now verify whether one can make the scalar kinetic terms invariant under the transformation of Eq. (A.3). The scalar covariant derivatives are given by

$$D^\mu = \partial^\mu + \frac{ig}{2}\sigma_i W_i^\mu + i\frac{g'}{2}B^\mu, \tag{A.5}$$

so that the scalar kinetic part of the Lagrangian can be written as

$$\mathcal{L}_k = (D_\mu\Phi_1)^\dagger(D^\mu\Phi_1) + (D_\mu\Phi_2)^\dagger(D^\mu\Phi_2). \tag{A.6}$$

The kinetic terms are invariant under the transformations of Eq. (A.3) if we combine them with

$$\begin{aligned} &\partial_\mu \to -i\partial_\mu, \\ &B_\mu \to iB_\mu, \\ &W_{1\mu} \to iW_{1\mu}, \quad W_{2\mu} \to -iW_{2\mu}, \quad W_{3\mu} \to iW_{3\mu}. \end{aligned} \tag{A.7}$$

We shall call this the extended r_0 transformation. Notice how the first of these corresponds to a transformation on the spacetime coordinates themselves,

$$x_\mu \to ix_\mu. \tag{A.8}$$

Strange as this transformation is, we observe that it leaves the spacetime measure d^4x invariant.

The imaginary transformations on the gauge fields of Eq. (A.7) would correspond to, for the gauge mass eigenstates,

$$\begin{aligned} &A_\mu \rightarrow i A_\mu, \\ &Z_\mu \rightarrow i Z_\mu, \\ &W_\mu \rightarrow i W_\mu^+, \quad W_\mu^+ \rightarrow i W_\mu, \end{aligned} \tag{A.9}$$

where $W_1^\mu = \frac{1}{\sqrt{2}}(W^{+\mu} + W^{-\mu})$, $W_2^\mu = \frac{i}{\sqrt{2}}(W^{+\mu} - W^{-\mu})$, $W_3^\mu = \cos\theta_W Z^\mu + \sin\theta_W A^\mu$ and $B^\mu = -\sin\theta_W Z^\mu + \cos\theta_W A^\mu$.[18]

The net effect of the extended r_0 transformation is that the covariant derivatives acting on the doublets transform according to

$$\begin{aligned} &D^\mu \Phi_1 \rightarrow i\left(D^\mu \Phi_2\right)^*, \quad \left(D^\mu \Phi_1\right)^\dagger \rightarrow -i\left(D^\mu \Phi_2\right)^T, \\ &D^\mu \Phi_2 \rightarrow -i\left(D^\mu \Phi_1\right)^*, \quad \left(D^\mu \Phi_2\right)^\dagger \rightarrow i\left(D^\mu \Phi_1\right)^T \end{aligned} \tag{A.10}$$

and then it is easy to see that the scalar kinetic terms are clearly invariant under these transformations.

Having found a gauge field transformation necessary to render invariant the scalar kinetic terms, we must then worry about the gauge kinetic terms themselves. These can be written compactly as

$$\mathcal{L}^B = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W_{i\mu\nu} W_i^{\mu\nu}, \tag{A.11}$$

where $B^{\mu\nu} = \partial^\nu B^\mu - \partial^\mu B^\nu$ and $W_i^{\mu\nu} = \partial^\nu W_i^\mu - \partial^\mu W_i^\nu + g\epsilon_{ijk} W_j^\mu W_k^\nu$. We find that under the extended r_0 transformation defined above in Eq. (A.7) we have

$$\begin{aligned} &B^{\mu\nu} \rightarrow B^{\mu\nu}, \\ &W_1^{\mu\nu} \rightarrow W_1^{\mu\nu}, \quad W_2^{\mu\nu} \rightarrow -W_2^{\mu\nu}, \quad W_3^{\mu\nu} \rightarrow W_3^{\mu\nu}, \end{aligned} \tag{A.12}$$

and it is then clear that $\mathcal{L}^B$ is invariant under the extended r_0 transformation. A generalization of these imaginary transformations to fermionic fields should also be possible. However, the rather extreme procedure we outlined here to reproduce the r_0-symmetry raises serious questions: could the imaginary transformations of Eqs. (A.3), (A.4), (A.7) and (A.8) be made compatible with the basic requirements of quantum field theory? In particular, (A.4) shows that each doublet and its complex conjugate transform differently than they should; so what will that imply when attempting to make a quantum field theory for the commutation relations of the annihilation and creation operators a and $a^\dagger$? We have no answer for this question, this matter requires further research.

[18] These are oddly consistent. It is curious to observe that, for the simple case in electromagnetism of the 4-potential produced by a moving point charge, if one makes $x_\mu \rightarrow i x_\mu$, we indeed obtain in that situation $A_\mu \rightarrow i A_\mu$.

Appendix B: Two loop fermionic beta-functions and the condition $m_{11}^2 + m_{22}^2 = 0$

For our purposes – demonstrating that the $m_{11}^2 + m_{22}^2 = 0$ condition is left invariant under RG running at two loops by CP2 and/or CP3 Yukawa matrices[19] – we do not need the *exact* form of the two-loop beta functions. All we need to do is analyse the structure of Yukawa couplings emerging from all contributions to the beta functions and deduce that they are such that $\beta_{m_{11}^2+m_{22}^2}$ ends up being proportional to $m_{11}^2 + m_{22}^2$. Let us show, through a partial calculation, how this comes about.

- Yukawa-only contributions

Using dimensional regularization, wherein the spacetime dimension is taken to be $4 - 2\epsilon$ with $\epsilon \rightarrow 0$, the two-loop beta functions for the scalar quadratic coefficients arise from Feynman diagrams such as those shown in Fig. 1, where we only considered contributions from quark interactions.[20] Specifically, each Feynman diagram will have a pole in $1/\epsilon$, the coefficient of which is the contribution to the beta function. In Fig. 1 the cross, "×", can be thought of as a "mass insertion", or rather as a "vertex" where a m_{11}^2 coefficient corresponds to a continuous Φ_1 line; a m_{22}^2 coefficient corresponds to a continuous Φ_2 line; and a $-m_{12}^2$ $(-m_{21}^2 = -m_{12}^{2*})$ coefficient turns a Φ_1 (Φ_2) line into a Φ_2 (Φ_1) one. Note that one should also consider a diagram analogous to "B" but with the fermionic lines flowing in the opposite way – the diagram topology of both possibilities is the same, but the Yukawa structures arising from each differ.

The crucial point for the argument that follows is that each of the diagrams of Fig. 1 has the same topology, and will therefore yield a given prefactor – A, B or C – containing the same pole coefficient, symmetry factor, sum on colour indices, etc, whatever the specific Yukawa couplings or m_{ij}^2 coefficients one may consider for that topology. As an example, consider diagram A. If one takes $m_{kl}^2 = m_{11}^2$, for instance, one is left with four possibilities for the fermion lines: (i) only "u" lines, (ii) only "d" lines, (iii) upper "u and lower "d" lines and (iv) vice-versa. These will correspond to combinations of Yukawa matrices given by, respectively, $\mathrm{Tr}(\Delta_1\Delta_1^\dagger\Delta_1\Delta_1^\dagger)$, $\mathrm{Tr}(\Gamma_1\Gamma_1^\dagger\Gamma_1\Gamma_1^\dagger)$, $\mathrm{Tr}(\Gamma_1\Delta_1^\dagger\Delta_1\Gamma_1^\dagger)$ and $\mathrm{Tr}(\Delta_1\Gamma_1^\dagger\Gamma_1\Delta_1^\dagger)$, but all will be multiplied by m_{11}^2 *and* the same factor A, characteristic of this

[19] To be precise, we should add that the conditions $\lambda_1 = \lambda_2$ and $\lambda_6 = -\lambda_7$ are also left invariant under RG running, but we already know that is the case for both CP2 and CP3 symmetries (with $\lambda_6 = \lambda_7 = 0$ in the latter case).

[20] The contributions from leptons would be simpler as we are not considering Dirac neutrino masses, and the argument would follow in the same way.

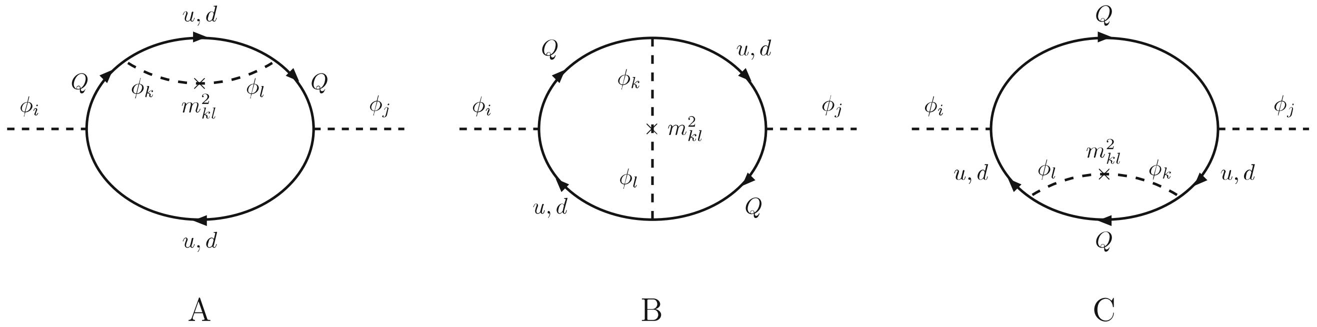

Fig. 1 Feynman diagrams contributing to the beta functions for the quadratic scalar coefficients involving only Yukawa interactions. The "×" symbol denotes a "mass insertion" corresponding to the m^2_{ij} coefficients

specific diagram topology. In this way, it is a simple exercise to write down the contributions from the "A" diagram to $\beta^{F,2L}_{m^2_{11}}$, obtaining

$$\begin{aligned}\beta^{F(A),2L}_{m^2_{11}} &= A\Big[\mathrm{Tr}(\Delta_1\Delta_1^\dagger\Delta_1\Delta_1^\dagger) + \mathrm{Tr}(\Gamma_1\Gamma_1^\dagger\Gamma_1\Gamma_1^\dagger)\\ &\quad + \mathrm{Tr}(\Gamma_1\Delta_1^\dagger\Delta_1\Gamma_1^\dagger) + \mathrm{Tr}(\Delta_1\Gamma_1^\dagger\Gamma_1\Delta_1^\dagger)\Big] m^2_{11}\\ &\quad + A\Big[\mathrm{Tr}(\Delta_1\Delta_2^\dagger\Delta_2\Delta_1^\dagger) + \mathrm{Tr}(\Gamma_1\Gamma_2^\dagger\Gamma_2\Gamma_1^\dagger)\\ &\quad + \mathrm{Tr}(\Gamma_1\Delta_2^\dagger\Delta_2\Gamma_1^\dagger) + \mathrm{Tr}(\Delta_1\Gamma_2^\dagger\Gamma_2\Delta_1^\dagger)\Big] m^2_{22}\\ &\quad - A\Big\{\Big[\mathrm{Tr}(\Delta_1\Delta_1^\dagger\Delta_2\Delta_1^\dagger) + \mathrm{Tr}(\Gamma_1\Gamma_1^\dagger\Gamma_2\Gamma_1^\dagger)\\ &\quad + \mathrm{Tr}(\Gamma_1\Delta_1^\dagger\Delta_2\Gamma_1^\dagger) + \mathrm{Tr}(\Delta_1\Gamma_1^\dagger\Gamma_2\Delta_1^\dagger)\Big] m^2_{12}\\ &\quad + \mathrm{h.c.}\Big\}. \end{aligned} \tag{B.1}$$

It is then trivial to obtain the contributions from the "A" diagram to $\beta^{F,2L}_{m^2_{22}}$, by taking the result above and performing the exchange $1 \leftrightarrow 2$ throughout, which results in

$$\begin{aligned}\beta^{F(A),2L}_{m^2_{22}} &= A\Big[\mathrm{Tr}(\Delta_2\Delta_1^\dagger\Delta_1\Delta_2^\dagger) + \mathrm{Tr}(\Gamma_2\Gamma_1^\dagger\Gamma_1\Gamma_2^\dagger)\\ &\quad + \mathrm{Tr}(\Gamma_2\Delta_1^\dagger\Delta_1\Gamma_2^\dagger) + \mathrm{Tr}(\Delta_2\Gamma_1^\dagger\Gamma_1\Delta_2^\dagger)\Big] m^2_{11}\\ &\quad + A\Big[\mathrm{Tr}(\Delta_2\Delta_2^\dagger\Delta_2\Delta_2^\dagger) + \mathrm{Tr}(\Gamma_2\Gamma_2^\dagger\Gamma_2\Gamma_2^\dagger)\\ &\quad + \mathrm{Tr}(\Gamma_2\Delta_2^\dagger\Delta_2\Gamma_2^\dagger) + \mathrm{Tr}(\Delta_2\Gamma_2^\dagger\Gamma_2\Delta_2^\dagger)\Big] m^2_{22}\\ &\quad - A\Big\{\Big[\mathrm{Tr}(\Delta_2\Delta_1^\dagger\Delta_2\Delta_2^\dagger) + \mathrm{Tr}(\Gamma_2\Gamma_1^\dagger\Gamma_2\Gamma_2^\dagger)\\ &\quad + \mathrm{Tr}(\Gamma_2\Delta_1^\dagger\Delta_2\Gamma_2^\dagger) + \mathrm{Tr}(\Delta_2\Gamma_1^\dagger\Gamma_2\Delta_2^\dagger)\Big]\\ &\quad \times m^2_{12} + \mathrm{h.c.}\Big\}. \end{aligned} \tag{B.2}$$

At this stage a direct calculation with the Yukawa structures from CP2 (Eq. (5.7)) or CP3 (Eq. (5.8)) shows that:

- The quantity in square brackets multiplying m^2_{11} in Eq. (B.1) is equal to the quantity in square brackets multiplying m^2_{22} in Eq. (B.2).
- The quantity in square brackets multiplying m^2_{22} in Eq. (B.1) is equal to the quantity in square brackets multiplying m^2_{11} in Eq. (B.2) – this equality may be seen even without using the specific CP2 or CP3 Yukawa structures, it is a direct consequence of the cyclical property of the trace of matrix products.
- If one sums Eqs. (B.1) and (B.2), the terms proportional to m^2_{12} cancel out – in fact, for the CP3 Yukawa textures, those terms are individually zero for each of the equations mentioned.

As a result, we obtain

$$\begin{aligned}\beta^{F(A),2L}_{m^2_{11}+m^2_{22}} &= A\Big[\mathrm{Tr}(\Delta_1\Delta_1^\dagger\Delta_1\Delta_1^\dagger) + \mathrm{Tr}(\Gamma_1\Gamma_1^\dagger\Gamma_1\Gamma_1^\dagger)\\ &\quad + \mathrm{Tr}(\Gamma_1\Delta_1^\dagger\Delta_1\Gamma_1^\dagger) + \mathrm{Tr}(\Delta_1\Gamma_1^\dagger\Gamma_1\Delta_1^\dagger)\\ &\quad + \mathrm{Tr}(\Delta_2\Delta_1^\dagger\Delta_1\Delta_2^\dagger) + \mathrm{Tr}(\Gamma_2\Gamma_1^\dagger\Gamma_1\Gamma_2^\dagger)\\ &\quad + \mathrm{Tr}(\Gamma_2\Delta_1^\dagger\Delta_1\Gamma_2^\dagger) + \mathrm{Tr}(\Delta_2\Gamma_1^\dagger\Gamma_1\Delta_2^\dagger)\Big]\\ &\quad \times \left(m^2_{11} + m^2_{22}\right) \end{aligned} \tag{B.3}$$

and again, as in Eq. (5.12), we see the proportionality to $(m^2_{11} + m^2_{22})$.

The same thing happens for the other diagrams from Fig. 1. For diagram "B", for instance – and summing both possibilities of direction of fermionic lines, as mentioned above –, one has

$$\begin{aligned}\beta^{F(B),2L}_{m^2_{11}} &= 2B\Big[\mathrm{Tr}(\Delta_1\Delta_1^\dagger\Delta_1\Delta_1^\dagger) + \mathrm{Tr}(\Gamma_1\Gamma_1^\dagger\Gamma_1\Gamma_1^\dagger)\\ &\quad + \mathrm{Tr}(\Gamma_1\Delta_1^\dagger\Delta_1\Gamma_1^\dagger) + \mathrm{Tr}(\Delta_1\Gamma_1^\dagger\Gamma_1\Delta_1^\dagger)\Big] m^2_{11}\\ &\quad + B\Big[\mathrm{Tr}(\Delta_1\Delta_2^\dagger\Delta_1\Delta_2^\dagger) + \mathrm{Tr}(\Gamma_1\Gamma_2^\dagger\Gamma_1\Gamma_2^\dagger)\\ &\quad + \mathrm{Tr}(\Gamma_1\Delta_2^\dagger\Delta_1\Gamma_2^\dagger) + \mathrm{Tr}(\Delta_1\Gamma_2^\dagger\Gamma_1\Delta_2^\dagger) + \mathrm{h.c.}\Big] m^2_{22}\\ &\quad - B\Big\{\Big[\mathrm{Tr}(\Delta_1\Delta_1^\dagger\Delta_1\Delta_2^\dagger) + \mathrm{Tr}(\Gamma_1\Gamma_1^\dagger\Gamma_1\Gamma_2^\dagger)\\ &\quad + \mathrm{Tr}(\Gamma_1\Delta_1^\dagger\Delta_1\Gamma_2^\dagger) + \mathrm{Tr}(\Delta_1\Gamma_1^\dagger\Gamma_1\Delta_2^\dagger)\Big] m^2_{12}\\ &\quad + \mathrm{h.c.}\Big\}. \end{aligned} \tag{B.4}$$

and

$$\beta^{F(B),2L}_{m^2_{22}} = B\Big[\mathrm{Tr}(\Delta_2\Delta_1^\dagger\Delta_2\Delta_1^\dagger) + \mathrm{Tr}(\Gamma_2\Gamma_1^\dagger\Gamma_2\Gamma_1^\dagger) + \mathrm{Tr}(\Gamma_2\Delta_1^\dagger\Delta_2\Gamma_1^\dagger) + \mathrm{Tr}(\Delta_2\Gamma_1^\dagger\Gamma_2\Delta_1^\dagger) + \mathrm{h.c.}\Big]\, m^2_{11} + B\Big[\mathrm{Tr}(\Delta_2\Delta_2^\dagger\Delta_2\Delta_2^\dagger) + \mathrm{Tr}(\Gamma_2\Gamma_2^\dagger\Gamma_2\Gamma_2^\dagger) + \mathrm{Tr}(\Gamma_2\Delta_2^\dagger\Delta_2\Gamma_2^\dagger) + \mathrm{Tr}(\Delta_2\Gamma_2^\dagger\Gamma_2\Delta_2^\dagger)\Big]\, m^2_{22} - B\Big\{\Big[\mathrm{Tr}(\Delta_2\Delta_1^\dagger\Delta_2\Delta_2^\dagger) + \mathrm{Tr}(\Gamma_2\Gamma_1^\dagger\Gamma_2\Gamma_2^\dagger) + \mathrm{Tr}(\Gamma_2\Delta_1^\dagger\Delta_2\Gamma_2^\dagger) + \mathrm{Tr}(\Delta_2\Gamma_1^\dagger\Gamma_2\Delta_2^\dagger)\Big]\, m^2_{12} + \mathrm{h.c.}\Big\}\,. \tag{B.5}$$

Once more, the terms proportional to m^2_{12} cancel when summing both beta functions, and one finds

$$\beta^{F(B),2L}_{m^2_{11}+m^2_{22}} = 2B\Big[\mathrm{Tr}(\Delta_1\Delta_1^\dagger\Delta_1\Delta_1^\dagger) + \mathrm{Tr}(\Gamma_1\Gamma_1^\dagger\Gamma_1\Gamma_1^\dagger) + \mathrm{Tr}(\Gamma_1\Delta_1^\dagger\Delta_1\Gamma_1^\dagger) + \mathrm{Tr}(\Delta_1\Gamma_1^\dagger\Gamma_1\Delta_1^\dagger) + \mathrm{Re}\Big\{\mathrm{Tr}(\Delta_2\Delta_1^\dagger\Delta_2\Delta_1^\dagger) + \mathrm{Tr}(\Gamma_2\Gamma_1^\dagger\Gamma_2\Gamma_1^\dagger) + \mathrm{Tr}(\Gamma_2\Delta_1^\dagger\Delta_2\Gamma_1^\dagger) + \mathrm{Tr}(\Delta_2\Gamma_1^\dagger\Gamma_2\Delta_1^\dagger)\Big\}\Big] \times \left(m^2_{11} + m^2_{22}\right). \tag{B.6}$$

A similar exercise may be undertaken for the diagram "C", resulting in

$$\beta^{F(C),2L}_{m^2_{11}+m^2_{22}} = C\Big[\mathrm{Tr}(\Delta_1\Delta_1^\dagger\Delta_1\Delta_1^\dagger) + \mathrm{Tr}(\Gamma_1\Gamma_1^\dagger\Gamma_1\Gamma_1^\dagger) + \mathrm{Tr}(\Gamma_1\Delta_1^\dagger\Delta_1\Gamma_1^\dagger) + \mathrm{Tr}(\Delta_1\Gamma_1^\dagger\Gamma_1\Delta_1^\dagger) + \mathrm{Tr}(\Delta_1\Delta_1^\dagger\Delta_2\Delta_2^\dagger) + \mathrm{Tr}(\Gamma_1\Gamma_1^\dagger\Gamma_2\Gamma_2^\dagger) + \mathrm{Tr}(\Delta_1\Gamma_1^\dagger\Gamma_2\Delta_2^\dagger) + \mathrm{Tr}(\Gamma_1\Delta_1^\dagger\Delta_2\Gamma_2^\dagger)\Big] \times \left(m^2_{11} + m^2_{22}\right). \tag{B.7}$$

Finally, we verified that diagrams like those of Fig. 1 with mass insertions on external lines instead of internal ones also yield Yukawa structures such that the conclusions reached above also hold: the beta function for $(m^2_{11} + m^2_{22})$ is proportional to that same quantity.

- Yukawa and quartic scalar coupling contributions

The beta functions for m^2_{11} and m^2_{22} also receive contributions involving Yukawa and scalar quartic interactions, such as those exemplified in the diagram of Fig. 2. All such contributions will be proportional to the quadratic combinations of Yukawa couplings appearing in Eqs. (5.10) and (5.11), multiplied by a λ_a quartic coupling and a common factor "D", in which we include the coefficient of the pole in $1/\epsilon$ and diagram symmetry and colour factors. As an example,

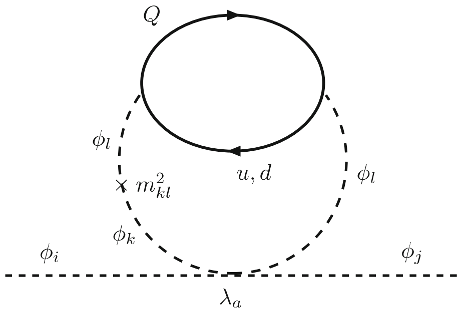

Fig. 2 Example of Feynman diagram contributing to the beta functions for the quadratic scalar coefficients involving both Yukawa and scalar quartic interactions. The "×" symbol denotes a "mass insertion" corresponding to the m^2_{ij} coefficients

consider the contribution to $\beta_{m^2_{11}}$ from this diagram which is proportional to λ_1: there will be a "mass insertion" m^2_{11}, which necessitates Yukawa interactions such as $\Gamma_1\Gamma_1^\dagger$, which preserve the scalar doublet index, and another "mass insertion" m^2_{12}, for which the Yukawa interactions must swap the doublet index from 1 to 2. This leads to

$$\beta_{m^2_{11}} = \ldots + D\,\lambda_1\left(\Big[3\,\mathrm{Tr}(\Delta_1\Delta_1^\dagger) + 3\,\mathrm{Tr}(\Gamma_1\Gamma_1^\dagger) + \mathrm{Tr}(\Pi_1\Pi_1^\dagger)\Big]\, m^2_{11} - \Big\{\Big[3\,\mathrm{Tr}(\Delta_1\Delta_2^\dagger) + 3\,\mathrm{Tr}(\Gamma_1\Gamma_2^\dagger) + \mathrm{Tr}(\Pi_1\Pi_2^\dagger)\Big]\, m^2_{12} + \mathrm{h.c.}\Big\}\right). \tag{B.8}$$

Analogously, $\beta_{m^2_{22}}$ will have a term proportional to λ_2, given by

$$\beta_{m^2_{22}} = \ldots + D\,\lambda_2\left(\Big[3\,\mathrm{Tr}(\Delta_2\Delta_2^\dagger) + 3\,\mathrm{Tr}(\Gamma_2\Gamma_2^\dagger) + \mathrm{Tr}(\Pi_2\Pi_2^\dagger)\Big]\, m^2_{22} - \Big\{\Big[3\,\mathrm{Tr}(\Delta_2\Delta_1^\dagger) + 3\,\mathrm{Tr}(\Gamma_2\Gamma_1^\dagger) + \mathrm{Tr}(\Pi_2\Pi_1^\dagger)\Big] \times\ m^2_{12} + \mathrm{h.c.}\Big\}\right). \tag{B.9}$$

Given the results of Eq. (5.11), the terms proportional to m^2_{12} vanish; and since for the r_0 symmetry one must have $\lambda_1 = \lambda_2$, given the results from Eq. (5.10) we see that once more the sum of these two contributions yields $\beta_{m^2_{11}+m^2_{22}} \propto m^2_{11}+m^2_{22}$.

Much in the same manner, it is simple to obtain the terms proportional to λ_3,

$$\beta_{m^2_{11}} = \ldots + D\,\lambda_3\left(\Big[3\,\mathrm{Tr}(\Delta_2\Delta_2^\dagger) + 3\,\mathrm{Tr}(\Gamma_2\Gamma_2^\dagger) + \mathrm{Tr}(\Pi_2\Pi_2^\dagger)\Big]\, m^2_{22} - \Big\{\Big[3\,\mathrm{Tr}(\Delta_2\Delta_1^\dagger) + 3\,\mathrm{Tr}(\Gamma_2\Gamma_1^\dagger) + \mathrm{Tr}(\Pi_2\Pi_1^\dagger)\Big]\, m^2_{12} + \mathrm{h.c.}\Big\}\right) \tag{B.10}$$

and

$$\beta_{m^2_{22}} = \ldots + D\,\lambda_3\left(\Big[3\,\mathrm{Tr}(\Delta_1\Delta_1^\dagger) + 3\,\mathrm{Tr}(\Gamma_1\Gamma1^\dagger) + \mathrm{Tr}(\Pi_1\Pi_1^\dagger)\Big]\, m^2_{11} - \Big\{\Big[3\,\mathrm{Tr}(\Delta_2\Delta_1^\dagger) + 3\,\mathrm{Tr}(\Gamma_2\Gamma_1^\dagger) + \mathrm{Tr}(\Pi_2\Pi_1^\dagger)\Big]\, m^2_{12} + \mathrm{h.c.}\Big\}\right), \tag{B.11}$$

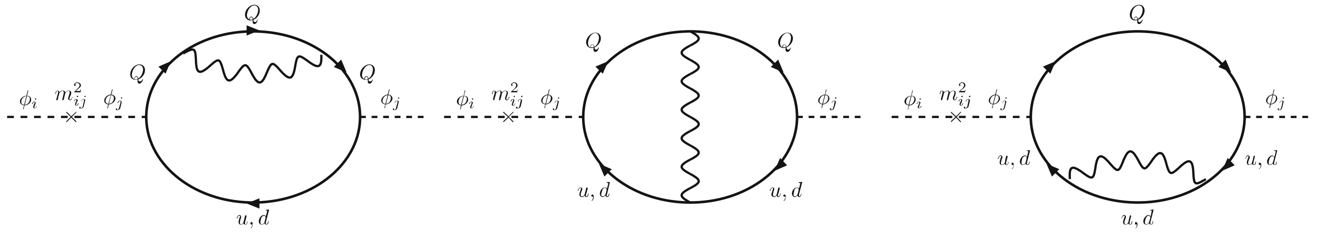

Fig. 3 Example of Feynman diagram contributing to the beta functions for the quadratic scalar coefficients involving both Yukawa and gauge interactions. The "×" symbol denotes a "mass insertion" corresponding to the m^2_{ij} coefficients

and again we see that the terms in m^2_{12} vanish and both contributions yield $\beta_{m^2_{11}+m^2_{22}} \propto m^2_{11} + m^2_{22}$. The same will be valid for terms involving the couplings λ_4 and λ_5. The couplings λ_6 and λ_7 are a more amusing situation, the former only contributing to $\beta_{m^2_{11}}$ and the latter only to $\beta_{m^2_{22}}$, in such a way that

$$\beta_{m^2_{11}} = \ldots + D\,\lambda_6 \left(\left[3\,\mathrm{Tr}(\Delta_1\Delta_2^\dagger) + 3\,\mathrm{Tr}(\Gamma_1\Gamma_2^\dagger) + \mathrm{Tr}(\Pi_1\Pi_2^\dagger) + \mathrm{h.c.} \right] (m^2_{11} + m^2_{22}) - \left\{ \left[3\,\mathrm{Tr}(\Delta_2\Delta_2^\dagger) + 3\,\mathrm{Tr}(\Gamma_2\Gamma_2^\dagger) + \mathrm{Tr}(\Pi_2\Pi_2^\dagger) \right] \times\, m^2_{12} + \mathrm{h.c.} \right\} \right) \tag{B.12}$$

and

$$\beta_{m^2_{22}} = \ldots + D\,\lambda_7 \left(\left[3\,\mathrm{Tr}(\Delta_2\Delta_1^\dagger) + 3\,\mathrm{Tr}(\Gamma_2\Gamma_1^\dagger) + \mathrm{Tr}(\Pi_2\Pi_1^\dagger) + \mathrm{h.c.} \right] (m^2_{11} + m^2_{22}) - \left\{ \left[3\,\mathrm{Tr}(\Delta_1\Delta_1^\dagger) + 3\,\mathrm{Tr}(\Gamma_1\Gamma_1^\dagger) + \mathrm{Tr}(\Pi_1\Pi_1^\dagger) \right] \times\, m^2_{12} + \mathrm{h.c.} \right\} \right). \tag{B.13}$$

We see that now it is the terms proportional to m^2_{11} and m^2_{22} that vanish due to Eq. (5.12), and the Yukawa coupling structures multiplying m^2_{12} are identical in both equations above. This then leads to

$$\beta_{m^2_{11}+m^2_{22}} = \ldots + D\, \left\{ (\lambda_6 + \lambda_7) \left[\text{Yukawa couplings} \right] \times\, m^2_{12} + \mathrm{h.c.} \right\} \tag{B.14}$$

which of course is equal to zero since the r_0 symmetry implies $\lambda_7 = -\lambda_6$.

Therefore, all contributions to $\beta_{m^2_{11}+m^2_{22}}$ involving Yukawa and quartic scalar couplings are proportional to $m^2_{11} + m^2_{22}$.

- Yukawa and gauge coupling contributions

Finally, the last contributions involve products of Yukawa and gauge couplings, and the demonstration is trivial: considering the mass insertions possible in each case and the Yukawa structures allowed for each case (see Fig. 3), we will have

$$\beta_{m^2_{11}} = \ldots + G_1 \left[3\,\mathrm{Tr}(\Delta_1\Delta_1^\dagger) + 3\,\mathrm{Tr}(\Gamma_1\Gamma_1^\dagger) + \mathrm{Tr}(\Pi_1\Pi_1^\dagger) \right] m^2_{11} - G_2 \left\{ \left[3\,\mathrm{Tr}(\Delta_1\Delta_2^\dagger) + 3\,\mathrm{Tr}(\Gamma_1\Gamma_2^\dagger) + \mathrm{Tr}(\Pi_1\Pi_2^\dagger) \right] \times\, m^2_{12} + \mathrm{h.c.} \right\}, \tag{B.15}$$

where we include all gauge, symmetry, pole factor in G_1 and G_2. For $\beta_{m^2_{22}}$ the result is quite simply

$$\beta_{m^2_{22}} = \ldots + G_1 \left[3\,\mathrm{Tr}(\Delta_2\Delta_2^\dagger) + 3\,\mathrm{Tr}(\Gamma_2\Gamma_2^\dagger) + \mathrm{Tr}(\Pi_2\Pi_2^\dagger) \right] m^2_{22} - G_2 \left\{ \left[3\,\mathrm{Tr}(\Delta_2\Delta_1^\dagger) + 3\,\mathrm{Tr}(\Gamma_2\Gamma_1^\dagger) + \mathrm{Tr}(\Pi_2\Pi_1^\dagger) \right] \times\, m^2_{12} + \mathrm{h.c.} \right\}. \tag{B.16}$$

Now, since Φ_1 and Φ_2 have exactly the same quantum numbers, gauge contributions to the beta functions of m^2_{11} and m^2_{22} will per force be identical, which justifies the fact that the factors G_1 and G_2 are repeated in the above equations. Equation (5.12) makes all terms proportional to m^2_{12} vanish, and Eq. (5.10) makes the terms in square brackets multiplying G_1 identical in both equations. Yet again, we obtain $\beta_{m^2_{11}+m^2_{22}} \propto m^2_{11} + m^2_{22}$.

To conclude, when one considers the CP2 or CP3 Yukawa matrices of Eqs. (5.7) and (5.8), the beta functions of the scalar squared mass coefficients are such that, at least to two-loop order, one has

$$\beta^{2L}_{m^2_{11}+m^2_{22}} = \left[\text{Scalar, gauge, Yukawa couplings} \right] \times (m^2_{11} + m^2_{22}), \tag{B.17}$$

so that the condition $m^2_{11} + m^2_{22} = 0$ is preserved under RG running.[21]

[21] Provided the relations $\lambda_1 = \lambda_2$ and $\lambda_7 = -\lambda_6$, which complete the r_0 symmetry conditions, are also obeyed, of course.

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