# VOLATILITY REGIMES FOR THE VIX INDEX


Jacinto Marabel Romo*

*BBVA*

*Vía de los Poblados s/n, 28033, Madrid*

*email: jacinto.marabel@grupobbva.com*

*and*

*University Institute for Economic and Social Analysis, University of Alcalá,*

*Plaza de la Victoria 2, 28802, Alcalá de Henares, Madrid*



## Abstract:

This article presents a Markov chain framework to characterize the behavior of the CBOE Volatility Index (VIX index). Two possible regimes are considered: high volatility and low volatility. The specification accounts for deviations from normality and the existence of persistence in the evolution of the VIX index. Since the time evolution of the VIX index seems to indicate that its conditional variance is not constant over time, I consider two different versions of the model. In the first one, the variance of the index is a function of the volatility regime, whereas the second version includes an autoregressive conditional heteroskedasticity (ARCH) specification for the conditional variance of the index.



**Keywords:** VIX index, Markov chain, realized volatility, implied volatility, volatility regimes.

**JEL:** C22, G12, G13.


* The content of this paper represents the author's personal opinion and does not reflect the views of BBVA.

## 1. INTRODUCTION

In the Black-Scholes (1973) model the instantaneous volatility corresponding to the underlying asset price process is assumed to be constant. However, Fisher Black (1976) stated that if we use the standard deviation of possible future returns on a stock as a measure of its volatility, then it is not reasonable to take that volatility as constant over time. In addition, empirical evidence shows that implied volatility, far from remain static through time, evolves stochastically. Examples of this fact can be found in Franks and Schwartz (1991), Avellaneda and Zhu (1997), Derman (1999), Bakshi, Cao and Chen (2000), Cont and da Fonseca (2001), Cont and da Fonseca (2002), Daglish, Hull and Suo (2007) and Carr and Wu (2009).

As evidenced by Carr and Lee (2009), in recent years new derivatives assets are emerging. These derivatives have some measure of volatility as underlying asset. In particular, in 2004 the Chicago Board Options Exchange (CBOE) introduced futures traded on the CBOE Volatility Index (VIX) and in 2006 options on that index. The VIX index started to be calculated in 1993 and was originally designed to measure the market's expectation of 30-day at-the-money implied volatility. But with the new methodology[1] implemented in 2003, the squared of the VIX index approximates the variance swap rate or delivery price of a variance swap, obtained from the European options corresponding to the Standard and Poor's 500 index with maturity within one month. The variance swap is a forward contract on the annualized realized variance of a certain asset. As with all forward contracts or swaps, the fair value of variance at any time is the delivery price that makes the swap currently have zero value. Therefore, the absence of arbitrage opportunities implies that the variance swap rate equals the expected value of the realized variance under the risk-neutral probability measure.

Carr and Wu (2006) find the existence of a negative strong correlation between the changes in the VIX volatility index and the performance corresponding to the Standard and Poor's s 500 index. This fact indicates that the volatility tends to be higher when the equity market falls.

[1] For a definition of the methodology of the VIX index, see CBOE (2009) and Carr and Wu (2006).

As said previously, the VIX index squared approximates the 30-day variance swap rate corresponding to the Standard and Poor's 500 index. This variable evolves stochastically through time and usually exhibits relatively persistent changes of level generated by news about the evolution of the economy and/or financial crisis. In this sense, Bali and Ozgur (2008) show that the existence of persistence and mean reversion is quite relevant in stock market volatility. To account for this persistence Grünbichler and Longstaff (1996) used the square root process to modelize the behavior of a standard deviation index such us the VIX index. Detemple and Osakwe (2000) proposed a log-normal Ornstein-Uhlenbeck process. Vasicek (1977) used Ornstein-Uhlenbeck process to describe the movement of short term interest rates and Phillips (1972) showed that the exact discrete model corresponding to this specification is given by a Gaussian first order autoregressive process (AR(1) process) if the variable is sampled at equally spaced discrete intervals.

The time evolution of the VIX index suggests that it could be possible to represent the behavior of this variable using a model in which the process for the VIX index can be in a regime of high volatility or alternatively, in a low volatility regime in such a way that the change between the two regimes is the result of a Markov chain process. Hamilton (1989) established a similar approach to represent the evolution of the economy. In his model the output mean growth rate depends on whether the economy is in a phase of expansion or in a phase of recession. The model postulates the existence of a discrete and unobservable variable, named state variable or regime variable, which determines the state of the economy at each point in time.

This article introduces a regime-switching framework to characterize the evolution of the VIX index that postulates the existence of two possible regimes: high volatility and low volatility and assumes that the state variable governing the transition between the two regimes is the result of a Markov process. The specification considers a t-distribution and, therefore, allows for deviations from normality in the distribution corresponding to VIX index. Note that the t-distribution includes the normal distribution as the limiting case where the degrees of freedom tend to infinity. To account of the observed persistence in the evolution of the VIX index, I postulate an AR(1) specification where the mean corresponding to that index depends on the state of the nature. Since the time evolution of the VIX index seems to indicate that its conditional variance is not constant over time, I consider two different versions of the model. In the first one, the variance of the index is also a function of the state of the nature, whereas the second version includes an autoregressive conditional heteroskedasticity (ARCH) specification for the conditional variance of the VIX index. For comparison, I also consider a standard AR specification for the mean of the VIX index that allows for ARCH effects in the conditional variance.

The regime-switching model allows estimating the average persistence of each regime and the probability of being in a particular regime. This information is a useful tool for investment decisions, as well as for hedging purposes regarding the volatility of a certain asset. The empirical results show that the model is able to characterize the volatility regimes corresponding to the VIX index quite accurately. Moreover, the estimated volatility corresponding to the VIX index is much higher in the high volatility regime.

Dueker (1997) points out that the volatility of financial assets usually exhibits discrete shifts and mean reversion. This author applies a GARCH/Markov-switching framework, using daily percentage changes in the Standard and Poor's 500 index, to characterize the evolution of the VIX index. The model presented in this article differs from the approach of Dueker (1997) in that I postulate a specification for the VIX index rather than for the returns of the stock market index.

The rest of the article is structured as follows. Section 2 focuses on the features and calculations of the VIX index and examines the data. Section 3 presents the specifications used in this article to characterize the evolution of the VIX volatility index. Section 4 shows the estimation results and provides in-sample and out-of-sample performance measures for each model considered. Finally, section 5 provides concluding remarks.

## 2. CONSTRUCTION OF THE VIX INDEX

### 2.1 THE VIX INDEX AND THE VARIANCE SWAP RATE

As Carr and Wu (2006) point out, the squared of the VIX index approximates the 30-day variance swap rate of variance swaps corresponding to the Standard and Poor's 500 index. Formally, a variance swap is a forward contract on the annualized historical variance. Its payoff at maturity is given by $N\left(\sigma_R^2 - VSR\right)$, where $\sigma_R^2$ represents the annualized realized variance during the lifetime of the contract, $VSR$ is the variance swap rate and $N$ is the notional amount expressed in currency units.

Let us assume that the underlying asset, whose time $t$ price is denoted by $S_t$, follows a geometric Brownian motion where the drift $\mu_t$, as well as the instantaneous volatility $\sigma_t$ may depend on time and other stochastic variables:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t^P$$

Where $W_t^P$ is a Wiener process associated to the real probability measure $P$. A particular case is the Black-Scholes (1973) model, where $\mu_t$ and $\sigma_t$ are assumed to be constant. The realized variance between the instant $t=0$ and the instant $t=T$ is defined by the following expression:

$$\Psi = \frac{1}{T}\int_0^T \sigma_t^2 dt$$

Let $vs_0$ denote the time $t=0$ value corresponding to the variance swap and let us assume that the notional amount equals one. It is possible to use the fundamental theorem of asset pricing to value this contract under the risk neutral probability measure $Q$:

$$vs_0 = P\left(0,T\right)E_Q\left[\Psi - VSR\right]$$

where $Q$ is the probability measure such that asset prices expressed in terms of the current account are martingales and $P\left(0,T\right)$ is the time $t=0$ price of a zero coupon bond which pays a currency unit at time $t=T$. The variance swap rate $VSR$ is chosen so that the net present value of the contract equals zero. Thus, the following condition must hold:

$$E_Q[\Psi] = \frac{1}{T} E_Q\left[\int_0^T \sigma_t^2 dt\right] = VSR$$

Therefore, we have to set a replication strategy that allows replicating the realized variance. Demeterfi et al. (1999) show that it is possible to obtain the following replicating portfolio corresponding to the variance swap rate[2]:

$$\begin{aligned} VSR = \frac{2}{T}\left[(r-q)T - \left(\frac{F_{o,T}}{S_*} - 1\right) - \ln\left(\frac{S_*}{S_0}\right)\right] + \\ \frac{1}{P(0,T)} \frac{2}{T}\left[\int_0^{S_*} \frac{P_{0T}(K)}{K^2} dK + \int_{S_*}^{\infty} \frac{C_{0T}(K)}{K^2} dK\right] \end{aligned} \quad (1)$$

Where the continuously compounded risk-free rate *r*, as well as the dividend yield *q*, are assumed to be constant. $F_{o,T} = \frac{S_0 e^{-qT}}{P(0,T)}$ is the time $t = 0$ value of a forward contract on the underlying asset with maturity $t = T$; $C_{0T}(K)$ is the time $t = 0$ value of a European call with maturity $t = T$ and strike price *K*, whereas $P_{0T}(K)$ denotes the price corresponding to a European put with the same features. Finally, $S_*$ denotes the strike price which represents the limit between liquid calls and puts.

The squared root of equation (1) can be interpreted as the continuous-time counterpart to the formula used by the CBOE for the VIX index calculation. This calculation is based on a weighted average of prices of European options corresponding to different strike prices which are associated with different implied volatility levels corresponding to the Standard and Poor's 500 index.

## 2.2 DATA

I consider monthly data corresponding to the VIX index during the period January 1990 to September 2010. The data are available at www.cboe.com/micro/vix/historical.aspx. As previously said, the methodology to calculate the VIX index was modified in 2003. The

[2] Although Demeterfi et al. (1999) do not consider the existence of dividends, in this article I present the variance swap rate obtained under the assumption of a continuous dividend yield for the underlying asset.

CBOE has used historical data corresponding to listed options on the Standard and Poor's 500 index, to generate historical prices for the VIX index with the new methodology.

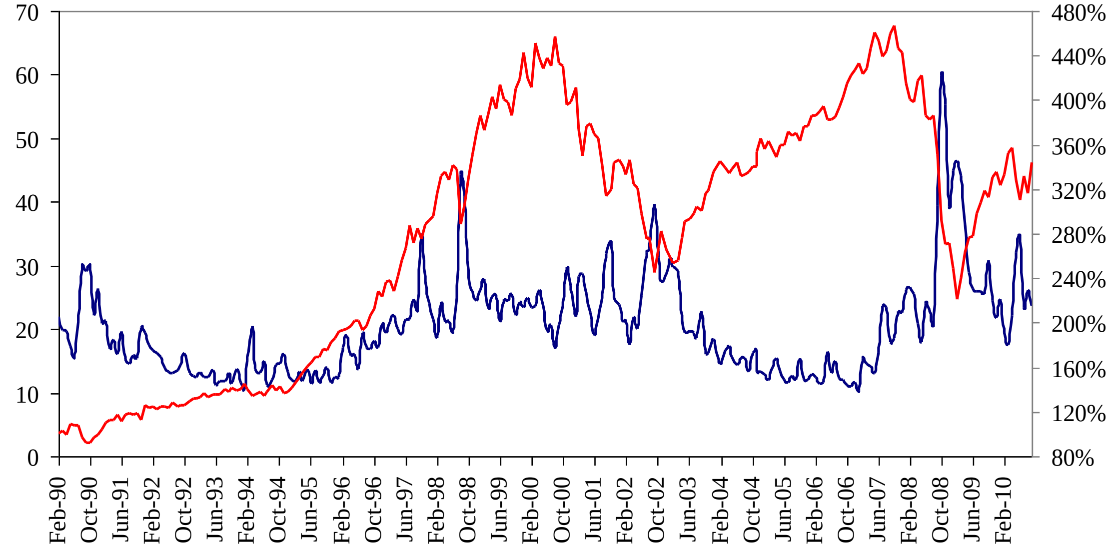


**Figure 1:** Monthly evolution of the VIX index (left axis) and the Standard and Poor's 500 index (right axis) during the period February 1990 to September 2010. The line denoted Standard and Poor's 500 index captures the month-end prices of the index, obtained from Bloomberg, as a percentage of the month-end value corresponding to February 1990. The data corresponding to the VIX index are available at www.cboe.com/micro/vix/historical.aspx.

Figure 1 shows the monthly evolution of the VIX index during the analyzed period, as well as the performance of the Standard and Poor's 500 index. The left axis accounts for the values of the VIX index, whereas the right axis captures the month-end values associated to the Standard and Poor's 500 index as a percentage of the month-end price corresponding to February 1990. As we can see from the figure, both indexes move in opposite directions. The existence of negative correlation between asset returns and volatilities accounts for the leverage effect introduced by Black (1976): for a given debt level, a decrease in the equity value implies greater leverage for the companies, which leads to an increase of the risk and volatility levels. Other explanations for the existence of this negative correlation can be found in Campbell and Kyle (1993) and Bekaert and Wu (2000).

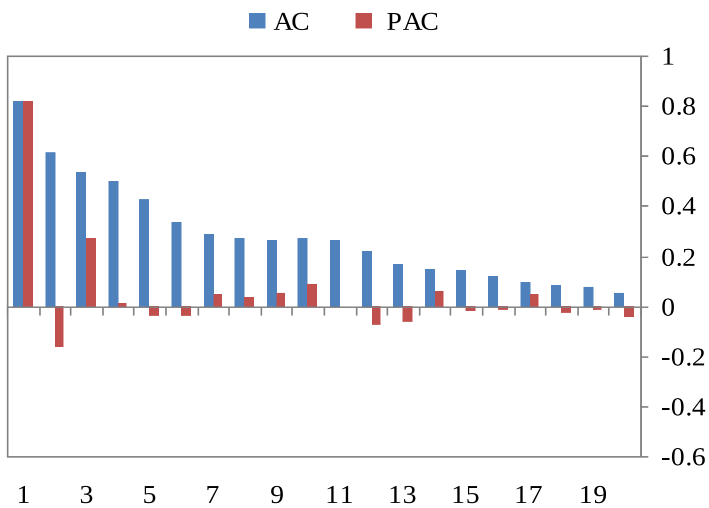


**Figure 2:** Autocorrelation (AC) and partial autocorrelation (PAC) functions corresponding to the VIX index squared.

Figure 1 also shows that the VIX index displays a relatively persistent switching of regime. Furthermore, this index seems to be more volatile in those periods in which the index reaches its highest values. These facts indicate that it might be appropriate to characterize de evolution of that index using a regime-switching model in which the variable that governs the transition between regimes is the result of a Markov chain.

Although not reported in the article for the sake of brevity, I carried out unit root tests and the null hypothesis of the existence of a unit root in the level of the VIX index was rejected. This result is in line with the empirical findings of Harvey and Whaley (1992) regarding the mean reversion of volatility.

On the other hand, figure 2 reports the sample autocorrelation and partial autocorrelation functions associated with the VIX index squared. The figure shows a decrease in the autocorrelation function, whereas the partial autocorrelation function tends quickly to zero for lags of order higher than one. In this sense, a Markov-switching specification for the mean of the VIX index combined with an ARCH model specification for its conditional variance can be a good candidate to modelize the evolution of this index. The next section presents the specifications of the models used in this article to represent the behavior of the VIX index.

## 3. MODEL SPECIFICATIONS FOR THE VIX INDEX

### 3.1 STANDARD SPECIFICATION

As starting point I consider an AR(1) specification to characterize the time evolution of the VIX index, based on the theoretical models postulated by Grünbichler and Longstaff (1996) and Detemple and Osakwe (2000). As figure 1 shows, the volatility of the VIX index seems to be time-varying and periods of high volatility tend to cluster. Moreover, figure 2 indicates the existence of serial correlation for the VIX index squared. To capture these effects, I also consider an ARCH(1) model as introduced by Engle (1982) and extended to generalized ARCH (GARCH) in Bollerslev[3] (1986).

Let $V_t$ represent the time *t* value of the VIX index. The first specification considered to characterize its evolution is given by the following equation:

$$\begin{aligned} V_t &= \mu + \phi\left(V_{t-1} - \mu\right) + \varepsilon_t \\ \varepsilon_{t|\Omega_{t-1}} &\sim N\left(0, \sigma_t^2\right) \\ \sigma_t^2 &\equiv E\left[\varepsilon_t^2 \mid \Omega_{t-1}\right] = \alpha + \theta\varepsilon_{t-1}^2 \end{aligned} \tag{2}$$

where $\Omega_{t-1}$ represents the observations obtained through date *t-1*. Under this model the unconditional mean corresponding to the VIX index is given by $\mu$, $\phi$ represents the degree of persistence and the unconditional variance is given by:

$$\sigma^2 = \frac{\alpha}{1-\theta} \tag{3}$$

I call the specification of equation (2) the standard ARCH model.

[3] Although not reported in the article, I also considered ARMA specifications for the mean, as well as GARCH specifications for the conditional variance but some of the coefficients were not significantly different from zero and the specifications did not provide improvements in the results.

### 3.2 REGIME-SWITCHING MODEL SPECIFICATIONS FOR THE VIX INDEX

I now consider a model in which the mean value of the index at every point in time depends on the state variable $z_t$. I consider two possible regimes or states: low volatility $(z_t = 1)$ and high volatility $(z_t = 2)$. Moreover, I postulate a model for the state variable $z_t$ in which the state of the world is the result of an unobservable Markov chain process, with $z_t$ and $\varepsilon_r$ independent for every *t* and *r*. The Markov process does not depend on the past values of $V_t$:

$$p\left(z_t = j \mid z_{t-1} = i, \Omega_{t-1}\right) = p\left(z_t = j \mid z_{t-1} = i\right) = p_{ij}$$

As Hamilton (1994) points out, the advantage of using a specification based on Markov chains is its great flexibility, since using different combinations of parameters it is possible to capture a broad range of patterns of behavior.

The model specification assumes a student-t error distribution. Note that, in case of normality, a large innovation in the low volatility period will lead to a switch to the high-volatility regime earlier, even if it is a single outlier in an otherwise tranquil period. Hence, this article considers a t-distribution that enhances the stability of the regimes and includes the normal distribution as the limiting case where the degrees of freedom tend to infinity. Therefore, the general specification of the model is:

$$\begin{aligned} V_t &= \mu_{z_t} + \phi\left(V_{t-1} - \mu_{z_{t-1}}\right) + \varepsilon_t \\ \varepsilon_{t|\Omega_{t-1}} &\sim Student - t\left(\mu = 0, \sigma_t^2, \nu\right) \end{aligned} \tag{4}$$

where the Student's t-distribution is given by:

$$\begin{gathered} \varphi\left(x_t \mid \mu, \sigma_t^2, \nu; \Omega_{t-1}\right) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\lambda_t}{\pi\nu}\right)^{\frac{1}{2}} \left(1 + \frac{\lambda_t\left(x_t - \mu\right)^2}{\nu}\right)^{-\frac{\nu+1}{2}} \\ \lambda_t = \frac{\nu}{\nu-2}\frac{1}{\sigma_t^2} \end{gathered} \tag{5}$$

where $\nu$ represents the degrees of freedom, $\mu$ is the location parameter and $\sigma_t^2$ denotes the scale parameter. The Student's t-distribution verifies that:

$$E\left[x_t \mid \Omega_{t-1}\right] = \mu \quad \text{for } \nu > 1$$
$$Var\left[x_t \mid \Omega_{t-1}\right] = \sigma_t^2 \quad \text{for } \nu > 2$$

Note that for the particular case of $\nu$ =1, the t-distribution reduces to the Cauchy distribution. The model of equation (4) considers an AR(1) specification for the level of the VIX index, where the mean value of the index is a function of the state of the nature. Regarding the specification for its conditional variance, I consider two alternative models. Under the first one, the conditional variance is also a function of the state of the nature, whereas the second model combines the Markov chain setting with mean reversion for the level of the VIX index and an ARCH(1) specification for its conditional variance.

Under the first version of the model, called Markov-switching in mean and variance (MSMV) model, the conditional variance is given by:

$$\sigma_t^2 = \sigma_{z_t}^2 \tag{6}$$

Under the second version of the model, denoted as Markov-switching in mean and ARCH in variance (MSM-ARCHV) model, the conditional variance takes the following form:

$$\sigma_t^2 = \alpha + \theta \varepsilon_{t-1}^2 \tag{7}$$

In both cases, it is possible to define a new regime variable $s_t$ as follows:

$$s_t = \begin{cases} 1 \text{ if } z_t = 1 \text{ and } z_{t-1} = 1 \\ 2 \text{ if } z_t = 2 \text{ and } z_{t-1} = 1 \\ 3 \text{ if } z_t = 1 \text{ and } z_{t-1} = 2 \\ 4 \text{ if } z_t = 2 \text{ and } z_{t-1} = 2 \end{cases}$$

Therefore, the state variable $s_t$ has the following transition matrix:

$$P = \begin{pmatrix} p_{11} & 0 & p_{11} & 0 \\ 1 - p_{11} & 0 & 1 - p_{11} & 0 \\ 0 & 1 - p_{22} & 0 & 1 - p_{22} \\ 0 & p_{22} & 0 & p_{22} \end{pmatrix}$$

Let $\omega$ denote the parameter vector. In the case of the MSMV model this vector will take the form $\omega = \left(\mu_1, \mu_2, \phi, \sigma_1^2, \sigma_2^2, p_{11}, p_{22}, \nu\right)'$, whereas in the case of the MSM-ARCHV model, the parameter vector is given by $\omega = \left(\mu_1, \mu_2, \phi, \alpha, \theta, p_{11}, p_{22}, \nu\right)'$. The regime-switching models can be estimated by maximum likelihood. The appendix A provides detailed derivations of the elements used in the estimation algorithm.

**Probability of being in each regime based on data obtained through the previous period**

Let $h_{t+1|t}^{j}$ denote the probability of being in regime *j* in period *t+1* given observations obtained though date *t*. This probability is given by:

$$h_{t+1|t}^{j} \equiv p\left(s_{t+1} = j \mid \Omega_t;\omega\right) = \sum_{i=1}^{4} p\left(s_{t+1} = j \mid s_t = i, \Omega_t;\omega\right) p\left(s_t = i \mid \Omega_t;\omega\right)$$

$$h_{t+1|t}^{j} = \sum_{i=1}^{4} p\left(s_{t+1} = j \mid s_t = i, \Omega_t;\omega\right) h_{t|t}^{i} \quad j=1,2,3,4.$$

In vector form the previous expression reduces to:

$$h_{t+1|t} = P h_{t|t} \tag{8}$$

where $h_{t+1|t}$ and $h_{t|t}$ are $(4\times 1)$ vectors.

**Log-likelihood function for** $V_t$

Let $\ln L(\omega)$ denote the log-likelihood function evaluated at the true parameter vector. Appendix A shows that this function takes the following form:

$$L(\omega) = \sum_{t=1}^{T} \ln\left[\, 1'\left(h_{t|t-1} \circ k_t\right)\right] \tag{9}$$

where $1$ is a $(4\times 1)$ vector of ones, the symbol $\circ$ represents element-by-element multiplication and $k_t$ is another $(4\times 1)$ vector, which includes the density functions corresponding to the VIX index given the four possible values for the state variable $s_t$. Hence, $k_t^j = f\left(V_t \mid s_t = j, \Omega_{t-1};\omega\right)$ is given by:

$$f\left(V_t \mid s_t = 1, \Omega_{t-1};\omega\right) = \varphi\left(V_t \mid \mu_1 + \phi\left(V_{t-1} - \mu_1\right), \sigma_t^2, \nu; \Omega_{t-1}\right)$$
$$f\left(V_t \mid s_t = 2, \Omega_{t-1};\omega\right) = \varphi\left(V_t \mid \mu_2 + \phi\left(V_{t-1} - \mu_1\right), \sigma_t^2, \nu; \Omega_{t-1}\right)$$
$$f\left(V_t \mid s_t = 3, \Omega_{t-1};\omega\right) = \varphi\left(V_t \mid \mu_1 + \phi\left(V_{t-1} - \mu_2\right), \sigma_t^2, \nu; \Omega_{t-1}\right)$$
$$f\left(V_t \mid s_t = 4, \Omega_{t-1};\omega\right) = \varphi\left(V_t \mid \mu_2 + \phi\left(V_{t-1} - \mu_2\right), \sigma_t^2, \nu; \Omega_{t-1}\right)$$

where $\sigma_t^2$ is given by equation (6) for the MSMV model and it is given by expression (7) for the MSM-ARCHV model.

**Probability of being in each regime based on data obtained through the current period**

Appendix A shows that it is possible to obtain the following expression for the probability of being in regime *j* in period *t*, given observations obtained though that date $h_{t|t}^j$:

$$h_{t|t}^j = \frac{h_{t|t-1}^j k_t^j}{f\left(V_t \,|\Omega_{t-1};\omega\right)} \quad j{=}1,2,3,4 \tag{10}$$

Equation (10) can be expressed in vector form as follows:

$$h_{t|t} = \frac{h_{t|t-1} \circ k_t}{1'\left(h_{t|t-1} \circ k_t\right)} \tag{11}$$

Note that, from the law of Total Expectations, the expected value of the VIX index based on data obtained through date $t-1$ is given by:

$$E\left[V_t \mid \Omega_{t-1}\right] = E_s\left[E\left[V_t \mid s_t, \Omega_{t-1}\right]\right] = \sum_{i=1}^{4} E\left[V_t \mid s_t = i, \Omega_{t-1}\right] h_{t|t-1}^i$$

with:

$$\begin{aligned}
E\left[V_t \mid s_t = 1, \Omega_{t-1}\right] &= \mu_1 + \phi\left(V_{t-1} - \mu_1\right) \\
E\left[V_t \mid s_t = 2, \Omega_{t-1}\right] &= \mu_2 + \phi\left(V_{t-1} - \mu_1\right) \\
E\left[V_t \mid s_t = 3, \Omega_{t-1}\right] &= \mu_1 + \phi\left(V_{t-1} - \mu_2\right) \\
E\left[V_t \mid s_t = 4, \Omega_{t-1}\right] &= \mu_2 + \phi\left(V_{t-1} - \mu_2\right)
\end{aligned}$$

Using equations (8), (9) and (10), as well as an initial value for the parameters of the model and for $h_{2|1}$, it is possible to estimate the unknown parameters corresponding to the regime-switching specifications.

## 4. EMPIRICAL RESULTS

This section applies the models presented in the previous section to the monthly data corresponding to the evolution of the VIX index during the period January 1990 to September 2010. I consider the data associated with the period January 1990 to October

2009 to estimate the parameters of the different models and I evaluate the out-of-sample empirical fit of the models over the period November 2009 to September 2010. In this period, it is possible to identify quite varied volatility patterns. In particular, it is possible to identify three volatility patterns. The first one includes a period of low volatility associated with the moments previous to the European debt crisis originated at the beginning of May. The second period coincides with the European debt crisis. Finally, we have a medium volatility pattern, which started after the publication of the stress tests corresponding to the European banks. These three different patterns offer a quite interesting testing environment to analyze the out-of-sample performance of the models considered in the article.

## 4.1 ESTIMATION RESULTS

Table 1 shows the maximum likelihood estimators, as well as its standard errors in parentheses, obtained from the numerical optimization of the conditional log-likelihood function for each of the models considered. In particular, the table reports the estimated parameters associated with the standard ARCH model of equation (2), the MSMV model of equations (4) and (6), and the MSM-ARCHV model of equations (4) and (7). In the case of the regime-switching specifications, the inverse of the degrees of freedom $\nu$ of the t-distribution is presented. Hence, testing for conditional normality is equivalent to testing whether $\nu^{-1}$ differs significantly from zero. The convergence to the maximum values reported in the table is robust with respect to a broad range of start-up conditions.

In all cases the parameters are significantly different from zero. In particular the estimated value for autoregressive coefficient $\phi$ indicates the existence of relative persistence in the term evolution of the VIX index. Importantly, the persistence coefficient $\phi$ corresponding to the regime-switching models is lower than the coefficient associated with the standard ARCH model. This result is in line with the findings of Perron (1989) that the existence of structural breaks in the mean make it more difficult to reject the null of a unit-root, that is, permanent persistence of shocks in the mean. In this sense, some part of the persistence included in $\phi$ under the standard ARCH model may be spurious reflecting the existence of two different regimes corresponding to the mean of the VIX index.

**Table 1: Estimation results**

Dependent variable: VIX index
$(V_t)$
Number of observations: 238
Sample period: January 1990 - October 2009

| | Standard ARCH | MSMV | MSM-ARCHV |
|---|---|---|---|
| $\mu$ | 17.868 | | |
| | (1.587) | | |
| $\mu_1$ | | 13.933 | 13.782 |
| | | (0.652) | (0.528) |
| $\mu_2$ | | 20.429 | 21.934 |
| | | (1.278) | (0.828) |
| $\phi$ | 0.807 | 0.749 | 0.649 |
| | (0.022) | (0.051) | (0.039) |
| $\alpha$ | 9.719 | | 6.423 |
| | (0.844) | | (1.918) |
| $\theta$ | 0.435 | | 0.676 |
| | (0.089) | | (0.269) |
| $\sigma_1^2$ | | 3.949 | |
| | | (1.216) | |
| $\sigma_2^2$ | | 20.782 | |
| | | (5.132) | |
| $p_{11}$ | | 0.962 | 0.985 |
| | | (0.022) | (0.009) |
| $p_{22}$ | | 0.973 | 0.989 |
| | | (0.018) | (0.010) |
| $1/\nu$ | | 0.260 | 0.277 |
| | | (0.066) | (0.068) |

*Notes. Standard errors in parentheses. Standard ARCH represents the model associated with equation (2). MSMV denotes the model corresponding to equations (4) and (6). Finally, MSM-ARCHV represents the model associated with equations (4) and (7).*

Regarding the regime-switching specifications, the estimation algorithm is able to identify the existence of the two volatility regimes. Furthermore, the estimated values corresponding to the mean values of the VIX index in each of the regimes, under the MSMV model and under the MSM-ARCHV model, are of the same order of magnitude. The estimated variance of the VIX index under the MSMV model is much higher in the high volatility regime than in the low volatility regime. This result is consistent with the monthly evolution of the VIX index as shown in figure 1, where the index is more volatile in those periods in which it reaches the maximum levels. Note that this result is also consistent with the existence of an upward sloping skew (positive skew) for the implied volatility corresponding to the VIX index options market, as reported by Sepp (2008).

Since in both specifications, the estimated values for $p_{11}$ and $p_{22}$ lie within the unit circle, the Markov chain corresponding to the state variable is irreducible and ergodic. Nevertheless, both regimes are particularly persistent.

Recall that, from equation (3), the unconditional variance under the standard ARCH model and under the MSM-ARCHV model is given by:

$$\sigma^2 = \frac{\alpha}{1-\theta}$$

Hence, the estimated unconditional variance under the standard ARCH model is 17.193, whereas in the case of the MSM-ARCHV model the estimated value is equal to 19.820.

## 4.2 EMPIRICAL PERFORMANCE

Table 2 reports the in-sample and out-of-sample root mean square errors (RMSE), as well as the mean absolute errors (MEA) corresponding to the three models considered in this article. Panel A of table 2 provides the in-sample performance measures and panel B reports the out-of-sample measures. The results show that the three models provide similar in-sample fit, whereas the MSM-ARCHV model exhibits better out-of-sample performance in terms of RMSE and in terms of MAE.

**Table 2: Comparing in-sample and out-of-sample empirical performance**

Dependent variable: VIX index

| | RMSE | MAE |
|---|---|---|
| Panel A | | |
| In-sample period: January 1990 - October 2009 | | |
| Standard ARCH model | 4.014 | 2.665 |
| MSMV model | 4.012 | 2.613 |
| MSM-ARCHV model | 4.054 | 2.578 |
| Panel B | | |
| Out-of-sample period: November 2009 - September 2010 | | |
| Standard ARCH model | 5.096 | 4.275 |
| MSMV model | 4.995 | 4.223 |
| MSM-ARCHV model | 4.763 | 4.047 |

One of the advantages of using a Markov chain to characterize the evolution of the state variable is that it is possible to estimate the probability of being in each regime given observations obtained through that date. In particular, the probability of being in the high volatility regime based on data obtained through the current period is given by:

$$p\left(z_t = 2 \,|\Omega_t;\omega\right) = p\left(s_t = 2 \,|\Omega_t;\omega\right) + p\left(s_t = 4 \,|\Omega_t;\omega\right)$$

Moreover, if we denote by $h_{t|\tau}$ the $\left(4\times 1\right)$ vector whose *i*th element is $p\left(s_t = i \,|\Omega_\tau;\omega\right)$. For $t > \tau$ this element represents a forecast about the regime for some future period, whereas for $t < \tau$ it denotes the smoothed inference about the regime the process was in at a date *t* based on data obtained through some later date $\tau$. Kim (1994) showed that, for the MSMV model and for the MSM-ARCHV model, it is possible to calculate the smoothed probabilities using the following algorithm:

$$h_{t|T} = h_{t|t} \circ \left[ P' \cdot \left(h_{t+1|T} \left(\div\right) h_{t+1|t}\right) \right] \qquad (12)$$

where the sign $(\div)$ represents element-by-element division. The smoothed probabilities can be then calculated iterating backward on the previous expression. Therefore, it is possible to evaluate equation (12) at the maximum likelihood estimators corresponding to the parameters of the models to obtain the smoothed probability of being in the high volatility regime. Figure 3 reports the estimated smoothed probability of being in the high volatility regime for the MSMV model corresponding to the period January 1990 to October 2009, whereas figure 4 exhibits the smoothed probability associated with the MSM-ARCHV model.

In general both models identify the changes of regime produced in the evolution of the VIX index. Nevertheless, the specification corresponding to the MSM-ARCHV model provides more stable regimes.

In this sense, figure 4 shows that the sample period starts in the low volatility regime which lasts until July 1996. The high volatility regime includes the Asian financial crisis which started in 1997, the Russian financial crisis of 1998, as well as the bursting of the IT bubble in 2000. This high volatility regime predominates until October 2003. In this month there is a new switch to the low volatility regime, but between July and August 2007 there is a sudden shift to the high volatility regime coinciding with the beginning of the international financial crisis, originated in the credit market and characterized by violent movements and epidemics of contagion from market to market affecting even the real economy.

Importantly, for the MSM-ARCHV model none of the estimated probabilities lie within the interval $[0.30, 0.70]$, while for the MSMV model this percentage is equal to 6.30%. This fact indicates that the algorithm is usually arriving at a fairly strong conclusion about the probability of being in a particular regime for the VIX index.

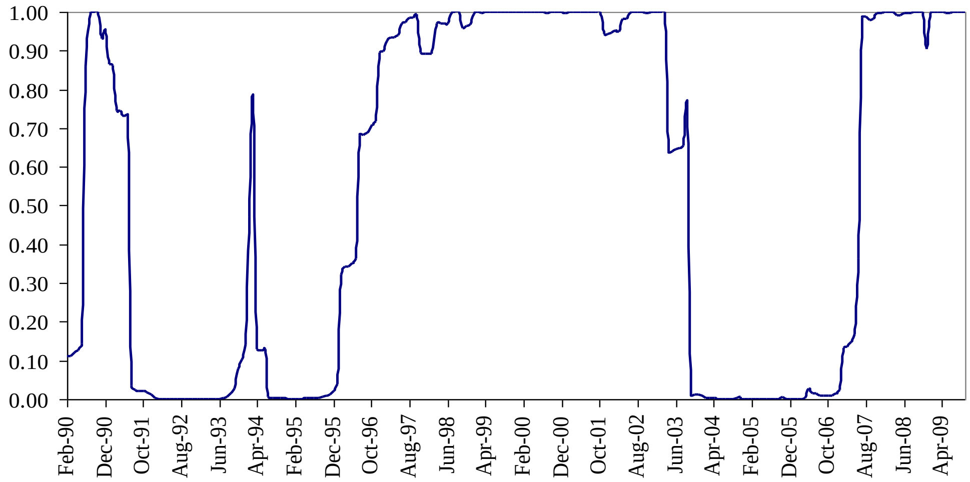


**Figure 3:** Estimated smoothed probability of being in the high volatility regime corresponding to the MSMV model.

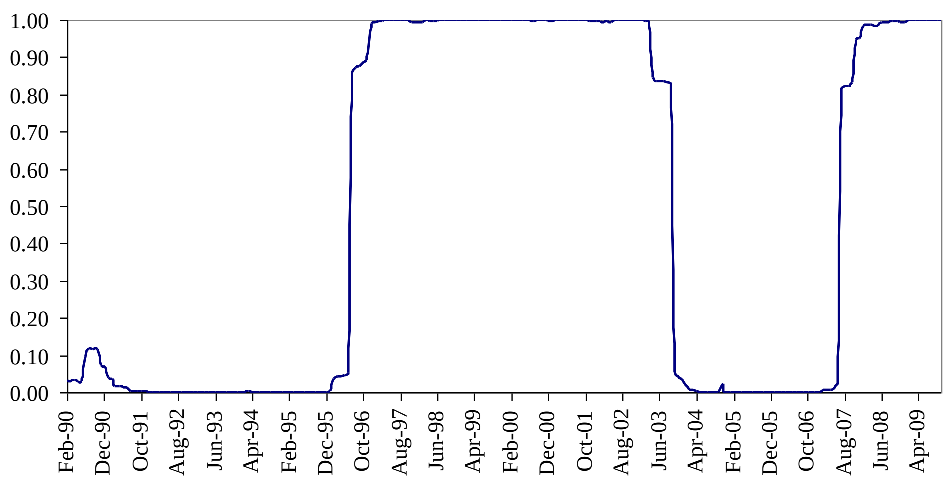


**Figure 4:** Estimated smoothed probability of being in the high volatility regime corresponding to the MSM-ARCHV model.

Another interesting feature of the algorithm is that it is possible to estimate the average persistence of each regime. Assume that the VIX index is in the low volatility regime

$\left(z_t = 1\right)$. The probability of staying in this regime is $p_{11}$, whereas the probability of switching to the high volatility regime $\left(z_t = 2\right)$ is given by $1 - p_{11}$. Let us consider the geometric variable $X$ as the number of months which are required to switch from the low volatility regime to the high volatility regime. The probability function is given by:

$$\Pr\left(x\right) = \left(1 - p_{11}\right) p_{11}^{x-1} \quad x=1,2,...$$

whereas the moment-generating function is:

$$g\left(t\right) = E\left[e^{tx}\right] = \sum_{x=1}^{\infty} \frac{1 - p_{11}}{p_{11}} \left[p_{11} e^{t}\right]^{x} = \frac{\left(1 - p_{11}\right) e^{t}}{1 - p_{11} e^{t}}$$

Therefore, we have the following expression for the average persistence of the low volatility regime:

$$\frac{dg\left(0\right)}{dt} = E\left[X\right] = \frac{1}{1 - p_{11}}$$

Let us consider the specification associated with the MSM-ARCHV model. Given the estimated value corresponding to $p_{11}$, the average persistence of the low volatility regime is 66.4225 months. Analogously the average persistence of the high volatility regime is equal to 90.969 months.

## 5. CONCLUSION

In recent years volatility has become an asset class and the derivatives on volatility have become quite common. Within this class of derivative assets, the variance swap or forward contract on the realized variance is one of the most popular. The Board Options Exchange (CBOE) calculates the VIX index. The squared of this index approximates the 30-day variance swap rate corresponding to the Standard and Poor's 500 index.

The VIX index evolves stochastically through time and it exhibits relatively persistent changes of level due to the existence of news and/or financial crisis. To take account of this behavior, in this article I have presented a regime-switching model to characterize the evolution of the VIX index. In this model the mean of the index depends on the state of the world (high volatility and low volatility) and the latent variable which determines the volatility regime is governed by an unobserved Markov Chain. The innovation is assumed to have a t-distribution allowing for deviations from normality in the distribution corresponding to VIX index. Note that, in case of normality, a large innovation in the low volatility period will lead to a switch to the high-volatility regime earlier, even if it is a single outlier in an otherwise tranquil period. The t-distribution enhances the stability of the regimes and includes the normal distribution as the limiting case.

To account for the observed persistence corresponding to the VIX index, I have considered an AR(1) specification for the evolution of this index where the mean is a function of the volatility regime. Since the time evolution of the VIX index seems to indicate that its conditional variance is not constant over time, I have considered two different versions of the model. Under the first one, called Markov-switching in mean and variance (MSMV) model, the variance of the index is a function of the state of the nature, whereas the second version, denoted as Markov-switching in mean and ARCH in variance (MSM-ARCHV) model, includes an ARCH specification for the conditional variance of the VIX index. For comparison, I also have considered a standard AR specification for the mean of the VIX index that allows for ARCH effects in the conditional variance.

The empirical results show that both regime-switching specifications are able to characterize the volatility regimes corresponding to the VIX index quite accurately. In particular, the high volatility regime identifies the Russian financial crisis in 1998, the bursting of the IT bubble in 2000, as well as the credit crisis starting in mid 2007. Moreover, the estimated volatility corresponding to the VIX index is much higher in the high volatility regime. Nevertheless, although all the models provide a similar in-sample fit, the MSM-ARCHV model provides better out-of-sample performance, as well as more stable regimes indicating the importance of considering the existence of regimes in the mean and ARCH effects in the conditional variance corresponding to the VIX index. The information provided by the model can be used as a useful tool for investment a hedging decisions regarding volatility. In particular, it is possible to set confidence intervals corresponding to the mean of the VIX index in each regime, so that if the index is above (bellow) the upper (lower) band corresponding to the mean in the high (low) volatility regime, it could be attractive to set a short (long) volatility position.

Finally, it could be of interest analyzing the joint dynamics of the VIX index and the Standard and Poor's 500 index and it is left for future research.

**APPENDIX A**

**Deriving the log-likelihood function for $V_t$:**

Let $\ln L(\omega)$ denote the log-likelihood function evaluated at the true parameter vector. This function takes the following form:

$$L(\omega) = \sum_{t=1}^{T} \ln\left[ f\left(V_t \mid \Omega_{t-1};\omega\right)\right]$$

where $f\left(V_t \mid \Omega_{t-1};\omega\right)$ is the density function associated with the VIX index based on data obtained through the previous period. Let $f\left(V_t \mid s_t = j, \Omega_{t-1};\omega\right) = k_t^j$ (for $j$=1,2,3,4) denote the density function of the VIX index given the current value of $s_t$. This function depends on the level of the index in the previous period and takes the following values:

$$\begin{aligned}
f\left(V_t \mid s_t = 1,\Omega_{t-1};\omega\right) &= \varphi\left(V_t \mid \mu_1 + \phi\left(V_{t-1} - \mu_1\right), \sigma_t^2, \nu; \Omega_{t-1}\right) \\
f\left(V_t \mid s_t = 2,\Omega_{t-1};\omega\right) &= \varphi\left(V_t \mid \mu_2 + \phi\left(V_{t-1} - \mu_1\right), \sigma_t^2, \nu; \Omega_{t-1}\right) \\
f\left(V_t \mid s_t = 3,\Omega_{t-1};\omega\right) &= \varphi\left(V_t \mid \mu_1 + \phi\left(V_{t-1} - \mu_2\right), \sigma_t^2, \nu; \Omega_{t-1}\right) \\
f\left(V_t \mid s_t = 4,\Omega_{t-1};\omega\right) &= \varphi\left(V_t \mid \mu_2 + \phi\left(V_{t-1} - \mu_2\right), \sigma_t^2, \nu; \Omega_{t-1}\right)
\end{aligned}$$

where $\sigma_t^2$ is given by equation (6) for the MSMV model and it is given by expression (7) for the MSM-ARCHV model. It is possible to express $f\left(V_t \mid \Omega_{t-1};\omega\right)$ as follows[4]:

$$\begin{aligned}
f\left(V_t \mid \Omega_{t-1};\omega\right) &= E_s\left[ f\left(V_t \mid s_t, \Omega_{t-1};\omega\right)\right] = \sum_{i=1}^{4} f\left(V_t \mid s_t = i, \Omega_{t-1};\omega\right) p\left(s_t = i \mid \Omega_{t-1};\omega\right) \\
f\left(V_t \mid \Omega_{t-1};\omega\right) &= \sum_{i=1}^{4} k_t^i h_{t|t-1}^i = 1'\left(k_t \circ h_{t|t-1}\right)
\end{aligned} \tag{13}$$

[4] To verify this result, consider the joint distribution of the variables $X$ and $Y$ given the variable $Z$. It is possible to obtain the marginal distribution of $Y$ given $Z$ integrating the joint conditional distribution with respect to the variable $X$:

$$f\left(y \mid z\right) = \int f\left(x, y \mid z\right) dx = \int f\left(y \mid x, z\right) f\left(x \mid z\right) dx = E_x\left[ f\left(y \mid x, z\right)\right]$$

where $1$ is a $(4\times 1)$ vector of ones, $k_t$ is another $(4\times 1)$ vector, which accounts for the density functions associated with the VIX index given the values corresponding to $s_t$. Finally, the symbol $\circ$ represents element-by-element multiplication. Therefore, the log-likelihood function is given by:

$$L(\omega)=\sum_{t=1}^{T}\ln\left[ f\left(V_t \mid \Omega_{t-1};\omega\right)\right]=\sum_{t=1}^{T}\ln\left[ 1'\left(h_{t|t-1}\circ k_t\right)\right]$$

**Probability of being in each regime based on data obtained through the current period:**

From the Bayes' theorem, it is possible to obtain the following expression for the probability of being in regime *j* in period *t*, given observations obtained though that date $h_{t|t}^{j}$:

$$h_{t|t}^{j}\equiv p\left(s_t=j \mid \Omega_t;\omega\right)=p\left(s_t=j \mid V_t,\Omega_{t-1};\omega\right)=\frac{p\left(s_t=j,V_t \mid \Omega_{t-1};\omega\right)}{f\left(V_t \mid \Omega_{t-1};\omega\right)}$$

$$h_{t|t}^{j}=\frac{p\left(s_t=j \mid \Omega_{t-1};\omega\right) f\left(V_t \mid s_t=j,\Omega_{t-1};\omega\right)}{f\left(V_t \mid \Omega_{t-1};\omega\right)}\quad j=1,2,3,4.$$

where $p\left(s_t=j \mid \Omega_{t-1};\omega\right)\equiv h_{t|t-1}^{j}$, $f\left(V_t \mid s_t=j,\Omega_{t-1};\omega\right)=k_t^{j}$ and $f\left(V_t \mid \Omega_{t-1};\omega\right)$ is given by equation (13). Hence, the previous equation can be expressed in vector form as follows:

$$h_{t|t}=\frac{h_{t|t-1}\circ k_t}{1'\left(h_{t|t-1}\circ k_t\right)}\quad . \tag{14}$$

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