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α-FRACTAL FUNCTION WITH VARIABLE PARAMETERS: AN EXPLICIT REPRESENTATION

T. M. C. PRIYANKA
Department of Mathematics, School of Advanced Sciences
Vellore Institute of Technology
Vellore 632 014, Tamil Nadu, India
priyankamohan195@gmail.com

C. SERPA
ISEL — Instituto Superior de Engenharia de Lisboa
Centro de Matemática Aplicações Fundamentais e Investigação Operacional
Faculdade de Ciências, Universidade de Lisboa
1749-016 Lisbon, Portugal
mcserpa@fc.ul.pt

A. GOWRISANKAR*
Department of Mathematics, School of Advanced Sciences
Vellore Institute of Technology
Vellore 632 014, Tamil Nadu, India
gowrisankargri@gmail.com

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*Corresponding author.

Abstract

In this paper, new results on the α-fractal function with variable parameters are presented. The Weyl–Marchaud variable order fractional derivative of an α-fractal function with variable parameters is examined by imposing certain conditions on the scaling factors. Following the investigation of fractional derivative, the definite integral of the α-fractal function with variable parameters is evaluated for various intervals in the prescribed domain. Finally, an explicit structure for the α-fractal function is provided using the base q representation of numbers.

Keywords: α-Fractal Function; Variable Parameters; Fractional Derivative; Definite Integral; Base Representation.

1. INTRODUCTION AND PRELIMINARIES

The fundamental construction of univariate fractal interpolation functions is developed from the theory of iterated function system (IFS) defined in $\mathbb{R}^2$. The unique feature of fractal functions is the self-referentiality, in a broader sense, fractals are finite union of transformed copies of themselves. Methods of generating types of fractal functions and investigating their properties comprise fascinating studies in the field of fractal geometry.[1,2] Following the seminal paper of Barnsley,[3] Navascués[4] has considered a particular kind of IFS to yield a family of parametrized self-referential functions for any given continuous map, so-called the α-fractal functions. The paper begins with the review of constructing α-fractal interpolation function as follows.

Let $\mathbb{N}_N = \{1, 2, \ldots, N\}$. Let $\{(x_k, y_k) \in I \times \mathbb{R} : k \in \mathbb{N}_N\}$ be the N data points. Set $I = [x_1, x_N]$ and its closed sub-interval $I_k = [x_k, x_{k+1}]$, $\forall\, k \in \mathbb{N}_{N-1}$. Let Δ be the partition of the interval I with $x_1 < x_2 < \cdots < x_N$. For $k \in \mathbb{N}_{N-1}$, consider the affine maps $L_k : I \to I_k$ satisfying

$$L_k(x_1) = x_k, \quad L_k(x_N) = x_{k+1},$$
$$|L_k(x) - L_k(y)| \le r_k|x - y|, \quad r_k \in (0, 1), \;\; x, y \in I.$$

For $k \in \mathbb{N}_{N-1}$, let F_k be the N real-valued continuous maps defined on $J := I \times \mathbb{R}$ satisfying

$$F_k(x_1, y_1) = y_k, \quad F_k(x_N, y_N) = y_{k+1},$$
$$|F_k(x, u) - F_k(x, v)| \le \alpha|u - v|, \quad x \in I, \;\; u, v \in \mathbb{R},$$

where $0 < \alpha_k < 1$. Now the maps $w_k : J \to I_k \times \mathbb{R}$ are defined by

$$w_k(x, y) = (L_k(x), F_k(x, y)), \quad k \in \mathbb{N}_{N-1}. \qquad (1)$$

The system $\{J; w_k : k \in \mathbb{N}_{N-1}\}$ becomes an IFS and its attractor is a unique invariant set G, satisfying $G = \bigcup_{k=1}^{N} w_k(G)$. This set G is the graph of a continuous function $f : I \to \mathbb{R}$ obeying $f(x_k) = y_k$, $\forall\, k \in \mathbb{N}_N$.

Denote by $\mathcal{C}^*(I)$, the space of all continuous functions equipped with the sup norm,

$$\|h\|_\infty = \max\{|h(x)| : x \in I\}.$$

Let $\mathcal{C}^{**}(I) = \{h \in \mathcal{C}^*(I) : h(x_1) = y_1 \text{ and } h(x_N) = y_N\}$ be the closed metric subspace of $\mathcal{C}^*(I)$. The Read–Bajraktarević (RB) operator T is defined on $\mathcal{C}^{**}(I)$ as

$$T(h(x)) = F_k(L_k^{-1}(x), h \circ L_k^{-1}(x)), \qquad (2)$$

for $x \in I_k$ and $k \in \mathbb{N}_{N-1}$. The operator T is a contraction mapping on the Banach space $(\mathcal{C}^{**}(I), \|\cdot\|_\infty)$ and the previously described fractal function f is its unique fixed point, such that $Tf(x) = f(x)$, $\forall\, x \in [x_1, x_N]$ and hence, f satisfies the following fixed point equation:

$$f(x) = F_k(L_k^{-1}(x), f \circ L_k^{-1}(x)),$$

for $x \in I_k$ and $k \in \mathbb{N}_{N-1}$. The commonly studied IFS is constructed using the maps L_k and F_k as defined in the following:

$$L_k(x) = a_k x + b_k,$$
$$F_k(x, y) = \alpha_k y + q_k(x), \quad k \in \mathbb{N}_{N-1}, \qquad (3)$$

where q_k are continuous functions chosen in such a way that the end point conditions of F_k are valid and the parameters α_k are called vertical scaling factors of the maps w_k obeying $0 < \alpha_k < 1$. In order to generate the α-fractal function, take

$$q_k(x) = g \circ L_k(x) - \alpha_k b(x), \quad k \in \mathbb{N}_{N-1} \qquad (4)$$

in Eq. (3), where $g : I \to \mathbb{R}$ is the given continuous function, also called germ function, b is a continuous map that agrees with g at the end points of the interval I such that $b(x_1) = g(x_1)$, $b(x_N) = g(x_N)$ and $b \neq g$.

Let g^α be the continuous function whose graph is the attractor for the IFS given in (3) and (4). Then, the function g^α obeying the functional equation

$$g^\alpha(x) = g(x) + \alpha_k (g^\alpha - b) \circ L_k^{-1}(x), \quad k \in \mathbb{N}_{N-1}, \tag{5}$$

is called the α-fractal interpolation function, shortly *α-fractal function* corresponding to the partition Δ and the base function b. The major advantage of this α-fractal function is that fractal analogues can be defined for any given continuous function g using g^α.

The class of functions, g^α, are the fractal perturbation of given function g and this process of perturbation depends on the partition Δ, the base function b and the vertical scaling parameters α_k. The scaling factors α_k give the degree of freedom to the fractal function either to preserve or to modify the properties of the continuous function g. Note that g^α coincides with g when the associated scaling factors α_k are taken as zero for each $k \in \mathbb{N}_{N-1}$. If the vertical scaling parameters are taken as continuous functions on I, $\alpha_k : [x_1, x_N] \to (0, 1)$ then Eq. (5) becomes

$$g^\alpha(x) = g(x) + \alpha_k(L_k^{-1}(x))\{(g^\alpha - b) \circ L_k^{-1}(x)\}, \tag{6}$$

for $k \in \mathbb{N}_{N-1}$, the function g^α satisfying Eq. (6) is referred as the α-fractal function with variable parameters. For more interesting results on α-fractal functions, the readers are encouraged to see Refs. 5–8.

Beginning with the univariate fractal interpolation functions, various kinds of fractal functions like bivariate, multivariate, fractal functions on the higher dimension, recurrent fractal functions and their properties like smoothness, stability and study of their fractal dimensions have been fruitfully carried out by many researchers (refer Refs. 9–17). The latest attractive research on fractal functions is to examine their classical as well as fractional calculus.[18–28] In this line, several sorts of fractional calculus methods have been applied on the fractal functions and the fractal dimension has also been investigated for the graphs of their fractional derivatives (and integrals). Among the types of fractal functions, α-fractal functions are some kind of special functions since any given continuous functions can be approximated rather than approximating the given data set. In particular, choosing the scaling factors as continuous functions make the α-fractal functions much more robust. Literature report clearly pictures that whenever the ordinary integral of α-fractal functions has been evaluated, only the case of indefinite integral is dealt. On the other hand, the fractional derivative is also studied only with the constant fractional order. Recently, a novel idea of representing the fractal functions in explicit form has been presented using the functional equation setting.[29,30] All the aforementioned results arise the following investigations: (i) the examination of definite integral of α-fractal function with variable parameters, (ii) the exploration of variable order fractional derivative of α-fractal function with variable parameters and (iii) an explicit representation of α-fractal function using the system of functional equations. With this motivation, the present work explores the Weyl–Marchaud (WM) variable order fractional derivative of the α-fractal function with variable parameters under the predefined conditions on the fractional derivatives of germ function g and the base function b. The definite integral of α-fractal function with variable parameters is examined for various sub-intervals of I. Further, the solution of system of functional equations is presented to provide an explicit structure for the α-fractal function using the base q representation of numbers.

2. WEYL–MARCHAUD VARIABLE ORDER FRACTIONAL DERIVATIVE

In this section, the WM variable order fractional derivative of an α-fractal function with variable parameters is evaluated.

Lemma 1. *Consider the α-fractal function, g^α, as defined in Eq.* (6), *then*

$$\int_{x_k}^{L_k(x)} g^\alpha(s)ds = a_k \int_{x_1}^{x} \{g(L_k(s)) + \alpha_k(s)(g^\alpha - b)(s)\}ds,$$

$$\int_{x_k}^{x_{k+1}} g^\alpha(s)ds = a_k \int_{x_1}^{x_N} \{g(L_k(s)) + \alpha_k(s)(g^\alpha - b)(s)\}ds,$$

$$\int_{x_k}^{x} g^\alpha(s)ds = a_k \int_{x_1}^{\frac{x-b_k}{a_k}} \{g(L_k(s)) + \alpha_k(s)(g^\alpha - b)(s)\}ds,$$

$$\int_{L_j(x_k)}^{L_j(x_{k+1})} g^\alpha(s)ds = a_j \int_{x_k}^{x_{k+1}} \{g(L_k(s)) + \alpha_k(s)(g^\alpha - b)(s)\}ds.$$

Proof. The result follows by performing a standard change of variable and considering the end point conditions. □

The WM variable order fractional derivative of a continuous function $g : I \to \mathbb{R}$ with variable fractional order is expressed by

$$(\mathbb{D}^{v(y)}g)(x) = \frac{v(y)}{\Gamma(1-v(y))}\int_0^\infty \frac{g(x)-g(x-t)}{t^{1+v(y)}}dt,$$

where $v : I \to (0,1)$.

Theorem 2. *Suppose the variable order fractional derivatives of g and b given by*

$$(\mathbb{D}_{x_1}^{v(y)}g)(x) = \frac{v(y)}{\Gamma(1-v(y))}\int_{x_1}^{x} \frac{g(x)-g(s)}{(x-s)^{1+v(y)}}ds,$$

$$(\mathbb{D}_{x_1}^{v(y)}b)(x) = \frac{v(y)}{\Gamma(1-v(y))}\int_{x_1}^{x} \frac{b(x)-b(s)}{(x-s)^{1+v(y)}}ds$$

exist and satisfy $\hat{y}_1 = (\mathbb{D}_{x_1}^{v(y)}g)(x_1) = (\mathbb{D}_{x_1}^{v(y)}b)(x_1) = 0$ and $(\mathbb{D}_{x_1}^{v(y)}g)(x_N) = (\mathbb{D}_{x_1}^{v(y)}b)(x_N)$. Let g^α be the α-fractal function with variable parameters associated to the continuous function g. If $\|\alpha_k(x)\|_\infty < |a_k^{v(y)}|$ and $\sum_{k=1}^N a_k^{-v(y)}\alpha_k(x_N) \neq 1$, then $\mathbb{D}_{x_1}^{v(y)}g^\alpha(x)$ is also a α-fractal function associated with the IFS $\{L_k(x), \hat{F}_k(x,\hat{y})\}_{k=1}^{N-1}$, where $\hat{F}_k(x,\hat{y}) = a_k^{-v(y)}\alpha_k(x)\hat{y} + \hat{q}_k(x)$ for $k \in \mathbb{N}_{N-1}$,

$$\hat{q}_k(x) = \hat{y}_{k-1} + g^\alpha_{k,v(y)}(x) + a_k^{-v(y)}(\mathbb{D}_{x_1}^{v(y)}g \circ L_k)(x) - a_k^{-v(y)}\alpha_k(x)(\mathbb{D}_{x_1}^{v(y)}b)(x),$$

$$g^\alpha_{k,v(y)}(x) = \frac{v(y)}{\Gamma(1-v(y))}\int_{x_1}^{x_{k-1}}\left(\frac{g^\alpha(L_k(x))-g^\alpha(s)}{(L_k(x)-s)^{1+v(y)}} - \frac{g^\alpha(x_{k-1})-g^\alpha(s)}{(x_{k-1}-s)^{1+v(y)}}\right)ds,$$

$$\hat{y}_k = \sum_{m=1}^{k}\big\{(g^\alpha_{m,v(y)}(x_N) + a_m^{-v(y)}\alpha_m(x_N)\hat{y}_N + a_m^{-v(y)}(\mathbb{D}_{x_1}^{v(y)}g\circ L_m)(x_N) - a_m^{-v(y)}\alpha_m(x_N)(\mathbb{D}_{x_1}^{v(y)}b)(x_N)\big\},$$

$$\hat{y}_N = \sum_{m=1}^{N}\big\{g^\alpha_{m,v(y)}(x_N) + a_m^{-v(y)}(\mathbb{D}_{x_1}^{v(y)}g\circ L_m)(x_N) - a_m^{-v(y)}\alpha_m(x_N)(\mathbb{D}_{x_1}^{v(y)}b)(x_N)\big\}\Big/\left\{1-\sum_{m=1}^{N}a_m^{-v(y)}\alpha_m(x_N)\right\}.$$

Proof. The WM variable order fractional derivative of g^α is given by

$$(\mathbb{D}_{x_1}^{v(y)}g^\alpha)(L_k(x)) = \frac{v(y)}{\Gamma(1-v(y))}\int_{x_1}^{L_k(x)}\frac{g^\alpha(L_k(x))-g^\alpha(s)}{(L_k(x)-s)^{1+v(y)}}ds.$$

Utilizing Lemma 1, the above equation is modified as

$$= \frac{v(y)}{\Gamma(1-v(y))}\int_{x_1}^{x_{k-1}}\frac{g^\alpha(x_{k-1})-g^\alpha(s)}{(x_{k-1}-s)^{1+v(y)}}ds - \frac{v(y)}{\Gamma(1-v(y))}\int_{x_1}^{x_{k-1}}\frac{g^\alpha(x_{k-1})-g^\alpha(s)}{(x_{k-1}-s)^{1+v(y)}}ds$$
$$+ \frac{v(y)}{\Gamma(1-v(y))}\int_{x_1}^{x_{k-1}}\frac{g^\alpha(L_k(x))-g^\alpha(s)}{(L_k(x)-s)^{1+v(y)}}ds + \frac{a_k v(y)}{\Gamma(1-v(y))}\int_{x_0}^{x}\frac{g^\alpha(L_k(x))-g^\alpha(L_k(u))}{(L_k(x)-L_k(u))^{1+v(y)}}du.$$

The following equation is obtained using the functional equation of g^α:

$$= \hat{y}_{k-1} + g^\alpha_{k,v(y)}(x) + \frac{a_k v(y)}{\Gamma(1-v(y))} \times \int_{x_0}^{x}\frac{\alpha_k(x)g^\alpha(x)+q_k(x)-\alpha_k(u)g^\alpha(u) - q_k(u)}{(a_k x + b_k - a_k u - b_k)^{1+v(y)}}du$$

$$= \hat{y}_{k-1} + g^{\alpha}_{k,v(y)}(x) + a_k^{-v(y)}\mathbb{D}^{v(y)}_{x_1}[\alpha_k(x)g^{\alpha}(x)]$$
$$+ a_k^{-v(y)}(\mathbb{D}^{v(y)}_{x_1} g \circ L_k)(x)$$
$$- a_k^{-v(y)}\mathbb{D}^{v(y)}_{x_1}[\alpha_k(x)b(x)].$$

On applying the Leibniz's rule, it is seen that

$$= \hat{y}_{k-1} + g^{\alpha}_{k,v(y)}(x) + a_k^{-v(y)}\sum_{j=0}^{\infty}\binom{v(y)}{j}(\mathbb{D}^{j}_{x_1}\alpha_k(x))$$
$$\times (\mathbb{D}^{v(y)-j}_{x_1} g^{\alpha}(x)) + a_k^{-v(y)}(\mathbb{D}^{v(y)}_{x_1} g \circ L_k)(x)$$
$$- a_k^{-v(y)}\sum_{j=0}^{\infty}\binom{v(y)}{j}(\mathbb{D}^{j}_{x_1}\alpha_k(x))(\mathbb{D}^{v(y)-j}_{x_1} b(x))$$
$$= \hat{y}_{k-1} + g^{\alpha}_{k,v(y)}(x) + a_k^{-v(y)}\alpha_k(x)(\mathbb{D}^{v}_{x_1} g^{\alpha}(x))$$
$$+ a_k^{-v(y)}\sum_{j=1}^{\infty}\binom{v(y)}{j}(\mathbb{D}^{j}_{x_1}\alpha_k(x))(\mathbb{D}^{v(y)-j}_{x_1} g^{\alpha}(x))$$
$$+ a_k^{-v(y)}(\mathbb{D}^{v(y)}_{x_1} g \circ L_k)(x) - a_k^{-v(y)}\alpha_k(x)(\mathbb{D}^{v}_{x_1} b(x))$$
$$- a_k^{-v(y)}\sum_{j=1}^{\infty}\binom{v(y)}{j}(\mathbb{D}^{j}_{x_1}\alpha_k(x))(\mathbb{D}^{v(y)-j}_{x_1} b(x))$$
$$= a_k^{-v(y)}\alpha_k(x)(\mathbb{D}^{v(y)}_{x_1} g^{\alpha})(x) + \hat{q}_k(x)$$
$$= \hat{F}_k(x, (\mathbb{D}^{v(y)}_{x_1} g^{\alpha})(x)).$$

Notations employed in the above equations are as follows:

$$\hat{q}_k(x) = \hat{y}_{k-1} + g^{\alpha}_{k,v(y)}(x)$$
$$+ a_k^{-v(y)}(\mathbb{D}^{v(y)}_{x_1} g \circ L_k)(x)$$
$$- a_k^{-v(y)}\alpha_k(x)(\mathbb{D}^{v(y)}_{x_1} b(x)),$$

$$g^{\alpha}_{k,v(y)}(x) = \frac{v(y)}{\Gamma(1-v(y))}\int_{x_1}^{x_{k-1}}\left(\frac{g^{\alpha}(L_k(x)) - g^{\alpha}(s)}{(L_k(x)-s)^{1+v(y)}} - \frac{g^{\alpha}(x_{k-1}) - g^{\alpha}(s)}{(x_{k-1}-s)^{1+v(y)}}\right)ds.$$

Consider the following equation to determine the new data points corresponding to the function $\mathbb{D}^{v(y)}_{x_1} g^{\alpha}$:

$$(\mathbb{D}^{v(y)}_{x_1} g^{\alpha})(L_k(x))$$
$$= \hat{y}_{k-1} + g^{\alpha}_{k,v(y)}(x) + a_k^{-v(y)}\alpha_k(x)[D^{v}_{x_1} g^{\alpha}(x)]$$
$$+ a_k^{-v(y)}(\mathbb{D}^{v(y)}_{x_1} g \circ L_k)(x)$$
$$- a_k^{-v(y)}\alpha_k(x)(\mathbb{D}^{v(y)}_{x_1} b)(x). \tag{7}$$

By taking $x = x_N$ and $L_k(x_N) = x_k$ in Eq. (7) and employing the system of equations $\hat{y}_k = \hat{y}_1 + \sum_{j=1}^{k}(\hat{y}_j - \hat{y}_{j-1})$, one can get

$$\hat{y}_k = \sum_{m=1}^{k}\big\{(g^{\alpha}_{m,v(y)}(x_N) + a_m^{-v(y)}\alpha_m(x_N)\hat{y}_N$$
$$+ a_m^{-v(y)}(\mathbb{D}^{v(y)}_{x_1} g \circ L_m)(x_N)$$
$$- a_m^{-v(y)}\alpha_m(x_N)(\mathbb{D}^{v(y)}_{x_1} b)(x_N))\big\}.$$

By substituting $k = N$, $\hat{y}_N$ is estimated as

$$\hat{y}_N = \sum_{m=1}^{N}\big\{(g^{\alpha}_{m,v(y)}(x_N) + a_m^{-v(y)}(\mathbb{D}^{v(y)}_{x_1} g \circ L_m)(x_N) - a_m^{-v(y)}\alpha_m(x_N)(\mathbb{D}^{v(y)}_{x_1} b)(x_N))\big\} \Big/ \left\{1 - \sum_{m=1}^{N} a_m^{-v(y)}\alpha_m(x_N)\right\}.$$

The scaling factors satisfy $\|\alpha_k(x)\|_{\infty} < |a_k^{v(y)}|$ and the join-up conditions of the function $(\mathbb{D}^{v(y)}_{x_1} g^{\alpha})$: $\hat{F}_k(x_1, \hat{y}_1) = \hat{y}_{k-1}$, $\hat{F}_k(x_N, \hat{y}_N) = \hat{y}_k$ are also satisfied. Hence, the WM variable order fractional derivative of a α-fractal function is again a fractal function of same kind corresponding to $(\mathbb{D}^{v(y)}_{x_1} g)(x)$.

Corollary 3. *Similar to Theorem* 1, *the WM variable order fractional derivative of an α-fractal function defined at the end point x_N can be explored if the fractional derivatives of g and b,*

$$(\mathbb{D}^{v(y)}_{x_N} g)(x) = -\frac{v(y)}{\Gamma(1-v(y))}\int_{x}^{x_N}\frac{g(x)-g(t)}{(x-t)^{1+v(y)}}dt,$$

$$(\mathbb{D}^{v(y)}_{x_N} b)(x) = -\frac{v(y)}{\Gamma(1-v(y))}\int_{x}^{x_N}\frac{b(x)-b(t)}{(x-t)^{1+v(y)}}dt,$$

exist and satisfy $(\mathbb{D}^{v(y)}_{x_N} g)(x_1) = (\mathbb{D}^{v(y)}_{x_N} b)(x_1)$ *and* $\hat{y}_N = (\mathbb{D}^{v(y)}_{x_N} g)(x_N) = (\mathbb{D}^{v(y)}_{x_N} b)(x_N) = 0$. *Then, the function* $\mathbb{D}^{v(y)}_{x_N} g^{\alpha}(x)$ *is again a α-fractal function* *satisfying* $\|\alpha_k(x)\|_\infty < |a_k^{v(y)}|$, *where* $\hat{F}_k(x,\hat{y}) = a_k^{-v(y)}\alpha_k(x)\hat{y} + \hat{q}_k(x)$ *and* $\sum_{k=1}^{N} a_k^{-v(y)}\alpha_k(x_1) \neq 1$, *for* $k \in \mathbb{N}_{N-1}$,

$$\hat{q}_k(x) = \hat{y}_k - g^{\alpha}_{k,v(y)}(x) + a_k^{-v(y)}(\mathbb{D}^{v(y)}_{x_N} g \circ L_k)(x) - a_k^{-v(y)}\alpha_k(x)(\mathbb{D}^{v(y)}_{x_1} b)x,$$

$$g^{\alpha}_{k,v(y)}(x) = \frac{v(y)}{\Gamma(1-v(y))} \times \int_{x_k}^{x_N} \left(\frac{g^{\alpha}(L_k(x)) - g^{\alpha}(s)}{(L_k(x)-s)^{1+v(y)}} - \frac{g^{\alpha}(x_k) - g^{\alpha}(s)}{(x_k - s)^{1+v(y)}} \right) ds,$$

$$\hat{y}_{k-1} = -\sum_{m=k}^{N} \big(g^{\alpha}_{m,v(y)}(x_1) - a_m^{-v(y)}\alpha_m(x_1)\hat{y}_1 - a_m^{-v(y)}(\mathbb{D}^{v(y)}_{x_N} g \circ L_m)(x_1) + a_m^{-v(y)}\alpha_m(x_1)(\mathbb{D}^{v(y)}_{x_N} b)(x_1) \big),$$

$$\hat{y}_1 = -\sum_{m=2}^{N} \big(g^{\alpha}_{m,v(y)}(x_1) - a_m^{-v(y)}(\mathbb{D}^{v(y)}_{x_N} g \circ L_m)(x_1) + a_m^{-v(y)}\alpha_m(x_1)(\mathbb{D}^{v(y)}_{x_N} b)(x_1) \big) \Big/ \left\{ 1 - \sum_{m=2}^{N} a_m^{-v(y)}\alpha_m(x_1) \right\}.$$

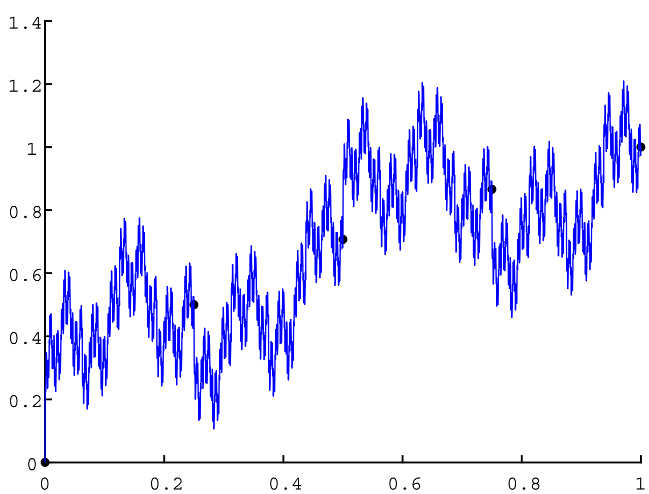
The proof of the corollary is a similar consequence of Theorem 2. □

Example 4. Let $g(x) = x^{1/3}$ be the given continuous function on the closed interval $[0,1]$. The α-fractal function is generated for the choice of scale vector α with the components $\alpha_k = (0.7, -0.7, 0.7, -0.7)$ for $k = 4$ and base function $b(x) = x$. Figures 1a and 1b represent the graph of α-fractal function g^{α} and its WM fractional derivative $\mathbb{D}^{0.2}g^{\alpha}$ with order 0.2 for the germ function g.

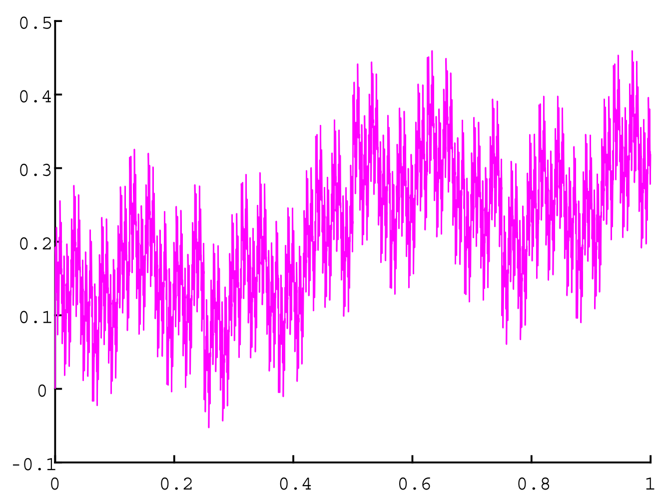
3. DEFINITE INTEGRAL OF α-FRACTAL FUNCTION

In this section, the scaling function $\alpha_k : [x_1, x_N] \to (0,1)$ is taken to be a continuous function such that $\|\alpha\|_\infty < 1$ and α' exists, and some results in relation to the definite integral of α-fractal function are established.

The following theorem gives the definite integral of α-fractal function for various intervals.



(a)



(b)

Fig. 1 Graphical representation of α-fractal function g^{α} and its WM fractional derivative $\mathbb{D}^{0.2}g^{\alpha}$.

Theorem 5. *Suppose g^α is an α-fractal function satisfying $\sum_{i=1}^{N-1} a_i\alpha_i(x_N) \neq 1$. Then*

$$\int_{x_1}^{x_N} g^\alpha(s)ds = \sum_{i=1}^{N-1} a_i\Bigg\{\int_{x_1}^{x_N}\{g(L_i(s)) - \alpha_i(s)b(s)\}ds - \Bigg\{\int_{x_1}^{x_N}\left(\alpha_i'(s)\int_{x_1}^{s} g(u)du\right)ds\Bigg\}\Bigg\}\Bigg/\Bigg\{1 - \sum_{i=1}^{N-1} a_i\alpha_i(x_N)\Bigg\},$$

$$\int_{x_1}^{x_k} g^\alpha(s)ds = \frac{\sum_{i=1}^{k-1} a_i\alpha_i(x_N)}{1-\sum_{i=1}^{N-1} a_i\alpha_i(x_N)}\sum_{i=1}^{N-1} a_i\Bigg\{\int_{x_1}^{x_N}\{g(L_i(s)) - \alpha_i(s)b(s)\}ds - \Bigg\{\int_{x_1}^{x_N}\left(\alpha_i'(s)\int_{x_1}^{s} g(u)du\right)ds\Bigg\}\Bigg\} + \sum_{j=1}^{k-1} a_j\int_{x_1}^{x_N}\{g(L_j(s)) - \alpha_j(s)b(s)\}ds - \sum_{i=1}^{k-1} a_i\int_{x_1}^{x_N}\left(\alpha_i'(s)\int_{x_1}^{s} g(u)du\right)ds,$$

$$\int_{x_k}^{x_{k+1}} g^\alpha(s)ds = a_k\alpha_k(x_N)\int_{x_1}^{x_N} g^\alpha(s)ds - a_k\Bigg\{\int_{x_1}^{x_N}\left(\alpha_k'(s)\int_{x_1}^{s} g(u)du\right)ds\Bigg\} + a_k\int_{x_1}^{x_N}\{g(L_k(s)) - \alpha_k(s)b(s)\}ds.$$

Proof. Consider

$$\begin{aligned}\int_{x_1}^{x_N} g^\alpha(s)ds &= \sum_{i=1}^{N-1} a_i\int_{x_1}^{x_N}\{g(L_i(s)) + \alpha_i(s)(g^\alpha - b)(s)\}ds\\ &= \sum_{i=1}^{N-1} a_i\alpha_i(x_N)\int_{x_1}^{x_N} g^\alpha(s)ds - \sum_{i=1}^{N-1} a_i\Bigg\{\int_{x_1}^{x_N}\left(\alpha_i'(s)\int_{x_1}^{s} g(u)du\right)ds\Bigg\} + \sum_{i=1}^{N-1} a_i\int_{x_1}^{x_N}\{g(L_i(s)) - \alpha_i(s)b(s)\}ds.\end{aligned}$$

The first formula follows from the last equality. Using the same argument, the second formula is computed as follows:

$$\begin{aligned}\int_{x_1}^{x_k} g^\alpha(s)ds &= \sum_{i=1}^{k-1} a_i\int_{x_1}^{x_N}\big\{\alpha_i(s)g^\alpha(s) + \{g(L_i(s)) - \alpha_i(s)b(s)\}\big\}ds\\ &= \sum_{i=1}^{k-1} a_i\alpha_i(x_N)\int_{x_1}^{x_N} g^\alpha(s)ds - \sum_{i=1}^{k-1} a_i\Bigg\{\int_{x_1}^{x_N}\left(\alpha_i'(s)\int_{x_1}^{s} g(u)du\right)ds\Bigg\} + \sum_{i=1}^{k-1} a_i\int_{x_1}^{x_N}\{g(L_i(s)) - \alpha_i(s)b(s)\}ds\\ &= \frac{\sum_{i=1}^{k-1} a_i\alpha_i(x_N)}{1-\sum_{i=1}^{N-1} a_i\alpha_i(x_N)}\sum_{i=1}^{N-1} a_i \times\Bigg\{\int_{x_1}^{x_N}\{g(L_i(s)) - \alpha_i(s)b(s)\}ds - \Bigg\{\int_{x_1}^{x_N}\left(\alpha_i'(s)\int_{x_1}^{s} g(u)du\right)ds\Bigg\}\Bigg\}\\ &\quad + \sum_{j=1}^{k-1} a_j\int_{x_1}^{x_N}\{g(L_j(s)) - \alpha_j(s)b(s)\}ds - \sum_{i=1}^{k-1} a_i\int_{x_1}^{x_N}\left(\alpha_i'(s)\int_{x_1}^{s} g(u)du\right)ds.\end{aligned}$$

Similarly, the third formula follows with the standard change of variable, $s = L_k(u)$. □

The following theorem is based on the fundamental results discussed in Ref. 3.

Theorem 6. *The hyperbolic IFS $\{I; L_k : k \in \mathbb{N}_{N-1}\}$ has a unique attractor I and for each $x \in I$,* *there exists a sequence* $\xi = \xi_1\xi_2\cdots\xi_n\cdots$ *such that* $\xi_n \in \{\mathbb{N}_{N-1}\}$ *and for all* $y \in I$,

$$x = L_\xi(y) = L_{\xi_1\xi_2\cdots\xi_n\cdots}(y) := L_{\xi_1}\circ L_{\xi_2}\circ\cdots\circ L_{\xi_n}\cdots(y).$$

For some x *in the interval* I, *this sequence is finite, with* $y = x_0$, *i.e.*

$$x = L_\xi(y) := L_{\xi_1}\circ L_{\xi_2}\circ\cdots\circ L_{\xi_n}(x_0),$$

where $n \in \mathbb{N}$.

Using this formulation and recursive procedure, the integration of g^α is obtained for any $x \in I$ as follows:

$$\int_{x_1}^{x_N} g^\alpha(s)ds = \sum_{k=1}^{N-1}\int_{x_k}^{x_{k+1}} g^\alpha(s)ds,$$

$$\int_{x_k}^{x_{k+1}} g^\alpha(s)ds = \sum_{j=1}^{N-1}\int_{L_j(x_k)}^{L_j(x_{k+1})} g^\alpha(s)ds,$$

$$\int_{L_{\xi_2}(x_k)}^{L_{\xi_2}(x_{k+1})} g^\alpha(s)ds = \sum_{\xi_1=1}^{N-1}\int_{L_{\xi_1\xi_2}(x_k)}^{L_{\xi_1\xi_2}(x_{k+1})} g^\alpha(s)ds.$$

For the integral of tiny sub-subintervals, general formulation is provided in the following lemma.

Lemma 7. *If* g^α *is an* α*-fractal function with variable parameters. Then,* $\forall\, n \in \mathbb{N}$,

$$\int_{L_{\xi_1\xi_2\cdots\xi_n}(x_1)}^{L_{\xi_1\xi_2\cdots\xi_n\xi_{n+1}}(x_1)} g^\alpha(s)ds$$

$$= \prod_{k=1}^{n} a_{\xi_k}\alpha_{\xi_k}(L_{\xi_2\cdots\xi_{n+1}}(x_1))\int_{x_1}^{x_{\xi_{n+1}}} g^\alpha(s)ds \quad (8)$$

$$- \sum_{j=1}^{n}\prod_{k=1}^{j-1} a_{\xi_k}\alpha_{\xi_k}(L_{\xi_2\cdots\xi_{n+1}}(x_1))a_{\xi_j} \times \left\{\int_{L_{\xi_{j+1}\cdots\xi_n}(x_1)}^{L_{\xi_{j+1}\cdots\xi_{n+1}}(x_1)}\left\{\alpha'_{\xi_j}(s) \times \int_{L_{\xi_{j+1}\cdots\xi_n}(x_1)}^{s} g^\alpha(u)du\right\}ds - \int_{L_{\xi_{j+1}\cdots\xi_n}(x_1)}^{L_{\xi_{j+1}\cdots\xi_{n+1}}(x_1)}\{g(L_{\xi_j}(s)) - \alpha_{\xi_{j}(s)}b(s)\}\right\}ds. \quad (9)$$

Proof. By mathematical induction, the proof follows. For $n = 1$,

$$\int_{L_{\xi_1}(x_1)}^{L_{\xi_1\xi_2}(x_1)} g^\alpha(s)ds = \int_{L_{\xi_1}(x_1)}^{L_{\xi_1}(x_{\xi_2})} g^\alpha(s)ds = a_{\xi_1}\int_{x_1}^{x_{\xi_2}} g^\alpha(L_{\xi_1}(s))ds$$

$$= a_{\xi_1}\alpha_{\xi_1}(x_{\xi_2})\int_{x_1}^{x_{\xi_2}} g^\alpha(s)ds - a_{\xi_1}\int_{x_1}^{x_{\xi_2}}\left\{\alpha'_{\xi_1}(s)\int_{x_1}^{s} g^\alpha(u)du\right\}ds + a_{\xi_1}\int_{x_1}^{L_{\xi_2}(x_1)}\{g(L_{\xi_1}(s)) - \alpha_{\xi_1}(s)b(s)\}ds.$$

Assume that Eq. (9) is valid for some $n \in \mathbb{N}$. Let

$$I = \int_{L_{\xi_1\xi_2\cdots\xi_{n+1}}(x_1)}^{L_{\xi_1\xi_2\cdots\xi_{n+2}}(x_1)} g^\alpha(s)ds = a_{\xi_1}\int_{L_{\xi_2\cdots\xi_{n+1}}(x_1)}^{L_{\xi_2\cdots\xi_{n+2}}(x_1)} g^\alpha(L_{\xi_1}(s))ds$$

$$= a_{\xi_1}\alpha_{\xi_1}(L_{\xi_2\cdots\xi_{n+2}}(x_1))\int_{L_{\xi_2\cdots\xi_{n+1}}(x_1)}^{L_{\xi_2\cdots\xi_{n+2}}(x_1)} g^\alpha(s)ds - a_{\xi_1}\int_{L_{\xi_2\cdots\xi_{n+1}}(x_1)}^{L_{\xi_2\cdots\xi_{n+2}}}\left\{\alpha'_{\xi_1}(s) \times \int_{L_{\xi_2\cdots\xi_{n+1}}(x_1)}^{s} g^\alpha(u)du\right\}ds + a_{\xi_1}\int_{L_{\xi_2\cdots\xi_{n+1}}(x_1)}^{L_{\xi_2\cdots\xi_{n+2}}(x_1)}\{g(L_{\xi_1}(s)) - \alpha_{\xi_1}(s)b(s)\}ds,$$

$$I = \prod_{k=1}^{n+1} a_{\xi_k}\alpha_{\xi_k}(L_{\xi_2\cdots\xi_{n+2}}(x_1))\int_{x_1}^{x_{\xi_{n+2}}} g^\alpha(s)ds - \sum_{j=1}^{n+1}\prod_{k=1}^{j-1} a_{\xi_k}\alpha_{\xi_k}(L_{\xi_2\cdots\xi_{n+2}}(x_1))a_{\xi_j} \times \left\{\int_{L_{\xi_{j+1}\cdots\xi_{n+1}}(x_1)}^{L_{\xi_{j+1}\cdots\xi_{n+2}}(x_1)}\left\{\alpha'_{\xi_j}(s)\right.\right.$$

$$\times \int_{L_{\xi_{j+1}\cdots\xi_{n+1}}(x_1)}^{s} g^{\alpha}(u)du \Bigg\} ds$$
$$- \int_{L_{\xi_{j+1}\cdots\xi_{n+1}}(x_1)}^{L_{\xi_{j+1}\cdots\xi_{n+2}}(x_1)} \{g(L_{\xi_j}(s)) - \alpha_{\xi_j}(s)b(s)\} \Bigg\} ds,$$

which concludes the proof. □

4. EXPLICIT SOLUTION OF α-FRACTAL FUNCTION

Consider the non-empty sets $\hat{X}$ and $\hat{Y}$ and an integer $q \geq 2$. Consider the following system of functional equations:

$$\phi(g_k(x)) = F_k(x, \phi(x)), \quad x \in \hat{X}_k, \tag{10}$$

for $k = 0, 1, \ldots, q-1$, where $\hat{X}_k \subset \hat{X}$, $g_k : \hat{X}_k \to W_k \subset X$, $F_k : \hat{X}_k \times Y \to \hat{Y}_k \subset Y$ are the provided functions and $\phi : \bigcup_{k=0}^{q-1} \hat{X}_k = \hat{X} \to \hat{Y}$ is the unknown continuous function. Suppose each map F_k is independent of x, then the system of equations (10) takes the form

$$\phi(g_k(x)) = F_k(\phi(x)), \quad x \in \hat{X}_k, \tag{11}$$

where $k = 0, 1, \ldots, q-1$.

Proposition 8. *Suppose $\phi : \hat{X} \to \hat{Y}$ is a solution of Eq.* (10), *then it obeys* $\forall\, x \in \hat{X}_j$, $\forall\, y \in \hat{X}_k$,

$$\begin{aligned} g_j(x) &= g_k(y), \\ F_j(x, \phi(x)) &= F_k(y, \phi(y)), \end{aligned} \tag{12}$$

for $j, k = 0, 1, \ldots, q-1$.

Proof. Assume that $g_j(x) = g_k(y)$, $\forall\, x \in \hat{X}_j$, $\forall\, y \in \hat{X}_k$. From Eq. (10),

$$\begin{aligned} \phi(g_j(x)) &= F_j(x, \phi(x)), \\ \phi(g_k(x)) &= F_k(x, \phi(x)), \end{aligned} \tag{13}$$

which implies $F_j(x, \phi(x)) = F_k(y, \phi(y))$ as stated. □

The conditions (12) are called the compatibility conditions (see Refs. 30 and 31) to the system (10) as they make sure that ϕ is well defined. Let $C = \{x \in X : \exists\, y \in X, \exists\, j, k = 0, 1, \ldots, q-1, j \neq k, g_j(x) = g_k(y)\}$. The elements of C are called the contact points for the system (10). The compatibility conditions are the necessary conditions for the existence of solutions as discussed in the following example if the images of ϕ for the contact points are estimated to the system (10).

Example 9. Let $\hat{X} = [0, 1]$. Consider the following system of fixed point equations,

$$\begin{aligned} \phi(g_0(x)) &= F_0(x, \phi(x)), \quad x \in [0, 1], \\ \phi(g_1(x)) &= F_1(x, \phi(x)), \quad x \in [0, 1]. \end{aligned} \tag{14}$$

Let $g_0(x) = \frac{x}{2}$, $g_1(x) = \frac{x+1}{2}$. Take $\hat{X}_0 = \hat{X}_1 = [0, 1]$. Note that

$$g_0(\hat{X}_0) \cap g_1(\hat{X}_1) = \frac{1}{2}$$

and

$$g_0(1) = \frac{1}{2} = g_1(0),$$

with the compatibility condition $F_0(1, \phi(1)) = F_1(0, \phi(0))$. Additionally, suppose

$$F_i(x, y) = \alpha_i(x)y + g \circ L_i(x) - \alpha_i(x)b(x),$$

then the corresponding compatibility condition is

$$\begin{aligned} &\alpha_0(1)\phi(1) + g \circ L_0(1) - \alpha_0(1)b(1) \\ &\quad = \alpha_1(0)\phi(0) + g \circ L_1(0) - \alpha_1(0)b(0). \end{aligned}$$

When the two equations in (14) are solved for $x = 0$ and $x = 1$, respectively, then the images of the contact points are obtained as

$$\begin{aligned} \phi(g_0(0)) &= F_0(0, \phi(0)) \\ &= \frac{g(0) - \alpha_0(0)b(0)}{1 - \alpha_0(0)} \end{aligned}$$

and

$$\begin{aligned} \phi(g_1(1)) &= F_1(1, \phi(1)) \\ &= \frac{g(1) - \alpha_1(1)b(1)}{1 - \alpha_1(1)}. \end{aligned}$$

The compatibility condition on the functions F_0 and F_1 is

$$\begin{aligned} &\alpha_0(1)\frac{g(1) - \alpha_1(1)b(1)}{1 - \alpha_1(1)} - \alpha_0(1)b(1) \\ &\quad = \alpha_1(0)\frac{g(0) - \alpha_0(0)b(0)}{1 - \alpha_0(0)} - \alpha_1(0)b(0). \end{aligned}$$

Theorem 10. *Let $q \in \{2, 3, 4, \ldots\}$. Let the continuous function $q_k : [0, 1] \to \mathbb{R}$ be defined by $q_k = g \circ L_k(x) - \alpha_k b(x)$, where g and b are real-valued continuous functions on* $[0, 1]$ *satisfying* $b(0) = g(0)$

and $b(1) = g(1)$ *with* $b \neq g$, $|\alpha_k| < 1$, *for* $0 \leq k \leq q-1$. *Suppose*

$$\frac{\alpha_{k-1}}{1-\alpha_{q-1}}(g \circ L_{q-1}(1) - \alpha_{q-1}b(1)) + g \circ L_{k-1}(1) - \alpha_{k-1}b(1) = \frac{\alpha_k}{1-\alpha_0}(g \circ L_0(0) - \alpha_0 b(0)) \tag{15}$$

$$+ g \circ L_k(0) - \alpha_k b(0). \tag{16}$$

Then there is a unique bounded $\phi : [0,1] \to \mathbb{R}$ *obeying the system*

$$\phi\left(\frac{x+k}{q}\right) = \alpha_k \phi(x) + g \circ L_k(x) - \alpha_k b(x), \quad x \in [0,1], \tag{17}$$

for $0 \leq k \leq q-1$. *The function* ϕ *is continuous and is expressed in terms of the base* q *expansion of* x *by*

$$\phi\left(\sum_{n=1}^{\infty}\frac{\xi_n}{q^n}\right) = \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n-1}\alpha_{\xi_k}\right)\left\{g \circ L_{\xi_n}\left(\sum_{k=1}^{\infty}\frac{\xi_{k+n}}{q^k}\right) - \alpha_{\xi_n} b\left(\sum_{k=1}^{\infty}\frac{\xi_{k+n}}{q^k}\right)\right\}. \tag{18}$$

Proof. This is given by Theorem 3 in Ref. 29, where $s_k = \alpha_k$, and $r_k(x) = g \circ L_k(x) - \alpha_k b(x)$. □

The following theorem provides the generalization of Theorem 10 by taking the constants α_k as the continuous functions on the interval $[0,1]$.

Theorem 11. *Let* $q \in \{2, 3, 4, \ldots\}$. *Let* q_k *and* α_k *be real-valued continuous functions on* $[0,1]$ *satisfying* $|\alpha_k(x)| < 1$, $\forall\, x \in [0,1]$, *where* $q_k = g \circ L_k(x) - \alpha_k(x)b(x)$, *satisfying* $b(0) = g(0)$ *and* $b(1) = g(1)$ *with* $b \neq g$. *Assume that the condition*

$$\frac{\alpha_k(1)}{1-\alpha_{q-1}(1)}(g \circ L_{q-1}(1) - \alpha_{q-1}(1)b(1)) + g \circ L_k(1) - \alpha_k(1)b(1) = \frac{\alpha_{k+1}(0)}{1-\alpha_0(0)}(g \circ L_0(0) - \alpha_0(0)b(0)) + g \circ L_{k+1}(0) - \alpha_{k+1}(0)b(0)$$

is satisfied, for any $k \in \{0, 1, \ldots, q-2\}$. *Then there is a unique bounded* $\phi : [0,1] \to \mathbb{R}$ *satisfying the system*

$$\phi\left(\frac{x+k}{q}\right) = \alpha_k(x)\phi(x) + g \circ L_k(x) - \alpha_k(x)b(x), \tag{19}$$

for $x \in [0,1], 0 \leq k \leq q-1$. *The function* ϕ *is continuous and is expressed in terms of the base* q *expansion of* x *by*

$$\phi\left(\sum_{n=1}^{\infty}\frac{\xi_n}{q^n}\right) = \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n-1}\alpha_{\xi_k}\left(\sum_{k=1}^{\infty}\frac{\xi_{k+m}}{q^k}\right)\right) \times \left\{g \circ L_{\xi_n}\left(\sum_{k=1}^{\infty}\frac{\xi_{k+n}}{q^k}\right) - \alpha_{\xi_n} \times \left(\sum_{k=1}^{\infty}\frac{\xi_{k+n}}{q^k}\right) b\left(\sum_{k=1}^{\infty}\frac{\xi_{k+n}}{q^k}\right)\right\}.$$

Proof. The proof is provided by Theorem 2 of Ref. 30, where the continuous functions $s_k(x)$ and $r_k(x)$ are, respectively, taken as $\alpha_k(x)$ and $g \circ L_k(x) - \alpha_k(x)b(x)$. □

This result is applied to determine the α-fractal function by considering the following problem:

$$\phi(L_k(x)) = \alpha_k(x)\phi(x) + g \circ L_k(x) - \alpha_k(x)b(x),$$

for $k = 0, 1, \ldots, q-1$, and

$$L_k(x) = \frac{x+k}{q}, \quad k = \{0, 1, \ldots, q-1\}, \ \ x \in [0,1].$$

Since $L_k(1) = L_{k+1}(0)$ and from the compatibility conditions

$$F_k(1, \phi(1)) = F_{k+1}(0, \phi(0))$$

$$\Leftrightarrow$$

$$\alpha_k(1)\phi(1) + g \circ L_k(1) - \alpha_k(1)b(1) = \alpha_{k+1}(0)\phi(0) + g \circ L_{k+1}(0) - \alpha_{k+1}(0)b(0),$$

for $k = 0, 1, \ldots, q-2$, we have

$$\phi(0) = \frac{g(0) - \alpha_0(0)b(0)}{1-\alpha_0(0)} \quad \text{and}$$

$$\phi(1) = \frac{g(1) - \alpha_{q-1}(1)b(1)}{1-\alpha_{q-1}(1)}$$

which imply

$$F_k\left(1, \frac{g(1) - \alpha_{q-1}(1)b(1)}{1-\alpha_{q-1}(1)}\right) = F_{k+1}\left(0, \frac{g(0) - \alpha_0(0)b(0)}{1-\alpha_0(0)}\right).$$

The endpoint conditions of F_k ensures that the compatibility conditions are satisfied. Further,

$$\frac{g(0) - \alpha_0(0)b(0)}{1-\alpha_0(0)} = y_0,$$

$$\frac{g(1) - \alpha_{q-1}(1)b(1)}{1-\alpha_{q-1}(1)} = y_q.$$

For the problem of general $L_k(x)$, the constructive formula is given by Theorem 4 in Ref. 30 as given in the following.

Theorem 12. *If ϕ is the α-fractal function solution of $\phi(L_k(x)) = F_k(x, \phi(X))$, $\forall\, k = 1, 2, \ldots, N$ with $L_k(x) = a_k x + x_{k-1}, 1 \le k \le N$ and $F_k(x, y) = \alpha_k(x)y + g \circ L_k(x) - \alpha_k(x)b(x)$ then ϕ is expressed in terms of Q-expansion*

$$x = \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n-1} a_{i_k} \right) x_{i_n - 1}$$

by

$$\begin{aligned}\phi(x) = \sum_{t=1}^{\infty} \left[\prod_{n=1}^{t-1} \alpha_{i_n} \left(\sum_{l=1}^{\infty} \left(\prod_{k=1}^{l-1} a_{i_{n+k}} \right) x_{i_{n+l}-1} \right) \right] \\ + g \circ L_{i_t} \left(\sum_{l=1}^{\infty} \left(\prod_{k=1}^{l-1} a_{i_{n+k}} \right) x_{i_{n+l}-1} \right) \\ - \alpha_{i_t} \left(\sum_{l=1}^{\infty} \left(\prod_{k=1}^{l-1} a_{i_{n+k}} \right) x_{i_{n+l}-1} \right) \\ \times b \left(\sum_{l=1}^{\infty} \left(\prod_{k=1}^{l-1} a_{i_{n+k}} \right) x_{i_{n+l}-1} \right).\end{aligned}$$

Proof. The proof follows from Theorem 4 of Ref. 30 with the application of Lemma 3.1 in Ref. 19, where $q_{i_t}(x) = g \circ L_{i_t}(x) - \alpha_{i_t}(x)b(x)$. □

This section has provided the explicit solution for the α-fractal functions with variable parameters in terms of base q representation of numbers.

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ORCID

T. M. C. Priyanka https://orcid.org/0000-0002-7473-9524
C. Serpa https://orcid.org/0000-0002-8561-118X
A. Gowrisankar https://orcid.org/0000-0002-5093-2805

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