Simulation of default events in a CDX and estimation of the spread

D.V. Boreiko, School of Economics and Management
Free University of Bozen-Bolzano
Piazza Università 1, 39100 Bolzano, Italy; e-mail: DBoreiko@unibz.it
Y.M. Kaniovski, School of Economics and Management
Free University of Bozen-Bolzano
Piazza Università 1, 39100 Bolzano, Italy; e-mail: YKaniovskyi@unibz.it
and
G.Ch.Pflug, Department of Statistics and Decision Support Systems
University of Vienna
Universitätstraße 5, 1090 Wien, Austria; e-mail: georg.pflug@univie.ac.at

Summary. The portfolio generating the iTraxx EUR index is modeled by coupled Markov chains. Each of the industries of the portfolio evolves according to its own Markov transition matrix. Using a variant of the method of moments, the model parameters are estimated from a data set of Standard and Poor's. Swap spreads are evaluated by Monte-Carlo simulations. Along with an actuarially fair spread, a least squares spread is considered.

Key words: Markov transition matrix, credit risk, credit events correlation, spread, tranche, recovery rate, percentile.

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1 Introduction

The recent financial crisis once again manifests the interdependence of business institutions worldwide. The exact mechanism of this interrelation is too complex to be documented in all its details. Nevertheless, investors need instruments for protection against its adverse consequences. A market index, like the Dow Jones iTraxx EUR, provides with a benchmark for measuring this risk. Having corresponding default swap spreads, an investor may choose a protection contract according to his risk preferences. Even being created by leading experts in the area, market indexes mimic only a segment of the economy, albeit representative enough. Simulating the evolution of an index portfolio allows to better understand the interdependence of its components as well as to justify the adopted pricing. The techniques are similar to what is known for the collateralized debt obligations (CDOs). They have been intensively studied from the early 1990s. Describing the corresponding model, we may use both names or even just CDO as an object with a longer history.

There are two possibilities. In a broader perspective, one is interested in modeling the portfolio evolution. A more narrow and pragmatic approach focuses only on obtaining the spreads. The phenomenon of correlation smile reported in the literature, see Amato and Gyntelberg (2005) for example, turns out to be a complication under the first view, but it does not create any problem under the second one.

The structural models have been a standard method for generating dependent defaults in a CDO. See Vasicek (1987) as an early example. They are based on Merton's (1974) representation for the asset position of a firm. The total risk is decomposed into an idiosyncratic part and a common factor relating different debtors. This construction, depending upon the distribution of the components, is referred to also as a Gaussian copula, a double t-distribution copula, etc. See Hull and White (2004), for example. The factor copulas approach by Laurent and Gregory (2005) is a further development of this class of models.

Duffie and Singleton (1999) model default intensities for different debtors as correlated stochastic processes. Willeman (2007) and Frey and Backhaus (2008) are among recent contributions in this direction.

Li (2000) pioneered the use of survival theory together with copulae in credit risk models. Choroś et al. (2009) represent the research frontier in this area.

The coupled Markov chain model by Kaniovski and Plug (2007) deals with a credit rating transition matrix and takes into account the dependence between the transitions in the simplest and the most natural way - using the corresponding credit event correlations. Hochreiter and Wozabal (2009) use the model to estimate the spreads generated by a portfolio that mimic the iTraxx EUR index. Allowing that every industry is governed by its own Markov transition matrix, we develop further this approach.

Calibration with the real data is a fundamental problem for all these models. In some cases, like Choroś et al. (2009) or Hochreiter and Wozabal (2009), the complexity of functions involved renders it difficult to guarantee that there is a unique global solution and that the employed algorithm does not lock in a local one. To avoid this complication, we use a method of moments.

Comparison of the assumptions employed by different approaches represents another problem. The structural models use typically asset correlations which, due to the employed construction, are also the default time correlations. Applying general copulae, the dependence of the marginals is typically reported as the Spearman rank correlation coefficient.

Recently the existing techniques for risk assessment have been severely criticized. Duffie et al. (2009, p. 2089) write: "At the high confidence levels at which portfolio default losses are typically estimated for meeting bank capital requirements and rating collateralized debt obligations (CDOs), our empirical results indicate that conventional estimators are downward biased by a full order of magnitude on typical test portfolios". Explaining the cause of this phenomenon, Jorion and Zhang (2009, p.2054) note that: "Unexpected default clustering is a major issue for traditional credit risk models because it generates greater dispersion, or fatter tails, in the distribution of credit losses."

The coupled Markov chain model simulates the evolution of a portfolio formed by interdependent debtors belonging to several industry sectors. The credit events correlations differs across the pool of debtors. As the correlations increase, the model, see Kaniovski and Plug (2007) or Boreiko et al. (2010), exhibits a skewed to the right cascade pattern. In fact, the support of the distribution of the number of defaults is disconnected. Being unobservable, the tendency variables are responsible for these atypically big losses. They are a particular case of frailty factors discussed in recent publications, see Duffie et al. (2009) or Koopman et al. (2008) among others.

Here we apply the model to valuate a CDO using Monte-Carlo simulations. Section 2 describes a modification, where every industry sector is governed by its own Markov transition matrix, of the basic scheme. Section 3 itroduces estimation schemes for parameters of this modification. The flow of payments generated by the model in a tranche of a CDO is defined in section 4. Next we discuss two possibilities to define the spread corresponding to a tranche of a CDO. Along with the actuarially fair spread used traditionally, section 5 treats a least squares spread. Section 6 contains estimates of the parameters and estimated spreads. A couple of observations useful for comparing our results with the known ones, are presented in section 7.

2 A modification of the coupled Markov chains model

We extend the coupling approach by Kaniovski and Pflug (2007) to a situation when the Markovian transition probabilities differ across the debtors. As a consequence, we have to modify the notion of a non-deteriorating move as well.

Consider a portfolio consisting of $\mathcal{N}(0)$ unit debt obligations. Following Nagpal and Bahar (2001), we distinguish only two non-default categories – investment grade and non-investment grade debtors. In terms of Standard and Poor's, the former contains ratings from AAA to BBB, while the latter covers BB or lower ones. Let there be $K \geq 1$ industry sectors. Set $\mathcal{N}^{(k)}(0)$ for the initial number of debtors belonging to industry sector k.

The evolution of a debtor belonging to industry k is governed by a 3×3 Markov transition matrix $P^{(k)}$ with elements $p^{(k)}_{i,j}$. States 1 and 2 are transient. They are occupied by investment grade and non-investment grade firms respectively, while 3 is an absorbing state. It hosts defaulted firms. These $\mathcal{N}(0)$ time homogeneous Markov chains $X^{(k)}_n(t)$, $t = 0, 1, \ldots$, are dependent. Conceptually, $X^{(k)}_n(t)$ is the credit rating of debtor n from industry sector k at year t. Because of time homogeneity, it is enough to describe the transition from time $t = 0$ to time $t = 1$.

A tendency variable χ_i takes on two values: 1 and 0. Let $\mathbb{P}\mathrm{r}\{\chi_i = 1\} = p_i$. Together with the coefficient of correlation $c_{1,2}$ between χ_1 and χ_2, the probabilities p_1 and p_2 define the distribution of the tendecy vector $\vec{\chi} = (\chi_1, \chi_2)$. It is not observable, but, affecting every debtor in the portfolio, it makes their evolutions dependent.

The strength of the dependence varies across credit classes and industries. The parameter $q^k_i \in [0, 1]$ measures the impact of the tendency on a debtor belonging to industry sector k and credit rating i. For a fixed k, we set

$$X^{(k)}_n(1) = [1 - \delta^{(k)}_n]\xi^{(k)}_n + \delta^{(k)}_n \eta^{(k)}_n, \; n = 1, 2, \ldots, \mathcal{N}^{(k)}(0). \tag{1}$$

The random variables $\delta^{(k)}_n$ and $\xi^{(k)}_n$ are independent in k and n. $\delta^{(k)}_n$ takes on two values: 1 and 0. $\mathbb{P}\mathrm{r}\{\delta^{(k)}_n = 1\} = q^k_{X^{(k)}_n(0)}$. $\xi^{(k)}_n$ assumes three values:

$$\xi^{(k)}_n = \begin{cases} 1 \text{ with probability } p^{(k)}_{X^{(k)}_n(0),1}, \\ 2 \text{ with probability } p^{(k)}_{X^{(k)}_n(0),2}, \\ 3 \text{ with probability } p^{(k)}_{X^{(k)}_n(0),3}. \end{cases}$$

The random variables $\eta^{(k)}_n$ do not depend upon all $\delta^{(l)}_s$ and $\xi^{(l)}_s$. $\eta^{(k)}_n$ takes on three values: 1, 2 and 3. Given a realization of the tendency vector, the random variables $\eta^{(k)}_n$ are independent in k and n. Set $p^{(k)}_{X^{(k)}_n(0),j}(\chi_{X^{(k)}_n(0)})$ for the corresponding conditional probabilities. We write $p^{(k)}_{X^{(k)}_n(0),j}(\chi_{X^{(k)}_n(0)})$ rather than $p^{(k)}_{X^{(k)}_n(0),j}(\vec{\chi})$ because the conditional distribution depends only upon the corresponding coordinate. Then

$$p^{(k)}_{i,j}(1) = \begin{cases} \frac{p^{(k)}_{i,j}}{p_i} \text{ for } j < i, \\ \frac{p_i - \pi^{(k)}_i}{p_i} \text{ for } j = i, \\ 0 \text{ for } j > i; \end{cases} \qquad p^{(k)}_{i,j}(0) = \begin{cases} \frac{p^{(k)}_{i,j}}{1-p_i} \text{ for } j > i, \\ \frac{1-p_i-\Pi^{(k)}_i}{1-p_i} \text{ for } j = i, \\ 0 \text{ for } j < i. \end{cases}$$

Here

$$\pi^{(k)}_i = \sum_{j<i} p^{(k)}_{i,j} \quad \text{and} \quad \Pi^{(k)}_i = \sum_{j>i} p^{(k)}_{i,j}.$$

Since $p_i - \pi^{(k)}_i + 1 - p_i - \Pi^{(k)}_i = p^{(k)}_{i,i}$, $\eta^{(k)}_n$ has the same (unconditional) distribution as $\xi^{(k)}_n$. Feasibility of this scheme requires that

$$p_i \geq \max_k \pi^{(k)}_i \quad \text{and} \quad 1 - p_i \geq \max_k \Pi^{(k)}_i. \tag{2}$$

Thus, the random variables $\eta_n^{(k)}$ are dependent through $\vec{\chi}$. $\chi_i = 1$ determines a non-deteriorating tendency for all debtors belonging to credit class i: the credit rating cannot worsen in this case. $\chi_i = 0$ corresponds to the opposite situation: none from credit class i improves its creditworthiness.

The counts $N_i^k(1)$ at time $t = 1$ of debtors in credit class i belonging to industry k are obtained by the following formula

$$N_i^k(1) = \sum_{n=1}^{\mathcal{N}^k(0)} \mathbb{1}_{\{X_n^k(1)=i\}}.$$

Here $\mathbb{1}_A$ denotes the indicator of a random event A. The number of defaults $D_i^k(1)$ at time $t = 1$ of debtors in credit class i belonging to industry k reads

$$D_i^k(1) = N_i^k(0) - N_i^k(1),$$

where $N_i^k(0)$ is the initial allocation of debtors to industry sectors and credit classes in the portfolio.

Formula (1) means that the next year credit rating of debtor n in industry sector k obtains by randomizing the purely idiosyncratic factor $\xi_n^{(k)}$ and the term $\eta_n^{(k)}$ dependent through all debtors in the portfolio. The Bernoulli random variable $\delta_n^{(k)}$ determines the weight of the respective addend in this stochastic convex combination. As a result the correlation $\rho_{i,j}^{k,s}$ between the random events that a debtor from industry sector k having credit rating i defaults during the same year as a debtor from industry sector s having credit rating j reads

$$\rho_{i,j}^{k,l} = c_{i,j} q_i^k q_j^l r_{i,j}^{k,l} s_{i,j}, \tag{3}$$

where $c_{1,1} = c_{2,2} = 1$, $s_{i,j}^2 = s_{i,i} s_{j,j}$,

$$r_{i,j}^{k,l} = \sqrt{\frac{p_{i,3}^{(k)} p_{j,3}^{(l)}}{[1 - p_{i,3}^{(k)}][1 - p_{j,3}^{(l)}]}}, \quad \text{and} \quad s_{i,j} = \sqrt{\frac{p_i p_j}{(1 - p_i)(1 - p_j)}}.$$

Since $c_{1,2}$ may take on both positive and negative values, the default event correlations of debtors belonging to different credit classes may be positive as well as negative. The positive values are easily justifiable: when the economy is doing well or poorly, this tendency should affect all firms, albeit with in varying degree, independently of their credit ratings. However, there might be also negative correlations when default of a firm allows its competitors to gain market share. Discussing this situation, Jorion and Zhang (2009, p. 2057) conclude that "the net of these two effects is intra-industry contagion" or, in our terms, a positive $c_{i,j}$. Then for $\min\left(\sqrt{\frac{s_{1,1}}{s_{2,2}}}, \sqrt{\frac{s_{2,2}}{s_{1,1}}}\right) \geq c_{1,2}$, the probabilities associated with the tendency vector are non-negative.

The conditional probability $DP_i^k(\chi_i)$ that a belonging to industry k debtor having credit rating i at the beginning of a year defaults by the end of this year reads

$$DP_i^k(\chi_i) = \begin{cases} (1 - q_i^k) p_{i,3}^{(k)} & \text{for } \chi_i = 1, \\ p_{i,3}^{(k)} (1 - q_i^k + \frac{q_i^k}{1 - p_i}) & \text{for } \chi_i = 0. \end{cases} \tag{4}$$

Since this probability depends only on the corresponding coordinate of $\vec{\chi}$, we denote it $DP_i^k(\chi_i)$ rather than $DP_i^k(\vec{\chi})$.

3 A scheme for parameter estimation

The values $r_{i,j}^{k,l}$ are known as long as the corresponding Markovian transition matrices have been estimated. Consequently, to specify the model, we have to find $c_{1,2}, s_{1,1}, s_{2,2}$ and q_i^k. The default correlations have been estimated. See Nagpal and Bahar (2001) or De Servigny and Renault (2002). When all transition matrices are identical, Boreiko et al (2008) suggest estimates for the model parameters $c_{1,2}$ and q_i^s. A generalization of this approach to our case is as follows.

Set $w_{i,j}^{k,l} = \frac{\rho_{i,j}^{k,l}}{r_{i,j}^{k,l}}$ and $z_i^k = q_i^k \sqrt{s_{i,i}}$. Then relations (3) become

$$w_{i,j}^{k,l} = c_{i,j} z_i^k z_j^l.$$

Summing up these relations for all possible k and l yields

$$w_{i,j} = c_{i,j} z_i z_j,$$

where

$$w_{i,j} = \sum_{k,l=1}^{K} w_{i,j}^{k,l} \quad \text{and} \quad z_i = \sum_{k=1}^{K} z_i^k.$$

Then $z_i = \sqrt{w_{i,i}}$ and

$$c_{1,2} = \frac{w_{1,2}}{\sqrt{w_{1,1} w_{2,2}}}. \tag{5}$$

Also both

$$z_i^k = \frac{w_{i,i}^k}{\sqrt{w_{i,i}}} \quad \text{and} \quad z_i^k = \frac{w_{i,j}^k \sqrt{w_{i,i}}}{w_{i,j}}.$$

Here

$$w_{i,j} = \sum_{k,l=1}^{K} w_{i,j}^{k,l} \quad \text{and} \quad w_{i,j}^k = \sum_{l=1}^{K} w_{i,j}^{k,l}.$$

Then

$$z_i^k = \lambda^{i,k} \frac{w_{i,i}^k}{\sqrt{w_{i,i}}} + (1 - \lambda^{i,k}) \frac{w_{i,j}^k \sqrt{w_{i,i}}}{w_{i,j}}, \; i, j = 1, 2, \; k = 1, 2, \ldots, K, \tag{6}$$

for every number $\lambda^{i,k}$. This parameter is chosen by the estimator.

Using consistent estimates $\hat{\rho}_{i,j}^{k,l}$ of $\rho_{i,j}^{k,l}$ and relations (5) and (6), gives consistent estimates of $c_{1,2}$ and z_i^k.

The values q_i^k and p_i cannot be simultaneously recovered from z_i^k. However, when p_i were known, q_i^k obtain from z_i^k. There are two possibilities: to estimate p_i or to consider a couple (p_1, p_2) as free parameters of the model and to run simulations for all admissible couples (with a certain grid).

Estimating $\rho_{i,j}^{l,s}$ required knowing the frequencies of simultaneous defaults of couples of debtors. Let us show that the frequencies of simultaneous defaults of triples of debtors allow to estimate p_i. Denote by $p_{i,i,j}^{k,l,s}$ the probability that during the same year default two firms from credit class i and a firm from credit class j belonging to industry sectors i, j and s, respectively. When $i \neq j$,

$$p_{i,i,j}^{k,l,s} = p_{i,3}^{(k)} p_{i,3}^{(l)} p_{j,3}^{(s)} \{1 + z_i^k z_i^l + c_{1,2}[z_i^k z_j^s + z_i^l z_j^s + \frac{1}{\sqrt{(1-p_i)p_i}} z_i^k z_i^l z_j^s)]\}.$$

Here $k \neq l$. When $i = j$

$$p_{i,i,i}^{k,l,s} = p_{i,3}^{(k)} p_{i,3}^{(l)} p_{i,3}^{(s)} [1 + z_i^k z_i^l + z_i^k z_i^s + z_i^l z_i^s + \frac{2p_i - 1}{\sqrt{(1-p_i)p_i}} z_i^k z_i^l z_j^s].$$

Here the indexes k, l, and s are all different.

Whenever $c_{1,2} \neq 0$ and $z_i^k \neq 0$ for all possible i and k, we can rewrite these formulas in the following way

$$v = \sqrt{(1-p)p} \quad \text{and} \quad r = \frac{\sqrt{(1-p)p}}{2p-1}.$$

Here we dropped the index i to simplify the notation. Since the right hand sides do not depend upon the industry sectors k, l, s, the left hand sides are also independent of them. Setting $d = p - \frac{1}{2}$, we get

$$d^2 = \frac{1}{4} - v^2 \quad \text{and} \quad d^2 = \frac{1}{4(1+4r^2)}.$$

The frequencies of simultaneous defaults of the triples k, l, s of debtors furnish consistent estimates $v_{k,l,s}$ and $r_{k,l,s}$ for v and r. There are $C_K^2 C_K^1$ such estimates for v and C_K^3 estimates for r. For any $C_K^2 C_K^1$ real numbers $\lambda_{k,l,s}^{(1)}$ and any C_K^3 real numbers $\lambda_{k,l,s}^{(2)}$ such that

$$\sum_{k \neq l} \lambda_{k,l,s}^{(1)} + \sum_{k \neq l, k \neq s, l \neq s} \lambda_{k,l,s}^{(2)} = 1,$$

the value

$$\hat{d^2} = \sum_{k \neq l} \lambda_{k,l,s}^{(1)} [\frac{1}{4} - v_{k,l,s}^2] + \frac{1}{4} \sum_{k \neq l, k \neq s, l \neq s} \lambda_{k,l,s}^{(2)} \frac{1}{1 + 4r_{k,l,s}^2} \tag{7}$$

will be a consistent estimate for d^2. Consequently, $\hat{p} = \frac{1}{2} + \hat{d}$ will be a consistent estimate for p. (Implying a negative $\hat{r}$ and, consequently, $\hat{p} < \frac{1}{2}$, the value $\frac{1}{2} - \hat{d}$ should be ignored.)

4 The flow of payments associated with a CDO tranche

A tranche i of a CDO is characterized by an attachment point A_i and a detachment point B_i, $0 \leq A_i < B_i \leq 1$. Consider a simplified situation when time is measured in quarters of a year and, consequently, all payments implied by the contract are done quarterly. A contract with maturity T starts at $t = 0$. At times 1, 2, ..., T the default events and the corresponding payments take place. Denote by $NV_i(t)$ the notional value allocated to tranche i at time t.

Since the above model generates $D_i^k(t)$ annually, that is $D_i^k(4s)$, $s = 0, 1, ...$, we have to use an approximation setting

$$D_i^k(t) = [\frac{D_i^k(4(s+1)) - D_i^k(4s)}{4}(t-4s)]$$

for $t = 4s+1,\ 4s+2,\ 4s+3$. Here $[a]$ stands for the integral part of a real number a, that is, the largest integer that does not exceed a. Also $D_i^k(0) = 0$ for all k and i.

Let $R_i^k(t) \in [0,1)$ be a recovery rate at t for a debtor belonging to industry sector k and credit class i. Since the model generates credit rating transitions annually, a recovery rate remains constant during the whole year. That is

$$R_i^k(4s+1) = R_i^k(4s+2) = R_i^k(4s+3) = R_i^k(4s).$$

Since the contribution to the CDO of every individual debtor is equal to 1, the loss $L(t)$ accumulated by and including time t reads

$$\sum_{s=1}^{t}\sum_{i=1}^{2}\sum_{k=1}^{K} D_i^k(s)[1 - R_i^k(s)].$$

Consequently, its share in the portfolio will be

$$SH_t = L(t)/\mathcal{N}(0).$$

Look at a tranche i' and a time instant t'. We have that

$$NV_{i'}(t') = NV_{i'}(t'-1) = ... = NV_{i'}(0), \text{ if } \mathrm{SH_{t'}} \leq \mathrm{A_{i'}}.$$

Also

$$NV_{i'}(t) = 0, \text{ if } \mathrm{SH_{t'}} \geq \mathrm{B_{i'}}.$$

When $SH_{t'} \in [A_{i'}, B_{i'}]$, we set

$$NV_{i'}(t') = NV_{i'}(0)[1 - (SH_{t'} - A_{i'})/(B_{i'} - A_{i'})].$$

Then

$$NV_{i'}(t'-1) - NV_{i'}(t') = NV_{i'}(0)(SH_{t'-1} - SH_{t'})/(B_{i'} - A_{i'})],$$

if both $SH_{t'-1}$ and $SH_{t'}$ belong to $[A_{i'}, B_{i'}]$.

Consider a tranche $i > 1$ and a time instant t. In the case of an unfunded version, see Hochreiter and Wozabal (2009) for a funded version, the protection seller pays

$$NV_i(t-1) - NV_i(t)$$

to the protection buyer. Over the life time of the contract, these payments amount to

$$\sum_{t=1}^{T} r_{T-t}[NV_i(t-1) - NV_i(t)].$$

Here r_{T-t} stands for a discount factor for $T-t$ time instants. Given a quarterly discount rate $r_{1/4}$, we obtain

$$r_{T-t} = (1 + r_{1/4})^{t-T}.$$

The protection buyer pays to the protection seller quarterly a coupon $NV_i(t)S_i$. Now the flow of payment, viewed from the prospective of the protection buyer, becomes

$$-S_i \sum_{t=1}^{T} r_{T-t} NV_i(t) + \sum_{t=1}^{T} r_{T-t} [NV_i(t-1) - NV_i(t)].$$

Defining the total index, we set the attachment point equal to zero and the detachment point equal to the largest of the B_is.

An alternative pricing scheme was applied traditionally only to the most junior tranche, defined by A_1 and B_1. Then, the protection buyer differs pays at the beginning an up-front payment of $NV_1(0)S_1$, $S_1 \in (0,1)$, and then a fixed coupon $C \in (0,1)$ of the remaining notional quarterly. The flow of payments in this case reads

$$-NV_1(0)S_1 - C \sum_{t=1}^{T} r_{T-t} NV_1(t) + \sum_{t=1}^{T} r_{T-t} [NV_1(t-1) - NV_1(t)].$$

For example, the Dow Jones iTraxx EUR index distinguishes the five tranches: an equity tranche corresponds to $A_1 = 0$ and $B_1 = 0.03$; a mezzanine tranche bears $A_2 = 0.03$ and $B_2 = 0.06$; a first and a second senior tranches account for $A_3 = 0.06$, $B_3 = 0.09$, $A_4 = 0.09$ and $B_4 = 0.12$, respectively; a super senior tranche holds $A_5 = 0.12$ and $B_5 = 0.22$. The pricing scheme with a fixed coupon applies now to all of the traches: $C = 0.05$ for the first three tranches and $C = 0.01$ for the remaining two. While the equity tranche has always been priced in this way, the remaining tranches received coupons first in March of 2009: $C = 0.03$ for the mezzanine and the first senior tranches, $C = 0.01$ for the second senior tranche and the super senior one.

5 Defining a spread

In all of the above cases, the flow of payments may be written as $\xi - S\zeta$. The constant S, we call it spread from now onward[1], has to be taken in such a way that the balance of payments equals zero. That is

$$0 = \xi - S\zeta$$

in some sense. Since we deal with random variables, there are several possibilities to interpret this relation.

1. An actuarially fair spread. Implicitly the protection seller and protection buyer are taken to be risk neutral. Then a spread S_{AC} is considered that equates to zero the expected flow of payments. That is

$$0 = \mathbf{E}\xi - S_{AC}\mathbf{E}\zeta$$

[1] This is a spread per se or an upfront payment, if a fixed coupon is used.

or

$$S_{AC} = \mathbf{E}\xi/\mathbf{E}\zeta.$$

The expectations involved in this relation cannot be evaluated analytically. A Monte-Carlo technique allows to approximate them. Let ξ_j and ζ_j denote realizations of the random variables ξ and ζ obtained by the jth run of the program. Set

$$\overline{\xi}_n = \frac{1}{n}\sum_{j=1}^{n}\xi_j \quad \text{and} \quad \overline{\zeta}_n = \frac{1}{n}\sum_{j=1}^{n}\zeta_j.$$

Since ξ_j and ζ_j are stochastically independent in j, $\overline{\xi}_n \approx \mathbf{E}\boldsymbol{\xi}$ and $\overline{\zeta}_n \approx \mathbf{E}\boldsymbol{\zeta}$. Moreover, these approximations improve as n increases. Consequently, the value

$$S_{AC}^{(n)} = \overline{\xi}_n/\overline{\zeta}_n$$

will be an estimate for S_{AC}.

2. A least squares spread spread. Having n realizations, ξ_j and ζ_j, of random variables ξ and ζ, we may try to find a spread as a value x such that all of the expressions $\xi_j - x\zeta_j$, $j = 1, 2, ..., n$, least deviate from zero. Considering as the measure of fit a sum of quadratic deviations $\sum_{j=1}^{n}(\xi_j - x\zeta_j)^2$ and minimizing it with respect to x gives a least squares solution:

$$S_{LS}^{(n)} = \sum_{j=1}^{n}\xi_j\zeta_j/\sum_{j=1}^{n}\zeta_j^2 = \frac{1}{n}\sum_{j=1}^{n}\xi_j\zeta_j/\frac{1}{n}\sum_{j=1}^{n}\zeta_j^2.$$

In terms of the random variables ξ and ζ,

$$S_{LS} = \mathbf{E}\xi\zeta/\mathbf{E}\zeta^2.$$

A given spread S^* does not equally affect the parties involved: the protection buyer benefit from positive values of $\xi - S^*\zeta$, while negative values of $\xi - S^*\zeta$ are better for the protection seller. The random variables $\xi_i - S^*\zeta_i$ are independent observations of $\xi - S^*\zeta$. They can be used for statistical inference concerning $\xi - S^*\zeta$.

3. A stochastic recovery rate. The recovery rate can depend upon the credit rating of the debtor and its industry sector. Altman et al. (2002) show that recovery rates tend to be negatively correlated with default rates. In other words, recovery rates a higher during the upswings of the economy and lower when it slows down. Since in our case the state of the economy is captured by the tendency vector, let us make recovery rates dependent upon it.

Consider a debtor whose credit rating is i and whose industry sector is k. Let $R_{i,k}$ be its average (or expected) recovery rate. When $R_{i,k} \in (0, 1)$, we may introduce a random recovery rate $RR_{i,k}$. It will on average equal to $R_{r,k}$, being higher than $R_{r,k}$ during the growth periods and lower than $R_{r,k}$ for the periods of decline. Moreover, the recovery rates for different debtors become dependent random variables. To this end, choose a positive number $\Delta_{i,k} \leq \min[(1 - R_{i,k})p_i, R_{i,k}(1 - p_i)]$. Set

$$RR_{i,k} = \begin{cases} R_{i,k} + \Delta_{i,k}/p_i, & \text{when } \chi_i = 1, \\ R_{i,k} - \Delta_{i,k}/(1 - p_i), & \text{when } \chi_i = 0. \end{cases}$$

Then $RR_{i,k}$ depends on χ_i. The coefficient of correlation between $RR_{i,k}$ and $RR_{j,s}$ equals $c_{i,j}$.

This parametrization implies as an extreme case a zero recovery rate for the periods of economic decline. The smaller is the corresponding p_i, the stronger will be the effect of a zero recovery rate on the spread.

Stochastic recovery rates have been considered in the literature. For example, Hull and White (2004) correlate a recovery rate with the common factor in a single parameter copula model.

6 Numerical estimators for the spread

A Standard and Poor's data set is used to estimate the parameters of a coupled Markov chain model. It mimics the portfolio of 125 investment grade European companies that defines the Dow Jones iTraxx EUR index. The four industry sectors involved are: 1 – manufacturing, 2 – transportation technology and utility, 3 – trade, 4 – finance. According to the Standard Industry Classification (SIC) of the US Department of Labor (Occupational Safety and Health Administration) first digit industry classification, their SIC codes are: $2000-3999$, $4000-4999$, $5000-5999$ and $6000-6999$. They are represented by 1648, 1574, 480 and 5583 firms in the data set.

The initial composition of the industry sectors is as follows: $N_1^1(0)=30$, $N_1^2(0)=40$, $N_1^3(0)=30$ and $N_1^4(0)=25$. That is, the portfolio initially consists of $\mathcal{N}(0)=125$ investment grade debtors. Having not enough default events for reliably estimating the required correlations, we are forced to consider fewer industry sectors than Choroś et al. (2009) or Hochreiter and Wazabal (2009). Modeling the same index, these authors distinguish six industries.

The transition matrices $P^{(k)}$ are based on the records from 1990 through 2006 (the row corresponding to the absorbing state is not quoted):

$$P^{(1)}=\begin{pmatrix} 0.9701 & 0.0292 & 0.0007 \\ 0.0291 & 0.9435 & 0.0274 \end{pmatrix}, \quad P^{(2)}=\begin{pmatrix} 0.9788 & 0.0190 & 0.0022 \\ 0.0428 & 0.8991 & 0.0581 \end{pmatrix},$$

$$P^{(3)}=\begin{pmatrix} 0.9584 & 0.0402 & 0.0014 \\ 0.0269 & 0.9469 & 0.0262 \end{pmatrix}, \quad P^{(4)}=\begin{pmatrix} 0.9737 & 0.0255 & 0.0008 \\ 0.1757 & 0.8106 & 0.0137 \end{pmatrix}.$$

Inequalities (2) imply that $0 \le p_1 \le 0.9584$ and $0.1757 \le p_2 \le 0.9419$. The values $\frac{w_{i,i}^k}{\sqrt{w_{i,i}}}$ and $\frac{w_{i,j}^k\sqrt{w_{i,i}}}{w_{i,j}}$ defining z_i^k in (6) are as follows

$$\begin{pmatrix} 1.5261 & 1.2941 & 1.6432 & 1.2144 \\ 1.0531 & 0.9061 & 1.1858 & 0.6876 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1.4809 & 2.0206 & 1.4231 & 0.7532 \\ 0.9996 & 1.3639 & 0.9606 & 0.15084 \end{pmatrix}.$$

The best match here is observed in the first column. Interestingly enough, it corresponds to the industry sector with the widest range of SIC codes. Setting $\lambda^{i,k}=\frac{1}{2}$ in (6) results in the following matrix

$$Z=\begin{pmatrix} 1.5035 & 1.6574 & 1.5332 & 0.9838 \\ 1.0264 & 1.1350 & 1.0732 & 0.5981 \end{pmatrix}.$$

Its i, kth element is z_i^k. Since $q_i^k \leq 1$, the obtained values z_i^k further limit the range for p_i: $p_1 \in [0.7331, 0.9584]$ and $p_2 \in [0.5630, 0.9419]$. The estimated coefficient of correlation $c_{1,2} = 0.9747$, through the feasibility constraint, $\min\left(\sqrt{\frac{s_{1,1}}{s_{2,2}}}, \sqrt{\frac{s_{2,2}}{s_{1,1}}}\right) \geq c_{1,2}$, further restricts these intervals to $[0.7331, 0.9446]$ for p_1 and $[0.7229, 0.9419]$ for p_2. Moreover, the constrain imposes an implicit relation between p_1 and p_2 that have to be taken into account running the model.

Taking in (7) with equal non-zero weights only admissible estimates $v_{k,l,s}$ and $r_{k,l,s}$, we obtain $p_1 = 0.9229$ based on 6 observations and $p_2 = 0.8818$ based on 8 observations. Here 'admissible' means that the estimate does not imply a relation that does not make sense, like $d^2 < 0$ in the corresponding formulas. These values belong to the intervals defined above. However, substituted into the feasibility constraint, they result in $0.7894 < 0.9747$.

To identify the sensitivity of the model to the values of p_i, we ran simulations for couples (p_1, p_2) such that $\sqrt{\frac{s_{2,2}}{s_{1,1}}} = 0.9747$. For a fixed duration and a tranche, we observed that

- the actuarially fair spread decreases in p_1;
- the least squares spread falls below the actuarially fair spread;
- a deterministic recovery rate implies a lower actuarially fair spread than a stochastic recovery rate with the same expected value.

To compare the model predictions with actually observed data, we ran 10000 times the program. For all of the simulations, the quarterly discount rate $r_{1/4}$ equals to 0.75 and a deterministic recovery rate R was 0.4 for all of the industries. The spread is quoted in percent.

First we used $p_1 = 0.9229$ and $p_2 = 0.9192$. Unlike the actually estimated $p_2 = 0.8818$, this value satisfies the feasibility constraint. The following estimates obtain:

Tranche	5 years	7 years	10 years
Equity	11.10	16.70	23.40
Mezzanine	3.76	4.47	5.15

Table 1: Estimated actuarially fair spread.

The values actually observed on the 31st of January of 2007 read:

Tranche	5 years	7 years	10 years
Equity	10.30	25.80	40.70
Mezzanine	0.42	1.12	3.29

Table 2: Actually observed spread, 31.01.2007.

For a given tranche, set P^* for the probability that an actually observed spread S^* favors the protection buyer. That is, $P^* = \mathbb{P}\mathrm{r}\{\xi - S^*\zeta > 0\}$. The next table contains estimates of P^* for the 31st of January of 2007.

Tranche	5 years	7 years	10 years
Equity	0.46	0.48	0.31
Mezzanine	0.31	0.41	0.55

Table 3: Probability that the actually observed spread favors the buyer.

Consequently, the risk seller seems to benefit from the pre-crisis risk pricing. This tendency strengthens as the crisis develops. As a result, the model does not match completely the spreads reported on the 25$^{\text{th}}$ of November of 2008. However, choosing $p_1 = 0.8180$ and $p_2 = 0.8110$, we are able to mimics the observed spread again. This is an effect of the frailty variables. In fact, p_i have been estimated for a period of time that did not contain a crisis comparable with the current one.

Hochreiter and Wazabal (2009) pioneered the use of coupled Markov chains in estimating spreads. The required parameters obtain by a maximum likelihood technique applied to a Standard and Poor's data set. Covering the period from 1985 till 2007, it differs from our data source. Since they deal with five non-default credit classes and six industry sectors, it is difficult to compare our estimates with what they reported. However, their 0.2131 for the one-year default probability for the lowest credit rating seems to be substantially higher than 0.0521 for the highest of our probabilities of default of non-investment grade debtors.

7 Conclusive remarks

Interpreting the results presented here and comparing them with what has been reported by other scholars, one has to keep in mind that the correlations in our case are the event correlations. That is, the correlations between credit rating transitions of the debtors presented in the portfolio. The existing literature typically deals with a particular case of them – the defaults. In partucular, our empirically measured default correlations are comparable with those reported by Nagpal and Bahar (2001). However, even in this case the populations of debtors are different: Nagpal and Bahar (2001) looks at American firms, while here the sample covers the OECD countries. To highlight the possible discrepancy between the default correlations typical to the common factor (or structural) models based on Merton's (1974) representation and the default correlations adopted here, consider a standard one-factor Gaussian copula model.

The asset value of debtor k is modeled by $\xi_k = \sqrt{\rho}\eta + \sqrt{1-\rho}\epsilon_k$, where η and ϵ_k are all independent Gaussian random variables with mean 0 and variance 1. Then $\mathbf{Corr}(\xi_k, \xi_j) = \rho$. That is $\rho \in [0,1]$ here is the coefficient of correlation between the asset positions of any two firms involved in the portfolio. Let D_k denote a default threshold of debtor k. That is, if the cash position of k reaches D_k or falls below this value, firm k defaults. Set χ_k for the indicator of this default event. That is $\chi_k = \mathbb{1}_{\{\xi_k \le D_k\}}$. Then χ_k are dependent Bernoulli random variables. The probability of a success of χ_k is $p_k = \mathbf{E}\chi_k = \Phi(D_k)$, where Φ denotes the standard normal distribution function. The default events correlation reads

$$\mathbf{Corr}(\chi_k, \chi_j) = \frac{\mathbf{Cov}(\chi_k, \chi_j)}{\sqrt{p_k(1-p_k)p_j(1-p_j)}}, \tag{8}$$

with

$$\mathbf{Cov}(\chi_k, \chi_j) = \mathbf{E}\chi_k\chi_j - p_k p_j =$$

$$= \int_{-\infty}^{\infty} \Phi\Big(\frac{\Phi^{-1}(p_k) - \sqrt{\rho}u}{\sqrt{1-\rho}}\Big)\Phi\Big(\frac{\Phi^{-1}(p_j) - \sqrt{\rho}u}{\sqrt{1-\rho}}\Big)\phi(u)du - p_k p_j.$$

Here Φ^{-1} denotes the inverse of Φ and ϕ stands for the density of Φ. In a special case when $p_k = p_j = p$ the default events correlation $\tilde{\rho}$ as a function of ρ and p looks like the following:

We see that, first, the event correlation $\tilde{\rho}$ is always smaller than the asset positions correlation ρ and, second, the event correlation slightly increases in p, the common default probability. Typically, the empirically measured default events correlations do not correspond to the values given by (8) for the asset positions correlations considered in the literature. For example, Laurent and Gregory (2005) use a Gaussian common factor model as above with $\rho = 0.3$. The corresponding value of the default events correlation $\tilde{\rho}$ as a function of p is as follows:

p	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.5
$\tilde{\rho}$	0.098	0.129	0.149	0.163	0.174	0.182	0.187	0.191	0.193	0.194

However the empirically measured default events correlations seldom exceed 0.1. See Nagpal and Bahar (2001) or Boreiko et al. (2008).

A further problem is the so-called correlation smile. Amato and Gyntelberg (2005) give a comprehensive discussion of this phenomenon. In short, the values of ρ providing satisfactory estimates for the spread differ across the tranches. "If the one-factor Gaussian model is indeed the correct description of joint default dependence, then the same implied correlation value should be inferred for all tranches" conclude Amato and Gyntelberg (2005, p. 83).

References

Altman, E., Brady, B., Resti, A., Sironi, A., 2005. The link between default and recovery rates. Journal of Business 78, 2203-2228.

Amato, J., Gyntelberg, J., 2005. CDS index tranches and the pricing of credit risk correlations. BIS Quaterly Review 2, 73-87.

Boreiko, D.V., Kaniovski, Y.M, Pflug, G. Ch., 2010. Dependent credit rating transitions and empirical estimates for event correlations. Mimeo, Free University of Bolzano-Bozen, Bolzano.

Choroś, B., Härdle, W. K., Okhrin, O., 2009. CDO Pricing with Multifactor and Copulae Models. Mimeo, Humboldt University, Berlin.

Credit Metrics, 1997. The Benchmark for Understanding Risk. Technical Document. J.P. Morgan Inc., New York, NY.

De Servigny, A., Renault, O., 2002. Default Correlation: Empirical Evidence. S&P'S Working Paper.

Duffie, D., Eckner, A., Horel, G., Saita, L., 2009. Frailty Correlated Default. Journal of Finance 64, 2089-2123.

Duffie, D., Singleton, K.J., 1999. Modeling term structures of defaultable bonds. Review of Financial Studies 12, 687-720.

Frey, R., Backhaus, J., 2008. Pricing and hedging of portfolio credit derivatives with interacting default intensities. International Journal of Theoretical and Applied Finance 11, 611-634.

Gregory, J., Laurent, J.-P., 2005. Basket default swaps, CDOs and factor copulas. Journal of Risk 7, 1-20.

Hochreiter, R., Wozabal, D., 2009. A Coupled Markov Chain Approach to Risk Analysis of Credit Default Swap Index Products. Mimeo, University of Vienna, Vienna.

Hull, J., White, A., 2004. Valuation of a CDO and an n-th to Default CDS Without Monte Carlo Simulation. Journal of Derivatives 2, 8-23.

Jorion, P., Zhang, G., 2009. Credit Contagion from Counterparty Risk. Journal of Finance 64, 2053-2087.

Kaniovski, Y.M, Pflug, G. Ch., 2007. Risk assessment for credit portfolios: A coupled Markov chain model. Journal of Banking and Finance 31, 2303-2323.

Koopman, S. J., Lucas, A., and Monteiro, A., 2008. The multi-state latent factor intensity model for credit rating transitions. Journal of Econometrics 142, 399-424.

Li, D. X., 2000. On default correlation: A copula approach. Journal of Fixed income 6, 43-54.

Merton, R., 1974. On the pricing of corporate debt: The risk structure of interest rates. Journal of Finance 29, 449-470.

Nagpal, K., Bahar, R., 2001. Measuring Default Correlation. Risk 14, 129–132.

Vasicek, O., 1987. Probability of loss on loan portfolio. Working paper, KMV Corporation.

Willeman, S., 2007. Fitting the CDO correlation skew: A tractable structural jump diffusion model. Journal of Credit Risk 3, 63-90.