

## PERIODIC ATTRACTORS OF NONAUTONOMOUS FLAT-TOPPED TENT SYSTEMS

LUÍS SILVA\*

ISEL - Instituto Superior de Engenharia de Lisboa, Mathematics Department and  
CIMA - Research Centre for Mathematics and Applications  
Rua Conselheiro Emídio Navarro, 1  
1959-007 Lisboa, Portugal

(Communicated by the associate editor name)

**ABSTRACT.** In this work we will consider a family of nonautonomous dynamical systems  $x_{k+1} = f_k(x_k, \lambda)$ ,  $\lambda \in [-1, 1]^{\mathbb{N}_0}$ , generated by a one-parameter family of flat-topped tent maps  $g_\alpha(x)$ , i.e.,  $f_k(x, \lambda) = g_{\lambda_k}(x)$  for all  $k \in \mathbb{N}_0$ . We will reinterpret the concept of attractive periodic orbit in this context, through the existence of some periodic, invariant and attractive nonautonomous sets and establish sufficient conditions over the parameter sequences for the existence of such periodic attractors.

**1. Introduction.** If a dynamic process is generated by a one-dimensional map, then insertion of a flat segment on the map will often lead to a stable periodic orbit. This mechanism has been widely used in the control of chaos on one-dimensional systems in areas as diverse as cardiac dynamics (see [2]), telecommunications or electronic circuits (see [8] and references therein). Families of flat-topped tent maps have also been used as models to study related families of differentiable maps, since they are closely related with symbolic dynamics and are rich enough to encompass in a canonical way all possible kneading data and all possible itineraries, see [3] or [7].

Parameters in real world situations very often are not constant with time. In that cases, the evolutionary equations have to depend explicitly on time, through time-dependent parameters or external inputs. Then the classical theory of autonomous dynamical systems is no longer applicable and we get into the field of nonautonomous dynamical systems. The time dependence may be periodic or not. Nonautonomous periodic dynamical systems can be used, for example, to model populations with periodic forcing, see [1].

In [5] we studied the bifurcation structure of a family of 2-periodic nonautonomous dynamical systems, generated by the alternate iteration of two flat-topped tent maps. However, when we get into the general (non periodic) nonautonomous context, usual notions from autonomous (and periodic nonautonomous) discrete dynamics, like fixed or periodic points, invariant sets, attractivity and repulsivity

---

2010 *Mathematics Subject Classification.* Primary: 37B55, 37E05, 37G35; Secondary: 37E15.

*Key words and phrases.* Nonautonomous dynamical systems, Attractors, symbolic dynamics.

We would like to thank the referee for several valuable suggestions.

The author was partially supported by FCT Fundação para a Ciência e a Tecnologia, Portugal, through the project UID/MAT/04674/2013, CIMA and ISEL.

\* Corresponding author: lfs@adm.isel.pt.

must be reinterpreted and reformulated. This is the core of nonautonomous bifurcation theory, that has been developed in recent years by various authors, see for example [4] and references therein.

In this work we will consider a family of nonautonomous dynamical systems  $x_{k+1} = f_k(x_k, \lambda)$ ,  $\lambda \in [-1, 1]^{\mathbb{N}_0}$ , generated by a one-parameter family of flat-topped tent maps  $g_\alpha(x)$ , i.e.,  $f_k(x, \lambda) = g_{\lambda_k}(x)$  for all  $k \in \mathbb{N}_0$ . We will reinterpret the concept of attractive periodic orbit in this context, through the existence of some periodic, invariant and attractive nonautonomous sets and establish sufficient conditions over the parameter sequences, based on symbolic dynamics, for the existence of such periodic attractors.

**2. Non autonomous flat-topped tent systems.** Consider the standard tent map

$$T(x) = \begin{cases} 2x + 1, & \text{if } -1 \leq x \leq 0 \\ -2x + 1, & \text{if } 0 \leq x \leq 1 \end{cases}$$

**Definition 2.1.** Define the symbolic address of a point  $x \in [-1, 1]$ , as

$$ad(x) = \begin{cases} L, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ R, & \text{if } x > 0 \end{cases}$$

and the itinerary of  $x$  under  $T$  as

$$I_T(x) = ad(x)ad(T(x))ad(T^2(x)) \dots ad(T^n(x)),$$

where  $n$  is the minimum such that  $T^n(x) = 0$ .

Let  $\Sigma$  be the set of sequences  $X_0 \dots X_n$  such that  $X_k \in \{L, R\}$  for all  $k < n$  and  $X_n = 0$  if  $n < +\infty$ . We say that  $X$  has length  $|X| = n$ .

To each symbol we associate a signal reflecting the monotonicity of  $T$  in the corresponding branch, i.e.,

$$\epsilon(L) = 1 \text{ and } \epsilon(R) = -1.$$

Now, for  $X \in \Sigma$  and  $0 \leq j < |X|$ , we define

$$\epsilon_j(X) = \prod_{k=0}^j \epsilon(X_k),$$

and

$$\epsilon(X) = \epsilon_{|X|-1}(X).$$

Define

$$-L = R \text{ and } -R = L.$$

Considering the natural order relation  $L < 0 < R$ , we will introduce an order structure in  $\Sigma$ . More precisely,  $X < Y$  if and only if there exists  $r \leq \min\{|X|, |Y|\}$ , such that  $X_j = Y_j$  for all  $j < r$  and  $\epsilon_r(X) < \epsilon_r(Y)$ .

The next lemma is standard in symbolic dynamics and states that the order in the interval is preserved by the itineraries.

**Lemma 2.2.** *Let  $x, y \in [-1, 1]$ , then  $I_T(x) < I_T(y)$  iff  $x < y$ .*

*Proof.* See [6]. □

Consider the map  $\Phi : \Sigma \rightarrow I$ , such that

$$\Phi(X) = - \sum_{i=0}^{|X|-1} \frac{\epsilon_i(X)}{2^{i+1}}.$$

**Lemma 2.3.** *For all  $X \in \Sigma$ ,  $X = I_T(\Phi(X))$ .*

*Proof.* If  $X_0 = L$ , then

$$\Phi(X) = -\frac{1}{2} - \sum_{i=1}^{|X|-1} \frac{\epsilon_i(X)}{2^{i+1}} < 0$$

and

$$\begin{aligned} T(\Phi(X)) &= - \sum_{i=1}^{|X|-1} \frac{\epsilon_i(X)}{2^i} \\ &= -\frac{\epsilon_1(X)}{2} - \sum_{i=2}^{|X|-1} \frac{\epsilon_i(X)}{2^i} \\ &= -\frac{\epsilon(X_1)}{2} - \sum_{i=2}^{|X|-1} \frac{\epsilon_i(X)}{2^i} \end{aligned}$$

so

$$T(\Phi(X)) \begin{cases} < 0 \text{ if } \epsilon(X_1) = 1, \text{ i.e., } X_1 = L \\ > 0 \text{ if } \epsilon(X_1) = -1, \text{ i.e., } X_1 = R \end{cases}$$

and

$$ad(T(\Phi(X))) = X_1.$$

Analogously, if  $X_0 = R$ , then

$$\Phi(X) = \frac{1}{2} - \sum_{i=1}^{|X|-1} \frac{\epsilon_i(X)}{2^{i+1}} > 0$$

and

$$\begin{aligned} T(\Phi(X)) &= \sum_{i=1}^{|X|-1} \frac{\epsilon_i(X)}{2^i} \\ &= \frac{\epsilon_1(X)}{2} + \sum_{i=2}^{|X|-1} \frac{\epsilon_i(X)}{2^i} \\ &= \frac{\epsilon(X_1)}{2} + \sum_{i=2}^{|X|-1} \frac{\epsilon_i(X)}{2^i}, \end{aligned}$$

so, again,

$$ad(T(\Phi(X))) = X_1.$$

Now the proof follows inductively.  $\square$

Define the shift map  $\sigma : \Sigma/\{0\} \rightarrow \Sigma$ , as  $\sigma(X_0 \dots X_{|X|-1} 0) = X_1 \dots X_{|X|-1} 0$ . From the previous Lemma we can conclude the following.

**Corollary 1.** *Let  $X \in \Sigma$ , then  $\Phi(\sigma(X)) = T(\Phi(X))$ .*

*Proof.* From the definition of  $I_T$  and the previous lemma we have that

$$\sigma(X) = I_T(T(\Phi(X))),$$

but then, applying Lemma 2.3 again, we get

$$\Phi(\sigma(X)) = T(\Phi(X)).$$

$\square$

**Definition 2.4.** Let  $u \in [-1, 1]$ , a flat-topped tent map  $f_u : [-1, 1] \rightarrow [-1, 1]$  is defined as

$$f_u(x) = \begin{cases} 2x + 1, & \text{if } -1 \leq x \leq (u-1)/2 \\ u, & \text{if } (u-1)/2 < x < (1-u)/2 \\ -2x + 1, & \text{if } (1-u)/2 \leq x \leq 1 \end{cases}.$$

**Remark 1.** Alternatively we could define  $f_u(x)$  as

$$f_u(x) = \min\{u, T(x)\}.$$

**Definition 2.5.** Let  $\lambda = (\lambda_n)_{n \in \mathbb{N}_0} \in [-1, 1]^{\mathbb{N}_0}$ , the nonautonomous flat-topped tent system associated to  $\lambda$  is defined by the nonautonomous difference equation

$$x_{k+1} = f_{\lambda_k}(x_k).$$

**Remark 2.** Although this article focuses on flat-topped tent maps, all results are valid for any suitable family of unimodal flat-topped maps, considering the corresponding map  $\Phi(X)$ .

For example, for the family of flat-topped quadratic maps  $f_h : I \rightarrow I$ ,

$$f_h(x) = \min\{h, 1 - 2x^2\},$$

we have

$$\Phi(X) = \frac{\epsilon(X_0)}{2} \sqrt{2 - \epsilon(X_1) \sqrt{2 - \epsilon(X_2) \sqrt{\dots - \epsilon(X_{|X|-1}) \sqrt{2}}}}.$$

**3. Periodic nonautonomous attractors.** Following [4], a nonautonomous set  $\mathcal{A}$  is a subset of the extended state space  $I \times \mathbb{N}_0$  and its  $k$ -fiber is defined to be

$$\mathcal{A}(k) = \{x \in I : (x, k) \in \mathcal{A}\}.$$

**Definition 3.1.** A nonautonomous set is called

- $p$ -Periodic if  $\mathcal{A}(i+p) = \mathcal{A}(i)$  for all  $i \in \mathbb{N}_0$ .
- $\lambda$ -Invariant, for  $\lambda \in I^{\mathbb{N}_0}$ , if  $f_{\lambda_k}(\mathcal{A}(k)) \subset \mathcal{A}(k+1)$  for all  $k \in \mathbb{N}_0$ .
- $\lambda$ -Attractive if, for each  $k$  there exists a neighborhood  $\mathcal{A}'(k)$  of  $\mathcal{A}(k)$  such that, for some  $n$ ,  $f_{\lambda_{k+n'}} \circ \dots \circ f_{\lambda_k}(\mathcal{A}'(k)) \subset \mathcal{A}(k+n')$  for all  $n' > n$ .

Comparing with the analogous concepts introduced in [4],  $\lambda$ -invariant corresponds to forward invariant and  $\lambda$ -attractive in this context corresponds to locally attractive in [4].

**Definition 3.2.** Let  $\lambda \in I^{\mathbb{N}_0}$ , then a proper subset  $\mathcal{A} \subsetneq I \times \mathbb{N}_0$  is a nonautonomous  $p$ -periodic  $\lambda$ -attractor if it is  $p$ -periodic,  $\lambda$ -invariant and  $\lambda$ -attractive.

For a finite sequence  $X \in \Sigma$ , define

$$X^R = (X_0 \dots X_{|X|-1} \hat{R})^\infty$$

and

$$X^L = X_0 \dots X_{|X|-1} \hat{L} X^R$$

where  $\hat{L} = \epsilon(X)L$ ,  $\hat{R} = \epsilon(X)R$  and  $(\cdot)^\infty$  denotes the infinite repetition of the finite word inside the parentheses.

**Remark 3.** From Lemma 2.3 and Corollary 1, we have that, for  $p = |X|$ ,

$$T^p(\Phi(X^L)) = T^p(\Phi(X^R)) = \Phi(X^R).$$

**Definition 3.3.** Let  $X \in \Sigma$  be a symbolic sequence with  $|X| = p < +\infty$ , then a sequence  $\lambda \in I^{\mathbb{N}_0}$  is an  $X$ -sequence if, for all  $n \in \mathbb{N}_0$ ,  $\Phi(X^L) < \lambda_{np} < \Phi(X^R)$  and, for all  $0 < j < p$ ,  $\lambda_{np+j} > \max\{\Phi(\sigma^j(X^L)), \Phi(\sigma^j(X^R))\}$ .

An  $X$ -sequence  $\lambda$  is a strict  $X$ -sequence if, in addition,  $\lambda_{np} = \lambda_0$ , for all  $n \in \mathbb{N}_0$ .

The nonautonomous systems associated to an  $X$ -sequence, may or may not be periodic, even in the strict case. However they may be seen as time-dependent perturbations of  $p$ -periodic systems where  $p = |X|$ . The main difference between the strict and the non strict case is that, in the strict case perturbations in  $np$  moments are not allowed.

Let  $X \in \Sigma$  be a symbolic sequence with  $|X| = p < +\infty$ ,  $\lambda$  be an  $X$ -sequence, and

$$J_i = \left] \frac{\lambda_i - 1}{2}, \frac{1 - \lambda_i}{2} \right[$$

be the constant plateau of  $f_{\lambda_i}$  for each  $i$ . Then we consider the generalized constant plateaus

$$\begin{aligned} \mathcal{P}(f_{\lambda_{(n+1)p-1}} \circ \dots \circ f_{\lambda_{np}}) &= J_{np} \cup f_{\lambda_{np}}^{-1}(J_{np+1}) \cup \dots \\ &\quad \cup (f_{\lambda_{(n+1)p-2}} \circ \dots \circ f_{\lambda_{np}})^{-1}(J_{(n+1)p-1}) \\ &\quad \vdots \\ \mathcal{P}(f_{\lambda_{(n+2)p-2}} \circ \dots \circ f_{\lambda_{(n+1)p-1}}) &= J_{(n+1)p-1} \cup f_{\lambda_{(n+1)p-1}}^{-1}(J_{(n+1)p}) \cup \dots \\ &\quad \cup (f_{\lambda_{(n+2)p-3}} \circ \dots \circ f_{\lambda_{(n+1)p-1}})^{-1}(J_{(n+2)p-2}) \end{aligned}$$

**Definition 3.4.** Let  $X \in \Sigma$  with  $|X| = p < +\infty$  and  $\lambda$  be an  $X$ -sequence.

Define the sets  $A_i = T^{i-1}(\mathcal{J}\Phi(X^L), \Phi(X^R)\mathcal{J})$ ,  $i = 1, \dots, p$  ( $T^0 \equiv id$ ).

**Lemma 3.5.** Let  $X \in \Sigma$  be a symbolic sequence with  $|X| = p < +\infty$ , and  $\lambda \in I^{\mathbb{N}_0}$  be an  $X$ -sequence. Then, for all  $n \in \mathbb{N}_0$ ,

$$\begin{aligned} A_p &\subset \mathcal{P}(f_{\lambda_{(n+1)p-1}} \circ \dots \circ f_{\lambda_{np}}), \\ A_1 &\subset \mathcal{P}(f_{\lambda_{(n+1)p}} \circ \dots \circ f_{\lambda_{np+1}}), \\ &\quad \vdots \\ A_{p-1} &\subset \mathcal{P}(f_{\lambda_{(n+2)p-2}} \circ \dots \circ f_{\lambda_{(n+1)p-1}}) \end{aligned}$$

*Proof.* Beware that, throw this proof the extremes of the interval may not be in the correct order, i.e., we may write  $]a, b[$  with  $b < a$ .

Now, since  $X_i^L = X_i^R$  for all  $i = 0, \dots, p-1$ , from Corollary 1

$$\begin{aligned} A_p &= T^{p-1}(\mathcal{J}\Phi(X^L), \Phi(X^R)\mathcal{J}) \\ &= \mathcal{J}\Phi(\sigma^{p-1}(X^L)), \Phi(\sigma^{p-1}(X^R))\mathcal{J} \\ &= \mathcal{J}\Phi(\hat{L}X^R), \Phi(\hat{R}X^R)\mathcal{J} \end{aligned}$$

and then, since  $\lambda_{np} < \Phi(X^R)$ ,

$$\mathcal{J}\Phi(\hat{L}X^R), \Phi(\hat{R}X^R)\mathcal{J} \subset J_{np} \subset \mathcal{P}(f_{\lambda_{(n+1)p-1}} \circ \dots \circ f_{\lambda_{np}}).$$

For  $i = 1, \dots, p-1$  and  $j < p-i$ , since  $\lambda$  is an  $X$ -sequence,

$$\begin{aligned} f_{\lambda_{np+i+j-1}} \circ \dots \circ f_{\lambda_{np+i}}(A_i) &= T^j(A_i) \\ &= T^{i+j-1}(\mathcal{J}\Phi(X^L), \Phi(X^R)\mathcal{J}) \\ &= \mathcal{J}\Phi(\sigma^{i+j-1}(X^L)), \Phi(\sigma^{i+j-1}(X^R))\mathcal{J} \end{aligned}$$

so, for  $j = p-i-1$ ,

$$f_{\lambda_{np+p-1}} \circ \dots \circ f_{\lambda_{np+i}}(A_i) = T^{p-1}(A_i) \subset J_{(n+1)p},$$

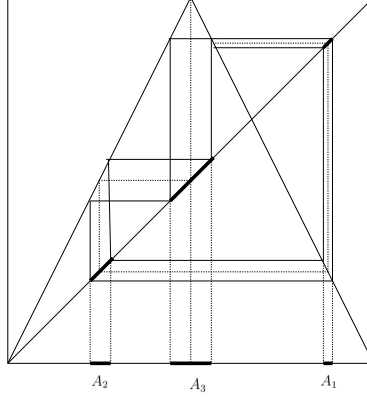


FIGURE 1. Construction of the sets  $A_i$  for  $X = RL0$ ,  $X^R = (RLL)^\infty$  and  $X^L = RLR(RLL)^\infty$ .

and so

$$A_i \subset (f_{\lambda_{np+p-1}} \circ \dots \circ f_{\lambda_{np+i}})^{-1}(J_{(n+1)p}) \subset \mathcal{P}(f_{\lambda_{np+i+p-1}} \circ \dots \circ f_{\lambda_{np+i}}).$$

□

**Lemma 3.6.** *Under the conditions of the previous lemma, if  $x \in A_i$  then, for all  $j \leq p - i$  and all  $n \in \mathbb{N}_0$*

$$(f_{\lambda_{np+i+j-1}} \circ \dots \circ f_{\lambda_{np+i}})(x) = T^j(x).$$

*Proof.* If  $x \in A_i$  then  $T(x) \in ]T^i(\Phi(X^L)), T^i(\Phi(X^R)[$ , so, since  $\lambda_{np+i} > \max\{\Phi(\sigma^i(X^L)), \Phi(\sigma^i(X^R))\}$ , then  $T(x) = f_{\lambda_{np+i}}(x)$ . Now, since  $T(x) \in A_{i+1}$  the proof follows inductively. □

It is immediate to conclude the following corollary.

**Corollary 2.** *Let  $X \in \Sigma$  be a symbolic sequence with  $|X| = p < +\infty$ , and  $\lambda \in I^{\mathbb{N}_0}$  be an  $X$ -sequence, then, for all  $n \in \mathbb{N}_0$  and  $j = 0, \dots, p - 1$ ,*

$$(f_{\lambda_{np+j}} \circ \dots \circ f_{\lambda_{np}})(0) = T^j(\lambda_{np}),$$

and

$$(f_{\lambda_{(n+1)p}} \circ \dots \circ f_{\lambda_{np+1}})(\lambda_{np}) = \lambda_{(n+1)p}.$$

**Definition 3.7.** Let  $X \in \Sigma$  with  $|X| = p < +\infty$  and  $\lambda$  be an  $X$ -sequence. Define the nonautonomous set  $\mathcal{A}_\lambda$ , where

$$\mathcal{A}_\lambda(np + j) = A_j \text{ for all } n \in \mathbb{N}_0 \text{ and } j = 1, \dots, p \text{ and } \mathcal{A}_\lambda(0) = A_p.$$

**Theorem 3.8.** *Let  $X \in \Sigma$  with  $|X| = p < \infty$  and  $\lambda \in I^{\mathbb{N}_0}$  be an  $X$ -sequence, then  $\mathcal{A}_\lambda$  is a nonautonomous  $p$ -periodic,  $\lambda$ -attractor.*

*Proof.*  $\mathcal{A}_\lambda$  is  $p$ -periodic by definition and it is  $\lambda$ -invariant from Lemma 3.6.

Finally we will see that  $\mathcal{A}_\lambda$  is  $\lambda$ -attractive.

We will prove that, for all  $n \in \mathbb{N}_0$  and all  $0 \leq j < p - 1$ , there is a neighborhood  $V_{n,j}$  of  $A_j$  such that

$$F_\lambda^p(V_{n,j}, np + j) \subset (A_j, (n+1)p + j)$$

and then the result follows from the  $p$ -periodicity and  $\lambda$ -invariance of  $\mathcal{A}_\lambda$ .

First we will consider  $j = 0$ . Since  $\Phi(X^L) < \lambda_{np} < \Phi(X^R)$ ,  $J_{np}$  is a neighborhood of  $A_{np} = A_0$  and, from Lemma 3.6,

$$\begin{aligned} F_\lambda^p(J_{np}, np) &= F_\lambda^p(A_0, np) \\ &= (T^{p-1}(\lambda_{np}), (n+1)p) \subset (A_0, (n+1)p) \end{aligned}$$

Now we will consider  $j > 0$ : From the proof of Lemma 3.5, for all  $n \in \mathbb{N}_0$  and  $0 \leq j < p$  we can take  $V_{n,j}$ , the connected component of  $(f_{\lambda_{np+p-1}} \circ \dots \circ f_{\lambda_{np+j}})^{-1}(J_{(n+1)p})$  that contains  $A_j$ . Moreover, since  $\lambda_{(n+1)p} < \Phi(X^R)$ ,  $V_{n,j}$  is a neighborhood of  $A_j$  and

$$\begin{aligned} (f_{\lambda_{(n+1)p+j-1}} \circ \dots \circ f_{\lambda_{(n+1)p}} \circ \dots \circ f_{\lambda_{np+j}})(V_{n,j}) &= \\ (f_{\lambda_{(n+1)p+j-1}} \circ \dots \circ f_{\lambda_{(n+1)p}} \circ \dots \circ f_{\lambda_{np+j}})(A_j) &= \cdot \\ \{T^{j-1}(\lambda_{(n+1)p})\} &\subset A_j \end{aligned}$$

□

**Definition 3.9.** Let  $X \in \Sigma$  with  $|X| = p < +\infty$  and  $\lambda$  be a strict  $X$ -sequence, define the nonautonomous set  $\mathcal{B}_\lambda$ , where

$$\mathcal{B}_\lambda(j + np) = \{(f_{\lambda_{j-1}} \circ \dots \circ f_{\lambda_0})(0)\} \text{ for all } n \in \mathbb{N}_0 \text{ and } j = 1, \dots, p$$

and

$$\mathcal{B}_\lambda(0) = \mathcal{B}_\lambda(p).$$

**Theorem 3.10.** *Let  $X \in \Sigma$  with  $|X| = p < \infty$  and  $\lambda \in I^{\mathbb{N}_0}$  be a strict  $X$ -sequence, then  $\mathcal{B}_\lambda$  is a nonautonomous  $p$ -periodic  $\lambda$ -attractor.*

*Proof.*  $\mathcal{B}_\lambda$  is  $p$ -periodic, by definition and, from Lemma 3.6, it is  $\lambda$ -Invariant.

From Corollary 2, since  $\lambda$  is strict then for all  $n$

$$(f_{\lambda_{np+j}} \circ \dots \circ f_{\lambda_{np}})(0) = T^j(\lambda_{np}) = T^j(\lambda_0),$$

so, from Lemma 3.5, for all  $n$

$$\mathcal{B}_\lambda(j + np) = T^j(\lambda_0) = T^j(\lambda_{np}) \subset \mathcal{P}(f_{\lambda_{(n+1)p+j}} \circ \dots \circ f_{\lambda_{np+j+1}}).$$

Consequently, if  $C_j$  is the connected component of  $\mathcal{P}(f_{\lambda_{(n+1)p+j}} \circ \dots \circ f_{\lambda_{np+j+1}})$  that contains  $\mathcal{B}_\lambda(j + np)$ , then, for all  $n' \geq p$

$$\begin{aligned} F_\lambda^{n'}(C_j, j + np) &= F_\lambda^{n'}(\mathcal{B}_\lambda(j + np), j + np) \\ &= (\mathcal{B}_\lambda(j + np + n'), j + np + n'). \end{aligned}$$

□

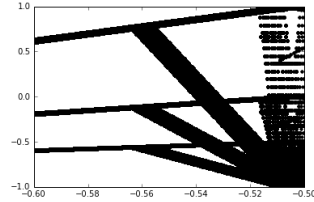


FIGURE 2. A bifurcation diagram with a sequence of nonautonomous  $p$ -periodic  $\lambda$ -attractors with period doubling periods  $p = 3, 6, \dots$ , for a sequence of sequences  $\lambda$ , such that  $\lambda_{3n+1} = \lambda_{3n+2} = 1$ ,  $\lambda_0$  varies from  $-0.6$  to  $-0.5$  and  $\lambda_{3n} = \lambda_0 + 0.01r_n$ , where  $r_n$  is a random integer between 0 and 9.

#### REFERENCES

- [1] J. Franke and A. Yakubu, Population models with periodic recruitment functions and survival rates, *Journal of Difference Equations and Applications*, **Vol. 11** (2005), 1169–1184.
- [2] L. Glass and W. Zeng, Bifurcations in flat-topped maps and the control of cardiac chaos, *International Journal of Bifurcation and Chaos*, **Vol. 4** (1994), 1061–1067.
- [3] J. Milnor and C. Tresser, On entropy and monotonicity for real cubic maps, *Comm. Math. Phys.*, **Vol. 209** (2000), 123–178.
- [4] C. Pötzsche, Bifurcations in Nonautonomous Dynamical Systems: Results and tools in discrete time, in *Proceedings of the International Workshop Future Directions in Difference Equations* (eds. E. Liz and V. Mañosa), Universidade de Vigo, Vigo, (2011), 163–212.
- [5] L. Silva, J. L. Rocha and M. T. Silva, Bifurcations of 2-periodic nonautonomous stunted tent systems, *Int. J. Bifurcation Chaos*, **Vol. 27** (2017), 1730020 [17 pages].
- [6] N. Franco, L. Silva and P. Simões, Symbolic dynamics and renormalization of nonautonomous  $k$  periodic dynamical systems, *Journal of Difference Equations and Applications*, **Vol. 19** (2013), 27–38.
- [7] A. Rădulescu, The connected isentropes conjecture in a space of quartic polynomials, *Discrete Contin. Dyn. Syst.*, **Vol 19** (2007), 139–175.
- [8] C. Wagner and R. Stoop, Renormalization Approach to Optimal Limiter Control in 1-D Chaotic Systems, *Journal of Statistical Physics*, **Vol. 106** (2002), 97–106.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: [lfs@adm.isel.pt](mailto:lfs@adm.isel.pt)