

Exponential discounting bias

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Received: 13 January 2013 / Accepted: 5 August 2013 / Published online: 17 August 2013
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Abstract We address intertemporal utility maximization under a general discount function that nests the exponential discounting and the quasi-hyperbolic discounting cases as particular specifications. Under the suggested framework, the representative agent adopts, at some initial date, an optimal behavior that shapes her consumption trajectory over time. This agent desires to take a constant discount rate to approach the optimization problem, but bounded rationality, under the form of a present bias, deviates the individual from the intended goal. As a result, decreasing impatience will end up dominating the agent's behavior. The individual will not be aware of her own time inconsistency and, therefore, she will not revise her plans as time elapses, what makes the problem relatively simple to address from a computational point of view.

Electronic supplementary material The online version of this article (doi:[10.1007/s00712-013-0363-3](https://doi.org/10.1007/s00712-013-0363-3)) contains supplementary material, which is available to authorized users.

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The general discounting framework is used to approach a standard optimal growth model in discrete time. Transitional dynamics and stability properties of the corresponding dynamic setup are studied. An extension of the standard utility maximization model to the case of habit persistence is also considered.

Keywords Intertemporal preferences · Exponential discounting · Quasi-hyperbolic discounting · Optimal growth · Habit persistence · Transitional dynamics

JEL Classification C61 · D91 · O41

1 Introduction

Typically, the benchmark utility maximization dynamic model takes a constant rate of time discounting and, thus, intertemporal discounting is modeled as being exponential. This is an analytically convenient assumption and it is logically consistent with the idea that a constant interest rate is often used to compare the value of money over time, for instance at the level of the evaluation of investment projects. However, there are psychological effects that must be taken into account when addressing intertemporal preferences. These effects may have a huge impact on how we perceive the behavior of the representative agent in the context of conventional economic models, relatively to what the trivial case of exponential discounting shows.

The most debated of such effects concerns the evidence on decreasing impatience: human beings tend to place much more weight on the difference between a reward to be received (or a cost to be incurred) today or tomorrow than on the difference between two consecutive dates in the far future. Thus, the rate of discount that we apply when measuring the present value of some near in time outcome is typically much larger than the discount rate applied to a distant in the future event (i.e., typically the discount rate decreases over time). Such type of phenomenon is associated with the notion of hyperbolic discounting.

The discussion about hyperbolic discounting, from an economic point of view, has started with [Strotz \(1956\)](#) and [Pollak \(1968\)](#) and received influential contributions in the 1990s, with the work, among others, of [Akerlof \(1991\)](#), [Laibson's \(1997, 1998\)](#) and [O'Donoghue and Rabin \(1999\)](#). These authors have raised some fundamental questions: Does the popularity of exponential discounting come from its time consistency or from analytical tractability? How can one incorporate into economic models an operational notion of decreasing impatience? If preferences are truly present-biased, how does this relate to important behavioral issues as self-control or procrastination? Are agents aware of their own intertemporal preferences, so that they adopt sophisticated plans of action or does unawareness lead to a naive interpretation about the future? These interrogations continue today to be a rich source of debate on behavioral economics and related fields.

This paper generalizes the most popular specification of hyperbolic discounting used in economics—Laibson's quasi-hyperbolic discounting concept—and applies the new framework of intertemporal preferences to a standard discrete time optimal growth problem. The setup differs from other approaches on the subject because we

relate the shape of the discount function to issues of financial literacy, following the analysis on the exponential growth bias as developed by [Stango and Zinman \(2009\)](#) and [Almenberg and Gerdes \(2011\)](#). Our argument is that in the same way people tend to underestimate future values of variables that grow at constant rates, individuals also tend to overestimate close in time values (relatively to the ones more distant in the future) when discounting them to the present. This reasoning allows us to present a discount function that is flexible enough to characterize different degrees of hyperbolic discounting and to nest the exponential discounting case as a possible limit outcome.

The proposed specification of intertemporal preferences is analytically convenient to address a discrete time optimal growth model. It enables us to derive explicit stability conditions and it serves to compare different degrees of deviation from the constant discount rate benchmark. Additionally, we extend the model to include habit persistence in consumption in order to demonstrate the flexibility of the exponential discounting bias concept when used in different settings.

As discussed in the literature that was mentioned above, intertemporal optimization under decreasing impatience typically yields a problem of time inconsistency. This means that the representative agent will benefit from a systematic re-optimization as time unfolds, and therefore she will not commit with her initial planning. Such time inconsistency and the consequent lack of commitment raises significant obstacles in the way one can approach the benchmark utility problem of the representative agent. In this paper, time inconsistency issues are circumvented because we assume that the agent faces a bounded rationality constraint; this constraint consists in the inability to perceive that impatience is in fact decreasing, since it is understood by the agent as being constant. If the agent is wrong but is not capable of recognizing the mistake, she will have no reason to think she needs to re-optimize and thus plans are never reviewed. It is this assumption, namely the presence of a reasoning anomaly, that makes the agent commit: she commits with a wrong perception of the reality that does not give her the incentive to ever review the planning problem. This has relevant implications in the optimal path of consumption that is formed and allows for a relatively simple approach to intertemporal utility maximization. Such implications are dissected in this paper.

The remainder of the paper is organized as follows. Section 2 approaches, briefly, the literature on non-exponential discounting and introduces the notion of exponential discounting bias. In Sect. 3, the discounting bias notion is explored from a formal point of view. Section 4 evaluates utility maximization under the general specification for intertemporal preferences provided by the notion of discounting bias. Section 5 sets up the growth model and analyzes the underlying dynamics. In Sect. 6, an extension is explored; namely, the model is adapted in order to account for habit persistence. Section 7 concludes. Technical details on the analytics of the discussed models are left to a final appendix.

2 Non-exponential discounting and the motivation for the discounting bias effect

2.1 Departures from exponential discounting

In [Xia \(2011\)](#), three types of behavioral evidence concerning time preference, that imply a deviation relatively to the standard exponential discounting setting, are

identified. These relate to the timing of the evaluation, the magnitude of the reward, and the sign of the reward. The sign effect was first highlighted by [Kahneman and Tversky \(1979\)](#) and basically states that gains are discounted more than losses. The magnitude effect is a matter that has received considerable attention on recent literature (see [Noor \(2011\)](#) and [Bialaszek and Ostaszewski \(2012\)](#)) and relates to the evidence that there is an inverse relation between the amount of the reward and the steepness of discounting over time, i.e., agents tend to be more patient when larger rewards are at stake. The central subject of discussion at this level, though, continues to be the one concerning the evidence on decreasing impatience and on how hyperbolic discounting should be approached.

Many authors argue that hyperbolic discounting is a time consistent and rational way of forming intertemporal preferences, more than exponential discounting. Formal approaches supporting this argument can be found, e.g., in [Dimitri \(2005\)](#), [Drouhin \(2009\)](#), [Farmer and Geanakoplos \(2009\)](#) and [Gollier \(2010\)](#). Other authors adopt a more skeptical view, claiming that although there is a tendency to search for analytical discount functions that may allow for an elegant treatment of economic models, modifying functional forms does not answer the main questions posed by the apparent lack of rationality in economic behavior (see [Rubinstein 2003](#); [Rasmussen 2008](#)). As stated by Ariel Rubinstein, a deeper understanding of intertemporal human decisions requires opening the black-box of decision making more than changing slightly the structure of the model used to address human behavior.

Other relevant contributions on the field of hyperbolic discounting relate the generalization of the concept and the exploitation of the corresponding implications. In [Bleichrodt et al. \(2009\)](#) the commonly used discount functions are modified in order to account for other kinds of time inconsistency on the formation of preferences besides decreasing impatience. Specifically, the proposed framework accommodates the possibilities of increasing impatience and strongly decreasing impatience. Also [Benhabib et al. \(2010\)](#) present a general version of the discount function, that contemplates the most common specifications of exponential and hyperbolic discounting found in the literature.

The powerful notion of hyperbolic discounting has been applied to study a wide range of relevant economic issues. Just to cite a few, we highlight the contributions of [Gong et al. \(2007\)](#), concerning consumption under uncertainty, [Groom et al. \(2005\)](#), [Dasgupta \(2008\)](#), [Gollier and Weitzman \(2010\)](#), and [Hepburn et al. \(2010\)](#) in the field of environment policy, namely when making the distinction between social and private discount rates, a paramount normative question in this field, [Graham and Snower \(2008\)](#) on short-run macroeconomics and inflation dynamics, and, [Barro \(1999\)](#) and [Coury and Dave \(2010\)](#) on the implications of non-exponential discounting to economic growth.

2.2 Hyperbolic and quasi-hyperbolic discount functions

In order to account for decreasing impatience, [Loewenstein and Prelec \(1992\)](#) proposed the following hyperbolic discount function: $D_H(t, s) = [1 + \alpha(t - s)]^{-\gamma/\alpha}$, where α and γ are two positive parameters and s is the time period in which the future is

being evaluated. This discount function implies a decreasing discount rate: short-term discount rates are higher than long-term discount rates. Empirical evidence suggests that this is, in fact, a much more appropriate and realistic way to approach intertemporal preferences than just considering a constant discount rate over time.

While empirically more suitable, hyperbolic discounting, considered as modeled above, is much less tractable from an analytical point of view than exponential discounting. Because of this, Laibson's (1997, 1998), based on a previous formalization by Phelps and Pollak (1968), suggested an approximation to hyperbolic discounting, that he dubbed quasi-hyperbolic discounting; this form of discounting is straightforward to apply to the standard dynamic optimization models of economists.

The quasi-hyperbolic discount function takes the following form,

$$D_{QH}(t, s) = \begin{cases} 1 & \text{if } t = s \\ \widehat{\beta}\widehat{\delta}^{t-s} & \text{if } t = s + 1, s + 2, \dots \end{cases} \quad (1)$$

with $\widehat{\beta}, \widehat{\delta} \in (0, 1)$. Note that in the limit case $\widehat{\beta} = 1$ we are back at exponential discounting.

As in the hyperbolic case, the quasi-hyperbolic discount function captures the idea that discount rates decline with the passage of time. Laibson proposes, in his studies, a small exercise to compare discount rates on each of the settings. He considers exponential discounting ($\widehat{\beta} = 1; \widehat{\delta} = 0.97$), quasi-hyperbolic discounting ($\widehat{\beta} = 0.6; \widehat{\delta} = 0.99$), and hyperbolic discounting ($\alpha = 10^5; \gamma = 5 \times 10^3$) and draws a graph where it is evident that $D_{QH}(t, s)$ generates a time trajectory that is considerably closer to $D_H(t, s)$ than the one originating in plain exponential discounting.

The presented quasi-hyperbolic discount function will be generalized below, in Sect. 3, in order to develop a formal structure capable of generating a more comprehensive discussion on the consequences of probable departures from the exponential discounting case. The new discount function will be economically supported on the evidence about the agents' misperception on how financial values accumulate over time, as explained in the following subsection.

2.3 Anomalies in financial evaluation

Recently, Stango and Zinman (2009) and Almenberg and Gerdes (2011) have carefully analyzed the evidence that points to a tendency to underestimate the future value of a given variable that grows at a constant rate. This exponential growth bias clearly exists in practice, for instance in what concerns household financial decision making.

The mentioned literature emphasizes the link between the extent of the bias and the degree of financial literacy. A poor ability to perform basic calculations and the lack of familiarity with elementary financial concepts and products will, in principle, imply a wider gap between individuals' calculations and the true future values, i.e., there is a negative correlation between financial literacy and the exponential growth bias.

Well informed agents will be able to understand the basic notion of capitalization and to perceive the exponential path followed by any value that accumulates over

time. However, many studies have been discovering serious flaws on the understanding, by the average citizen, of simple financial concepts and mechanisms. This was highlighted by Lusardi (2008) and Japelli (2010), among others. Financial literacy or, more precisely, the lack of it, can explain the kind of deficiency that consists in linearizing an exponential series in time.

The important argument concerning the lack of ability on accurately addressing the value of money in time is that incorrect answers are biased. As emphasized by Almenberg and Gerdes (2011), individuals are almost twice as likely to underestimate the correct amount than to overestimate it. Thus, on the aggregate it makes sense to state that in a society where a given degree of financial illiteracy exists, the future values of a series that grows at a constant rate will be underestimated. Exponential growth bias will then be common when assessing the future value of an investment that offers a return at a given annual constant interest rate.

It is reasonable to conceive the existence of a link between the interest rate and the rate of time preference. In Farmer and Geanakoplos (2009, pages 1,2), this link is explained in simple terms,

A natural justification for exponential discounting comes from financial economics and the opportunity cost of foregoing an investment. A dollar at time s can be placed in the bank to collect interest at rate r , and if the interest rate is constant, it will generate $\exp(r(t - s))$ dollars at time t . A dollar at time t is therefore equivalent to $\exp(-r(t - s))$ dollars at time s . Letting $\tau = t - s$, this motivates the exponential discount function $D_s(\tau) = D(\tau) = \exp(-r\tau)$, independent of s .

The above sentence establishes a possible direct connection between the interest rate and the discount rate of intertemporal preferences. Nevertheless, there is a substantial difference between the two. While the interest rate is obtained as a market outcome, and might not vary if market conditions do not change, the rate of intertemporal choice is a matter of perception and preferences. Agents may want to adopt a rate of time preference that is close to the interest rate, but if they fail in understanding how future values accumulate, discounting will possibly deviate from the benchmark exponential case for which the discount rate is constant.

To understand how financial illiteracy might contribute to deviate agents' preferences from exponential discounting, we just need to make the inverse path to the one that is present in the evaluation of the exponential growth bias, i.e., if individuals tend to underestimate future values when assessing them in the present, they will certainly overestimate current values when thinking about them as if they were taking decisions at some future time moment. In analytical terms, the idea of exponential growth bias is commonly presented as $FV = PV(1 + r)^{(1-\theta)t}$, where FV is the future value, PV the present value, r the interest rate, t is time and $\theta \in (0, 1)$ measures the magnitude of the bias. If one wants to address the present value given the future value, we just need to rearrange the previous expression and write it as $PV = FV/(1 + r)^{(1-\theta)t}$.

The mentioned reasoning implies decreasing impatience. Far in the future outcomes are much less valued than the ones occurring in the near future. We can call this effect **exponential discounting bias**, and we might define it as the tendency to overestimate close in time values of a variable that grows at a constant rate.

The exponential discounting bias will be as much larger as the wider is the extent of financial illiteracy and it constitutes an alternative explanation about why preferences in time tend to imply hyperbolic discounting: agents *want* to select a constant rate of time preference, namely a rate of time preference that follows the interest rate path, but their ability to undertake the proper computations is biased, in such a way that far in time values are less considered than the ones near the current period.

To take into consideration the notion of exponential discounting bias can be an analytically convenient way of approaching departures from strict exponential discounting. According to the distinction that, in this context, might be taken between naive and sophisticated agents (see, e.g., O'Donoghue and Rabin 1999), the discussed bias puts us closer to the naive evaluation of intertemporal preferences in Akerlof (1991) than to the sophisticated behavior that is implicit in Laibson's (1997, 1998) analysis. A sophisticated person will know exactly what the respective future selves' preferences will be, while naive individuals are not able to realize that as time evolves, preferences will evolve as well.

As a result of the understanding that a bias on discounting cannot be perceived by the agent, since it is the outcome of an anomaly on an otherwise intended constant discounting behavior, the representative agent in the models of the following sections will display a clearly naive behavior. Therefore, she will not be concerned with the possibility of tomorrow selves choosing options that are different from the ones chosen today. Since people are not aware of their own time inconsistency, it is legitimate to consider a dynamic optimal control problem where the representative agent maximizes at a given date $t = 0$ her future utility, and thus to design an optimal plan where the present bias exists but the agent acts as if it did not exist.

In short, the analysis in this paper finds support on two logical arguments:

- First — Individuals desire to turn intertemporal preferences compatible with the opportunity cost of money. This is the benchmark time consistent behavior that the rational agent would like to adopt;
- Second — Lack of a solid financial literacy eventually introduces a biased evaluation of intertemporal preferences, that makes the representative agent to act as if she was an exponential discounter, when in fact she is not.

3 The analytics of the discounting bias

In the previous section, it was stated that the absence of a stable impatience level over time may be interpreted as an anomaly, something similar to the tendency that individuals have to linearize a series of values that accumulate at a constant rate (and, hence, truly exhibit an exponential path). In the proposed setting, this anomaly should be considered in the reverse way, i.e., if individuals tend to linearize exponential trajectories for the future, when discounting values to the present, they will exacerbate the exponential nature of the series under analysis.

The above idea is analytically translated into the following reasoning. We will consider exponential discounting, $D_E(t, s) = \beta^{t-s}$, $\beta \in (0, 1)$, but we add the possibility of an error of evaluation that increases short-run impatience, generating a kind of hyperbolic discounting. Let $\theta(t, s)$ be the anomaly term, which transforms $D_E(t, s)$

into a discount function with an exponential bias, i.e.,

$$D_{EB}(t, s) = \beta^{[1+\theta(t,s)](t-s)} \quad (2)$$

Function $\theta(t, s)$ will take the following form:

$$\theta(t, s) = \begin{cases} 0 & \text{if } t = s \\ \frac{\theta_1}{t-s} - \theta_0 & \text{if } t = s + 1, s + 2, \dots \end{cases} \quad (3)$$

In Eq. (3), parameters are such that $\theta_0 \in [0, 1]$ and $\theta_1 \geq 0$.¹ Naturally, exponential discounting holds for $\theta_0 = \theta_1 = 0$, while quasi-hyperbolic discounting is also a particular case of the more general setting provided by $D_{EB}(t, s)$, for $\hat{\beta} = \beta^{\theta_1}$ and $\hat{\delta} = \beta^{(1-\theta_0)}$.

The assumption of $D_{EB}(t, s)$ as the discount function has two advantages. On one hand, it allows for an intuitive explanation on why we depart from exponential discounting. There is an error of evaluation by the agents; perhaps they want to adopt a constant discount rate but, relatively to the periods that are closer in time they do not have the capacity to make an objective evaluation of their priorities. As time goes by, such ability evolves and, in the long-run, the error in evaluation is much smaller. On the other hand, we introduce a more general and flexible approach to time discounting than the one underlying $D_{QH}(t, s)$; as we will see below, the values of β , θ_0 , and θ_1 can be chosen in such a way that we obtain an approximation to $D_H(t, s)$ that is undoubtedly better than the one provided by quasi-hyperbolic discounting.

Except for the particular case of exponential discounting, constraints on parameters guarantee the existence of decreasing impatience. This is in fact our goal: to discuss the implications of a decreasing discount rate. However, the advanced specification is flexible enough to accommodate additional circumstances, namely the presence of the opposite and less plausible case of increasing impatience. Increasing impatience requires $\theta_1 < 0$. To confirm this claim, note that the discount rate at date t is given by $\rho(t, s) = \sqrt[t-s]{\frac{1}{D_{EB}(t, s)}} - 1$ and that the necessary relation for increasing impatience is $\rho(t, s + 1) > \rho(t, s)$. Analyzing this inequality, $\forall t, s$, it is straightforward to arrive to the result that it holds as long as $\theta_1 < 0$.

Recover Laibson's example and consider the following parameter values for the exponential bias discount function: $\beta = 0.97$, $\theta_0 = 0.95$, and $\theta_1 = 23$. Figure 1 displays a graph similar to the one in the original Laibson's analysis (50 periods are considered and hyperbolic and quasi-hyperbolic discount functions are displayed; pure exponential discounting is, in this case, ignored). To this figure, we add the exponential bias case for the chosen parameter values.

¹ As it evident, instead of presenting the discount function through (2) and (3), we can equivalently display it under the more compact form $D_{EB}(t, s) = \begin{cases} 1 & \text{if } t = s \\ \beta^{\theta_1 + (1-\theta_0)(t-s)} & \text{if } t = s + 1, s + 2, \dots \end{cases}$. The first presentation is preferred for reasons that have to do with the clarity of exposition.

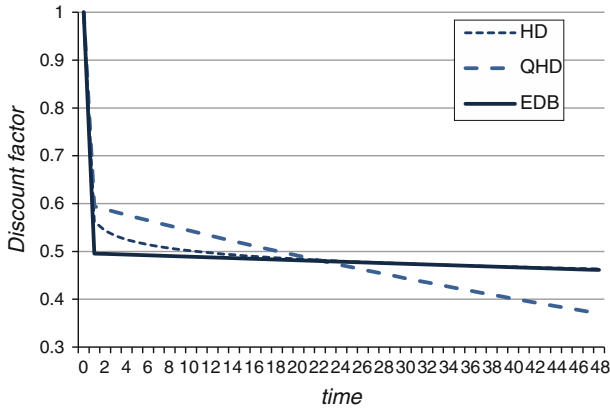


Fig. 1 Discount factors for hyperbolic discounting, quasi-hyperbolic discounting and exponential discounting bias

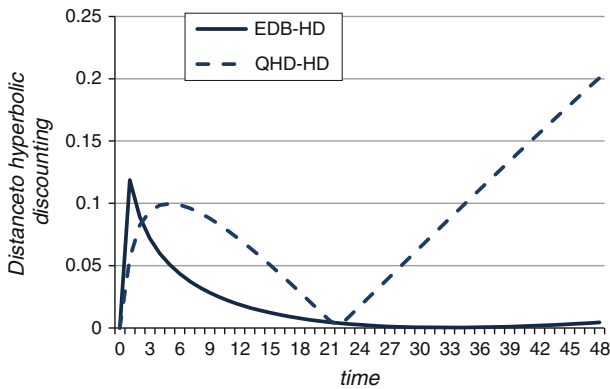


Fig. 2 Comparison between hyperbolic discounting and the two approximations (quasi-hyperbolic discounting and exponential discounting bias)

It is evident that the new function generates results that offer a much better fit with the hyperbolic discount function than the ones generated by the quasi-hyperbolic case. After 15 periods there is almost a perfect match between $D_{EB}(t, s)$ and $D_H(t, s)$ (although, if we introduced additional periods - after 50 - we would start to see a departure of one of the series relatively to the other; nevertheless, this widening gap would never be as pronounced as the one regarding quasi-hyperbolic discounting).

Figure 2 allows for a closer look on this issue. The figure represents the distance (in percentage and in absolute value) between $D_{EB}(t, s)$ and $D_H(t, s)$ and between $D_{QH}(t, s)$ and $D_H(t, s)$. Only in three of the 50 time periods ($t = 1, t = 2$ and $t = 22$), the distance between $D_{EB}(t, s)$ and $D_H(t, s)$ exceeds the distance between $D_{QH}(t, s)$ and $D_H(t, s)$. It is notorious that the present proposal is well suited to address decreasing impatience and it is also well founded on the idea that agents lack the information, literacy, or ability to maintain a constant discount rate over time.

4 Exponential discounting bias and intertemporal utility

In many economic settings, discount functions are used to construct intertemporal utility functions. Their typical presentation is as follows,

$$U_s(c) = u(c_s) + \sum_{t=s+1}^{\infty} D(t, s)u(c_t) \quad (4)$$

Equation (4) represents the utility in the current period, s , from consuming today and in all future moments from $s + 1$ to an undefined future date. The term $u(c_s)$ is current consumption utility; the instantaneous utility function obeys conventional properties of continuity, smoothness, and concavity. Future utility is taken into account for all possible time moments but discounting implies that a larger weight is put on closer in time consumption opportunities. The discount function that we will consider is the one involving the exponential bias, $D(t, s) = D_{EB}(t, s)$.

The same sequence of utility functions, initiating one period later is, accordingly,

$$U_{s+1}(c) = u(c_{s+1}) + \sum_{t=s+2}^{\infty} D(t, s+1)u(c_t) \quad (5)$$

Taking functions $U_s(c)$ and $U_{s+1}(c)$ as displayed above, we can address intertemporal utility under a recursive form. The following expression is obtained by simultaneously taking (4) and (5) under exponential discounting bias,²

$$U_t(c) = u(c_t) + \beta^{1-\theta_0} [U_{t+1}(c) - (1 - \beta^{\theta_1})u(c_{t+1})] \quad (6)$$

Consider a trivial budget constraint, according to which a representative agent accumulates financial wealth (a_t) at a constant rate (r), besides receiving a constant labor income w . This constraint takes the form

$$a_{t+1} = w + (1 + r)a_t - c_t, a_0 \text{ given.} \quad (7)$$

The problem the representative agent will want to solve consists in maximizing utility subject to (7). It is crucial to remark, at this stage, that the intertemporal problem is solved under the implied assumption that the representative agent is naive. As discussed in Sect. 2, we are not concerned with the tendency to procrastinate that an individual with decreasing impatience might display, because she will never realize that her intertemporal preferences are, in fact, not constant over time.

However, one must also highlight that the inability to understand how the future is effectively being discounted does not constitute an obstacle to the adoption of an optimal behavior; the agent solves an optimality problem and chooses the consumption path that best serves her purpose, which is the maximization of intertemporal utility. Putting it in other words, the fact that the agent is naive may be interpreted as a literacy

² Appendix A describes the steps required to arrive to this relation.

constraint that affects the evaluation of time discounting; despite this, the agent is able to act rationally and to solve the dynamic optimization problem she faces.

Appendix B characterizes, in detail, the steps required to solve the optimization problem. A salient feature is that one will be able to find a solution for the problem only because of the mentioned second constraint, the one on the ability to discount the future. The computation of first-order conditions allows for finding the following difference equation that describes the motion of consumption over time,³

$$c_{t+1} = \frac{\beta^{3-2\theta_0}(1 - \beta^{\theta_1})(1 + r)^2 c_{t-1} c_t}{\beta^{1-\theta_0} [1 + \beta(1 - \beta^{\theta_1})] (1 + r) c_{t-1} - c_t} \tag{8}$$

For clarity of exposition, we rewrite Eq. (8) considering as the endogenous variable the ratio $\psi_t := c_t/c_{t-1}$,

$$\psi_{t+1} = \frac{\beta^{3-2\theta_0}(1 - \beta^{\theta_1})(1 + r)^2}{\beta^{1-\theta_0} [1 + \beta(1 - \beta^{\theta_1})] (1 + r) - \psi_t} \tag{9}$$

Rearranging the terms of (9), it is straightforward to confirm that our general specification contemplates the exponential discounting case under constraint $\theta_0 = \theta_1 = 0$. For these values of parameters, $\psi_t = \beta(1 + r)$, as expected. In what follows, the stability features of the consumption equation are addressed; the particular case of exponential discounting will serve as a benchmark to interpret the results.

From Eq. (9), we can determine the steady-state of the ratio between two consecutive values of consumption.

Proposition 1 *The typical intertemporal optimization problem of the representative agent under exponential discounting bias has two steady-state points: $\psi_1^* = \beta^{1-\theta_0}(1 + r) \vee \psi_2^* = \beta^{2-\theta_0}(1 - \beta^{\theta_1})(1 + r)$.*

Proof See Appendix C □

Under the constraint that implies exponential discounting, the steady-state values are, as expected, reduced to $\psi_1^* = \beta(1 + r) \vee \psi_2^* = 0$.

In the exponential discounting case, only the first solution is economically meaningful, since $\psi_2^* = 0$ implies a long-term scenario where the level of consumption drops to zero. In the discounting bias case, the two solutions are positive and, thus, they might be considered, both, as having economic meaning. However, the first solution is always preferable, because it represents a higher rate of growth of consumption ($\psi_1^* > \psi_2^*$) and, thus, a faster increase in utility levels. Figure 3 represents a phase diagram that makes this argument clear: the steady-state ψ_1^* is the one on the right and any path in its direction guarantees faster consumption growth than the one that would be obtainable when there is convergence towards ψ_2^* .

Figure 3 also reveals that the two solutions have a different nature: ψ_1^* is locally unstable and ψ_2^* is locally stable. This can be easily confirmed by computing derivatives of the right hand side of (9) in the vicinity of the corresponding steady-state

³ This equation is derived under the assumption that the utility function is logarithmic, $u(c) = \ln c$.

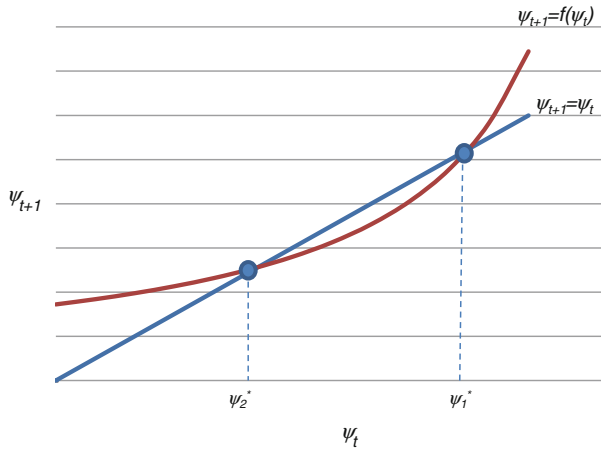


Fig. 3 Phase diagram and the steady-state values

values and observing that the first one falls outside the unit circle, while the second remains inside the unit circle:

$$\left. \frac{d\psi_{t+1}}{d\psi_t} \right|_{\psi_1^*} = \frac{1}{\beta(1 - \beta^{\theta_1})} > 1;$$

$$\left. \frac{d\psi_{t+1}}{d\psi_t} \right|_{\psi_2^*} = \beta(1 - \beta^{\theta_1}) \in (0, 1).$$

The unstable solution is the one that is preferable from a welfare point of view (it guarantees faster consumption growth and utility depends solely on consumption). Given that consumption is a control variable, the representative agent has the possibility of selecting this solution as the long-run outcome that will effectively prevail.

Finally, observe that the steady-state values are as much larger as the wider is the discounting bias, meaning that the deviation from exponential discounting promotes a faster steady-state growth of consumption. This is the obvious result of taking a discount function with a corresponding steady-state value, $\beta^{1-\theta_0}$, that is higher than the benchmark constant discount factor β .

The results that we have accomplished in this section are the direct outcome of the assumption about the discounting bias: the agent acts as if she were an exponential discounter, although in fact she is not. Therefore, she solves a problem that involves a non-constant discount rate but she thinks she is solving a problem with a constant discount rate. The main implication of this observation is that the agent will never re-evaluate her plans and she will never abandon the optimal consumption path that the solution of the optimization problem generates. Time inconsistency is not, under this view, a relevant issue, because the household will not be aware of the necessity to accommodate her behavior; this household will never repudiate her past plans.

5 Exponential discounting bias in the neoclassical growth model

5.1 The setup

We now characterize the dynamics of a neoclassical growth model under exponential discounting bias. The maximization problem is the same as in the previous section (and the same remarks on naive intertemporal preferences and on the ability to optimize even under eventual financial literacy flaws continue to be valid). However, the constraint on the problem differs. We take capital accumulation and a production function involving decreasing marginal returns. Let k_t represent the capital stock and assume that capital used in production at each time period fully depreciates in that same time period. The resource constraint takes the form

$$k_{t+1} = Ak_t^\eta - c_t, k_0 \text{ given} \tag{10}$$

In Eq. (10), $A > 0$ is a technology index and $\eta \in (0, 1)$ represents the output-capital elasticity.

As in the previous section, we need to compute first-order conditions to encounter an optimal dynamic equation for consumption. If $\eta = 1$ (endogenous growth model with an AK production function), we end up with exactly the same dynamics as in the previous section (with $1 + r = A$). Under decreasing marginal returns, the calculus leads to the following difference equation for consumption,⁴

$$c_{t+1} = \frac{\beta^{3-2\theta_0}(1 - \beta^{\theta_1})(\eta A)^2 k_t^{-(1-\eta)} k_{t+1}^{-(1-\eta)} c_{t-1} c_t}{\beta^{1-\theta_0} [1 + \beta(1 - \beta^{\theta_1})] \eta A k_t^{-(1-\eta)} c_{t-1} - c_t} \tag{11}$$

Because, on the present setting, one cannot address consumption dynamics independently of capital accumulation, the dynamic analysis requires the study of a system of three difference equations with three endogenous variables, namely the following set of relations

$$\begin{cases} k_{t+1} = Ak_t^\eta - c_t \\ c_{t+1} = \frac{\beta^{3-2\theta_0}(1 - \beta^{\theta_1})(\eta A)^2 k_t^{-(1-\eta)} (Ak_t^\eta - c_t)^{-(1-\eta)} z_t c_t}{\beta^{1-\theta_0} [1 + \beta(1 - \beta^{\theta_1})] \eta A k_t^{-(1-\eta)} z_t - c_t} \\ z_{t+1} = c_t \end{cases} \tag{12}$$

Note that variable z_t is a lag variable; it corresponds to consumption in the period immediately before the current one.

Next, we proceed to the characterization of the dynamics of system (12). This requires finding the steady-state and looking at local dynamics.

⁴ See Appendix D for the derivation of this equation.

Proposition 2 *The steady-state of the neoclassical optimal growth problem under exponential discounting bias is a pair of long-term equilibrium points:*

$$\begin{aligned}
 (k_1^*, c_1^*) &= \left[\left(\beta^{1-\theta_0} \eta A \right)^{1/(1-\eta)} ; \left(\beta^{1-\theta_0} \eta A \right)^{1/(1-\eta)} \left(\frac{1}{\beta^{1-\theta_0} \eta} - 1 \right) \right] \\
 (k_2^*, c_2^*) &= \left\{ \left[\beta^{2-\theta_0} (1 - \beta^{\theta_1}) \eta A \right]^{1/(1-\eta)} ; \left[\beta^{2-\theta_0} (1 - \beta^{\theta_1}) \eta A \right]^{1/(1-\eta)} \right. \\
 &\quad \left. \times \left[\frac{1}{\beta^{2-\theta_0} (1 - \beta^{\theta_1}) \eta} - 1 \right] \right\}
 \end{aligned}$$

Proof See Appendix C □

As in the case discussed in Sect. 4, two steady-state points were found. Similarly to that case, the two points have different welfare implications. Note that $k_1^* > k_2^*$ and $c_1^* > c_2^*$ and, therefore, the first steady-state is preferable relatively to the second. The representative agent will want to guarantee that (k_1^*, c_1^*) are the capital and consumption levels in the long-run and will concentrate efforts in achieving this goal. For this reason, the stability analysis will focus in this first point; in the next subsection, one will want to know whether, under the model’s settings, convergence towards this steady-state point is feasible.

For now, note the role of the technology index and of the discount factor in shaping the long-term outcome: a larger value of A benefits the economy regarding the long-term accumulated capital and consumption levels. The same is true for the discount factor, meaning that it pays to be patient, under a long-term perspective.

5.2 Local dynamics

Two steady-state points exist, but the representative agent wants to converge to one of them, (k_1^*, c_1^*) , as discussed previously. Capital and consumption are variables with a different nature. Consumption is a non pre-determined or control variable, that the representative agent is capable of manipulating; capital is a pre-determined or state variable that falls outside the control of the household. Therefore, to guarantee the possibility of convergence towards the mentioned steady-state, one needs to ensure the existence of at least one stable dimension, the one concerning the variable capital, in the three-dimensional space under evaluation. In technical terms, this signifies that at least one of the eigenvalues of the Jacobian matrix of the system should fall inside the unit circle. This is the point that we now discuss.

In order to address local stability properties, system (12) is linearized in the vicinity of the steady-state. The result is a set of relations that can be displayed under matricial form, as follows,

$$\begin{bmatrix} k_{t+1} - k_1^* \\ c_{t+1} - c_1^* \\ z_{t+1} - c_1^* \end{bmatrix} = \begin{bmatrix} \chi_1 & -1 & 0 \\ (\chi_2 - \chi_1)\chi_3 & 1 + \chi_2 + \chi_3 & -\chi_2 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} k_t - k_1^* \\ c_t - c_1^* \\ z_t - c_1^* \end{bmatrix} \quad (13)$$

with $\chi_1 = \frac{1}{\beta^{1-\theta_0}}$; $\chi_2 = \frac{1}{\beta(1-\beta^{\theta_1})}$; $\chi_3 = (1-\eta) \left(\frac{1}{\beta^{1-\theta_0}\eta} - 1 \right)$. The 3×3 squared matrix in (13) is the Jacobian matrix; each of its elements corresponds to the differentiation of each equation with respect to each variable, after replacing each variable by its steady-state value.

Proposition 3 *The linearized system obtained from the neoclassical growth model is saddle-path stable. There exists one stable dimension, in the three dimensional space of the model.*

Proof See Appendix C □

With saddle path stability, one concludes that the achieved result is qualitatively similar to the one of the original Ramsey model with a constant discount rate. There is convergence towards the steady-state point (k_1^*, c_1^*) for any initial state (k_0, c_0) in the vicinity of the equilibrium, with the two variables, capital and consumption, following a one-dimensional stable path. The representative agent adapts her initial consumption level in order to place it over the stable path, given the nature of consumption as a control variable. The expression of the stable trajectory is presentable in general form, as follows.

Proposition 4 *Consider a point (k_0, c_0) in the vicinity of the steady-state (k_1^*, c_1^*) . In the convergence from the initial point to the steady-state, contemporaneous values of consumption and capital evolve following the stable path*

$$c_t = (1 - \eta)c_1^* + \left(1/\beta^{1-\theta_0} - \eta \right) k_t$$

Proof See Appendix C □

As in the constant discounting case, the convergence relation between capital and consumption is of positive sign; the two variables evolve in the same direction as they adjust to the values in which they will remain in the long-term. The position of the saddle-path is influenced by the value of parameter θ_0 in the discount function. If one takes two values for this parameter, θ_0^L and θ_0^H , such that $\theta_0^H > \theta_0^L$, the steady-state values of capital and consumption are relatively larger for θ_0^H ; the stable trajectory is relatively steeper when we take θ_0^L . Therefore, the stronger the bias relatively to the exponential discounting benchmark, the larger are the steady-state values of the model’s endogenous variables and the less consumption changes for a given change in k_t , in the process of adjustment to the steady-state following the stable path. Figure 4 illustrates this outcome.

One should note that the introduction of the capital constraint does not change the nature of the stability result. While in the precedent section, consumption was the only endogenous variable and we had a preferable steady-state that was unstable, with the growth model we have a two-dimensional system for which we find a stable dimension, associated with the state variable, and an unstable dimension that, again, relates to the control variable, i.e., to consumption.

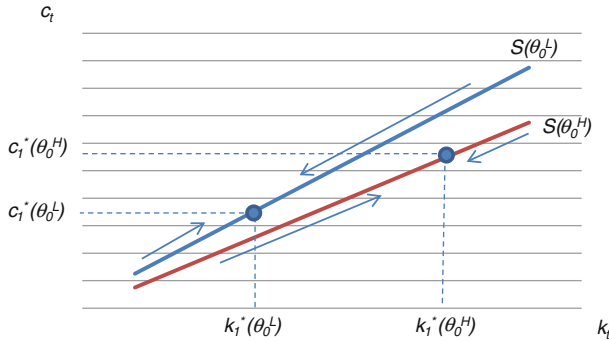


Fig. 4 Stable trajectories for different values of θ_0

6 Habit persistence

In this section, we extend the discounting bias analysis to a setting where consumption choices are subject to habit persistence. This extension serves the purpose of showing that the adaptation of the standard intertemporal optimization model in order to include the discounting bias is flexible enough to approach other meaningful issues concerning the analysis of utility dynamics. The analysis is focused on the case where the household’s constraint relates to financial wealth accumulation, as in Sect. 4. One could extend, as well, the discussion to the capital accumulation case; we leave this as a suggestion for possible future work.

Habit persistence is modeled through the following utility function,⁵

$$u(c_t, c_{t-1}) = \ln(c_t - bc_{t-1}), \quad b \in [0, 1) \tag{14}$$

According to (14), when $b = 0$ we have the conventional version of the model without habit persistence; a positive b indicates that utility is directly dependent on how much more the individual consumes today, relatively to consumption in the previous period. As it is obvious, the larger the value of b the stronger is the habit persistence effect. Constraint $c_t > bc_{t-1}$ must hold, in order to guarantee a feasible solution.

Consider the problem in Sect. 4, relating utility maximization under the exponential discounting bias, subject to resource constraint (7). The dynamic equation for consumption is, in this case,⁶

$$c_{t+1} = bc_t + \frac{\beta^{2(1-\theta_0)}(1 - \beta^{\theta_1})(1+r)[b + \beta(1+r)](c_{t-1} - bc_{t-2})(c_t - bc_{t-1})}{\beta^{1-\theta_0} \{ (1+r) + (1 - \beta^{\theta_1})[b + \beta(1+r)] \} (c_{t-1} - bc_{t-2}) - (c_t - bc_{t-1})} \tag{15}$$

Evidently, Eq. (15) reduces to (8) for $b = 0$.

⁵ See Heer and Maussner (2005) for a more general presentation of the utility function with habit persistence and corresponding analytical treatment.

⁶ See Appendix E for the derivation of this equation.

Recovering ratio $\psi_t := c_t/c_{t-1}$, we rearrange Eq. (15) in order to display it under the form

$$\psi_{t+1} = b + \frac{\beta^{2(1-\theta_0)}(1 - \beta^{\theta_1})(1 + r) [b + \beta(1 + r)]}{\beta^{1-\theta_0} \{ (1 + r) + (1 - \beta^{\theta_1}) [b + \beta(1 + r)] \} \frac{1}{1-b/\psi_t} - \frac{\psi_t}{1-b/\psi_{t-1}}} \tag{16}$$

Proposition 5 *The optimal control problem of the representative household under exponential discounting bias and habit persistence has two steady-state points: $\psi_1^* = \beta^{1-\theta_0}(1 + r) \vee \psi_2^* = \beta^{1-\theta_0}(1 - \beta^{\theta_1})[b + \beta(1 + r)]$.*

Proof See Appendix C □

Comparing with the problem without habit persistence, we verify that the first of the two steady-state points coincides with the one found for case $b = 0$; the second steady-state consumption ratio is different, because it incorporates the habit persistence parameter. In opposition to what occurred previously, now ψ_1^* is not necessarily preferable to ψ_2^* ; relation $\psi_1^* > \psi_2^*$ holds, but only under constraint

$$b < \frac{1 - \beta(1 - \beta^{\theta_1})}{1 - \beta^{\theta_1}}(1 + r) \tag{17}$$

According to inequality (17), the magnitude of habit persistence, measured by b , matters in terms of selecting the preferable long-term steady-state; ψ_1^* is the one preferred for a relatively low level of habit persistence; this might change as b acquires a larger value. For instance, let $\beta = 0.97, r = 0.05, \theta_0 = 0.95$ and $\theta_1 = 50$. In this case, if, for instance, $b = 0.25$, then $\psi_1^* = 1.0185$ and $\psi_2^* = 0.9676$; if $b = 0.35$, then $\psi_1^* = 1.0185$ and $\psi_2^* = 1.0457$. The second steady-state value is, in this last case, desired relatively to the first, since it guarantees a larger long-term growth rate for consumption.

Habit persistence has changed the nature of the conclusions: while in its absence, one steady-state point was dominant and the representative household would act in order to attain it whatever the values of parameters, now this depends on the degree of habit persistence in relation with the other parameter values of the model.

Again, because consumption is a control variable, it does not matter if the difference equation for consumption involves a stability or an instability result, in the vicinity of each one of the steady-states. Nevertheless, Appendix F discusses the stability features of ψ_1^* and ψ_2^* .

7 Conclusion

People do not evaluate future outcomes as if they were computers or calculators. Measuring the future value of some current event or the present value of some future event is many times an intuitive process in which individuals engage. In the same way there is evidence of an exponential growth bias, according to which individual agents

tend to linearize the sequence of accumulated future outcomes, we can conceive a kind of exponential discounting bias, according to which we may explain the evidence that points to decreasing impatience and that is analytically translated in the concept of hyperbolic discounting.

The notion of exponential discounting bias is more general than the one commonly used by economists to characterize observed intertemporal preferences, i.e., the notion of quasi-hyperbolic discounting. This allows for a flexible analysis, where we can shape the trajectory of the discount factor in the way we find more reasonable in order to be as close as possible to what evidence reveals.

Furthermore, the new specification has appealing features from an analytical tractability point of view: because the bias originates on a misperception about how to evaluate the future that does not introduce any kind of sophistication on individual behavior, i.e., any kind of ability to understand that the perception of the future will change as time evolves, the optimization model can be approached similarly to what is done in the exponential discounting case.

Results point to the existence of two steady-state points, with one of them being dominant in terms of faster consumption growth. Given that in the current setting consumption is a control variable, the representative agent can direct her effort to the goal of attaining such desirable long-term outcome.

When assessing the dynamics of an optimal growth model in discrete time, the assumption on the exponential discounting bias allowed to construct a three dimensional dynamic system, from which it is straightforward to analyze steady-state properties and transitional dynamics. The analysis makes it possible to proceed with a thorough characterization of how different intertemporal preferences may shape the optimal relation between capital accumulation and consumption. Stability results are qualitatively similar to the ones obtained under the more restrict case of constant discounting, although the slope of the saddle trajectory depends on the extent of the bias.

The exponential discounting bias concept is adaptable to other features of the benchmark utility analysis. Specifically, in this paper, one has explored the implications of introducing habit persistence into the utility function; this new feature may change the previously found outcome, namely in terms of the steady-state result that the representative agent aims to accomplish.

Acknowledgments We thank participants and organizers of the 6th Annual Meeting of the *Portuguese Economic Journal* and the comments of colleagues from the Business Research Unit of the Lisbon University Institute. We gratefully acknowledge, as well, the valuable and insightful comments of two anonymous referees. Alexandra Ferreira-Lopes and Orlando Gomes acknowledge financial support from PEst-OE/EGE/UI0315/2011. Alexandra Ferreira-Lopes and Tiago Neves Sequeira acknowledge support from FCT, Project PTDC/EGE-ECO/102238/2008.

Appendix A: Computation of equation (6)

To derive Eq. (6), observe that (4) and (5) might be presented, respectively, as

$$U_s(c) = u(c_s) + D(s+1, s)u(c_{s+1}) + D(s+2, s)u(c_{s+2}) + \dots \quad (18)$$

and

$$U_{s+1}(c) = u(c_{s+1}) + D(s + 2, s + 1)u(c_{s+2}) + D(s + 3, s + 1)u(c_{s+3}) + \dots \tag{19}$$

Under the assumption of exponential discounting bias, (18) and (19) come

$$U_s(c) = u(c_s) + \beta^{1-\theta_0+\theta_1}u(c_{s+1}) + \beta^{2(1-\theta_0)+\theta_1}u(c_{s+2}) + \dots \tag{20}$$

and

$$U_{s+1}(c) = u(c_{s+1}) + \beta^{1-\theta_0+\theta_1}u(c_{s+2}) + \beta^{2(1-\theta_0)+\theta_1}u(c_{s+3}) + \dots \tag{21}$$

Now, we rearrange the terms in Eq. (20) in order to display an equivalent expression,

$$\begin{aligned}
 U_s(c) &= u(c_s) \\
 &+ \beta^{1-\theta_0} \left[u(c_{s+1}) + \beta^{1-\theta_0+\theta_1}u(c_{s+2}) + \beta^{2(1-\theta_0)+\theta_1}u(c_{s+3}) \right. \\
 &\left. + \dots - (1 - \beta^{\theta_1})u(c_{s+1}) \right]
 \end{aligned} \tag{22}$$

Replacing $U_{s+1}(c)$ as defined in (21) into (22), we finally arrive to

$$U_s(c) = u(c_s) + \beta^{1-\theta_0} [U_{s+1}(c) - (1 - \beta^{\theta_1})u(c_{s+1})] \tag{23}$$

Equation (23) is identical to Eq. (6). In the main text, we replace s by t in order to make clear that the function holds for any two consecutive periods. After displaying this equation, all the subsequent analysis uses t as the time subscript.

Appendix B: Solution of the optimization problem

In order to maximize intertemporal utility subject to the budget constraint, we resort to a conventional dynamic programming procedure.⁷ This requires recovering Eq. (6) and defining function $V(a_t)$, such that,

$$V(a_t) = \max_c \left\{ u(c_t) + \beta^{1-\theta_0} [V(a_{t+1}) - (1 - \beta^{\theta_1})u(c_{t+1})] \right\} \tag{24}$$

Note that, in (24), $V(a_{t+1})$ is a function of a_t and c_t , given the budget constraint.

The corresponding first order conditions are

⁷ See [Walde \(2011\)](#) for an extensive discussion on intertemporal optimization techniques, including dynamic programming.

$$\frac{\partial V(a_t)}{\partial c_t} = 0 \Rightarrow u'(c_t) + \beta^{1-\theta_0} \left[V'(a_{t+1}) \frac{\partial a_{t+1}}{\partial c_t} - (1 - \beta^{\theta_1}) u'(c_{t+1}) \frac{dc_{t+1}}{dc_t} \right] = 0 \quad (25)$$

and

$$\begin{aligned} \frac{\partial V(a_t)}{\partial a_t} &= \beta^{1-\theta_0} \frac{\partial V(a_{t+1})}{\partial a_t} \Rightarrow \\ V'(a_t) &= \beta^{1-\theta_0} V'(a_{t+1}) \frac{\partial a_{t+1}}{\partial a_t} \end{aligned} \quad (26)$$

The transversality condition of the problem is given by

$$\lim_{t \rightarrow \infty} a_t D_{EB}(t, s) V(a_t) = 0 \quad (27)$$

To proceed with the analysis, one must compute the derivatives that are present in the expressions of the first-order conditions. Note that $\frac{\partial a_{t+1}}{\partial c_t} = -1$ and $\frac{\partial a_{t+1}}{\partial a_t} = 1 + r$; these values are straightforward to obtain from the resource constraint (7). Relatively to the derivative $\frac{dc_{t+1}}{dc_t}$, that appears in condition (25), it indicates the expected change on the value of consumption from date t to date $t + 1$. If we recall that our central assumption is that the representative agent intends to discount the future at a constant rate, although she is unable to follow her plan, the expected change in consumption will be the effective change that would be obtained under constant discounting. Constant discounting implies a well known rule of motion for consumption, $c_{t+1} = \beta(1 + r)c_t$, which holds for every value of t . Thus, the expected change in consumption to be used in (25) is $\frac{dc_{t+1}}{dc_t} = \beta(1 + r)$.

The previous reasoning allows to rewrite the first-order conditions as follows,

$$\beta^{1-\theta_0} V'(a_{t+1}) = u'(c_t) - \beta^{2-\theta_0} (1 - \beta^{\theta_1}) (1 + r) u'(c_{t+1}) \quad (28)$$

and

$$V'(a_t) = \beta^{1-\theta_0} (1 + r) V'(a_{t+1}) \quad (29)$$

Combining the two optimality conditions, (28) and (29), one obtains an equation of motion for consumption. Observe that (28) is equivalent to

$$V'(a_{t+1}) = \frac{1}{\beta^{1-\theta_0}} u'(c_t) - \beta(1 - \beta^{\theta_1}) (1 + r) u'(c_{t+1}) \quad (30)$$

or, one period earlier,

$$V'(a_t) = \frac{1}{\beta^{1-\theta_0}} u'(c_{t-1}) - \beta(1 - \beta^{\theta_1})(1 + r)u'(c_t) \tag{31}$$

Replacing (30) and (31) into (29) and taking a logarithmic utility function, one easily arrives to Eq. (8) in the main text.

An alternative way to approach the optimization problem, that is similar to the previous one, consists in solving it through the Pontryagin’s principle. This requires writing the corresponding current-value Hamiltonian function, which takes the form

$$H(a_t, c_t, p_t) = u(c_t) + \beta^{1-\theta_0} [p_{t+1}(w + ra_t - c_t) - (1 - \beta^{\theta_1})u(c_{t+1})] \tag{32}$$

In function (32), p_t represents the shadow-price of the state variable a_t , also known as the problem’s costate variable. First-order conditions are, in this case,

$$\begin{aligned} \frac{\partial H}{\partial c_t} &= 0 \Rightarrow \\ \beta^{1-\theta_0} p_{t+1} &= u'(c_t) - \beta^{2-\theta_0}(1 - \beta^{\theta_1})(1 + r)u'(c_{t+1}) \tag{33} \\ \beta^{1-\theta_0} p_{t+1} - p_t &= -\frac{\partial H}{\partial a_t} \Rightarrow p_t = \beta^{1-\theta_0}(1 + r)p_{t+1} \tag{34} \end{aligned}$$

The transversality condition holds, as presented in (27). Combining (33) and (34) and, again, taking the logarithmic utility function, we arrive to the same difference equation for consumption, (8). The two methods had to deliver the same outcome, what becomes clear once we note that there is a correspondence between $V'(a_t)$ in the dynamic programming problem and the shadow-price p_t used to write the Hamiltonian function.

Appendix C: Proofs of Propositions

Proof of Proposition 1

To prove this proposition, one just needs to solve (9), for $\psi^* := \psi_{t+1} = \psi_t$. A quadratic equation is generated and, as a result, two solutions exist. The equation is

$$(\psi^*)^2 - \beta^{1-\theta_0} [1 + \beta(1 - \beta^{\theta_1})] (1 + r)\psi^* + \beta^{3-2\theta_0}(1 - \beta^{\theta_1})(1 + r)^2 = 0 \tag{35}$$

Simple algebra allows to arrive to the pair of solutions of (35) that are given in the proposition.

Proof of Proposition 2

The steady-state is defined as the pair of values (k^*, c^*) such that $k_{t+1} = k_t$ and $c_{t+1} = c_t = c_{t-1}$. Applying these conditions to the equations in system (12), one calculates the values in the proposition.

The evaluation of the consumption’s equation in the steady-state yields the following quadratic equation that can be solved with respect to the term $\eta A (k^*)^{-(1-\eta)}$,

$$\beta^{3-2\theta_0}(1-\beta^{\theta_1}) \left[\eta A (k^*)^{-(1-\eta)} \right]^2 - \beta^{1-\theta_0} [1+\beta(1-\beta^{\theta_1})] \eta A (k^*)^{-(1-\eta)} + 1 = 0 \tag{36}$$

The solution of (36) is

$$\eta A (k^*)^{-(1-\eta)} = \frac{1}{\beta^{1-\theta_0}} \vee \eta A (k^*)^{-(1-\eta)} = \frac{1}{\beta^{2-\theta_0}(1-\beta^{\theta_1})} \tag{37}$$

From the above expressions, one withdraws the steady-state values of capital, k_1^* and k_2^* that are presented in the proposition. To obtain the steady-state values of consumption, one just needs to replace k_1^* and k_2^* in expression $c^* = A (k^*)^\eta - k^*$; this expression follows directly from constraint (10).

Proof of Proposition 3

The existence of one stable dimension implies that one of the eigenvalues of the Jacobian matrix locates inside the unit circle, while the other two fall outside the unit circle. The eigenvalues are straightforward to obtain once we solve the corresponding characteristic equation; they are $\lambda_1 = \eta$, $\lambda_2 = \frac{1}{\beta(1-\beta^{\theta_1})}$, $\lambda_3 = \frac{1}{\beta^{1-\theta_0}\eta}$. Observing that $\lambda_1 \in (0, 1)$, $\lambda_2, \lambda_3 > 1$, one confirms the statement in the proposition.

Proof of Proposition 4

The saddle-path stable trajectory can be obtained by computing the eigenvector associated to the eigenvalue inside the unit circle. This requires solving the system

$$\begin{bmatrix} \chi_1 - \lambda_1 & -1 & 0 \\ (\chi_2 - \chi_1)\chi_3 & 1 + \chi_2 + \chi_3 - \lambda_1 & -\chi_2 \\ 0 & 1 & -\lambda_1 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{38}$$

Letting $p_1 = 1$, the eigenvector might be written as

$$P = \begin{bmatrix} 1 \\ \chi_1 - \lambda_1 \\ \chi_1/\lambda_1 - 1 \end{bmatrix} \tag{39}$$

or, equivalently,

$$P = \begin{bmatrix} 1 \\ 1/\beta^{1-\theta_0} - \eta \\ 1/(\eta\beta^{1-\theta_0}) - 1 \end{bmatrix} \tag{40}$$

The slope of the contemporaneous relation between consumption and capital is given by the ratio p_2/p_1 , i.e., $c_t - c_1^* = (p_2/p_1)(k_t - k_1^*)$, which corresponds to expression

$$c_t - c_1^* = \left(1/\beta^{1-\theta_0} - \eta\right)(k_t - k_1^*) \tag{41}$$

Taking into consideration the steady-state for which the analysis is being undertaken, (41) is equivalent to the equation in the proposition.

Proof of Proposition 5

By taking $\psi^* := \psi_{t+1} = \psi_t = \psi_{t-1}$, one transforms (16) into the quadratic equation $(\psi^*)^2 - \ell_2\psi^* + \ell_1 = 0$, with $\ell_1 := \beta^{2(1-\theta_0)}(1 - \beta^{\theta_1})(1 + r)[b + \beta(1 + r)]$ and $\ell_2 := \beta^{1-\theta_0} \{(1 + r) + (1 - \beta^{\theta_1})[b + \beta(1 + r)]\}$. The solutions of this equation are the steady-state values $\psi_1^* = \beta^{1-\theta_0}(1 + r) \vee \psi_2^* = \beta^{1-\theta_0}(1 - \beta^{\theta_1})[b + \beta(1 + r)]$.

Appendix D: Computation of equation (11)

Begin by defining function

$$V(k_t) = \max_c \left\{ u(c_t) + \beta^{1-\theta_0} [V(k_{t+1}) - (1 - \beta^{\theta_1})u(c_{t+1})] \right\} \tag{42}$$

First-order conditions are,

$$\begin{aligned} \frac{\partial V(k_t)}{\partial c_t} = 0 \Rightarrow \\ u'(c_t) + \beta^{1-\theta_0} \left[V'(k_{t+1}) \frac{\partial k_{t+1}}{\partial c_t} - (1 - \beta^{\theta_1})u'(c_{t+1}) \frac{dc_{t+1}}{dc_t} \right] = 0 \end{aligned} \tag{43}$$

and

$$\frac{\partial V(k_t)}{\partial k_t} = \beta^{1-\theta_0} \frac{\partial V(k_{t+1})}{\partial k_t} \Rightarrow V'(k_t) = \beta^{1-\theta_0} V'(k_{t+1}) \frac{\partial k_{t+1}}{\partial k_t} \tag{44}$$

The transversality condition also applies. As in section 4, the derivative $\frac{dc_{t+1}}{dc_t}$ represents the expected change on consumption, under constant discounting. For the constraint assumed in the current case,

$$\frac{dc_{t+1}}{dc_t} = \beta\eta Ak_{t+1}^{-(1-\eta)} \tag{45}$$

Replacing (45) into (44), one obtains relation

$$V'(k_{t+1}) = \frac{1}{\beta^{1-\theta_0}} u'(c_t) - \beta(1 - \beta^{\theta_1})\eta Ak_{t+1}^{-(1-\eta)} u'(c_{t+1}) \tag{46}$$

or, one period earlier,

$$V'(k_t) = \frac{1}{\beta^{1-\theta_0}} u'(c_{t-1}) - \beta(1 - \beta^{\theta_1}) \eta A k_t^{-(1-\eta)} u'(c_t) \quad (47)$$

The second optimal condition is, in the current model's specification,

$$V'(k_t) = \beta^{1-\theta_0} \eta A k_t^{-(1-\eta)} V'(k_{t+1}) \quad (48)$$

Now, we replace (46) and (47) into (48); as a result, the equation of motion for consumption takes the form

$$u'(c_{t+1}) = \frac{\beta^{1-\theta_0} [1 + \beta(1 - \beta^{\theta_1})] \eta A k_t^{-(1-\eta)} u'(c_t) - u'(c_{t-1})}{\beta^{3-2\theta_0} (1 - \beta^{\theta_1}) (\eta A)^2 k_t^{-(1-\eta)} k_{t+1}^{-(1-\eta)}} \quad (49)$$

Taking utility function $u(c) = \ln c$, it is straightforward to realize that Eq. (49) is equivalent to Eq. (11) in the main text.

Appendix E: Computation of equation (15)

Let us start by analyzing the problem under constant discounting. Define $V(a_t)$ in the current circumstance as

$$V(a_t) = \max_c [u(c_t, c_{t-1}) + \beta V(a_{t+1})] \quad (50)$$

First-order conditions are:

$$u'(c_t, c_{t-1}) + \beta V'(a_{t+1}) \frac{\partial a_{t+1}}{\partial c_t} \Rightarrow \frac{1}{c_t - b c_{t-1}} = \beta V'(a_{t+1}) \quad (51)$$

and

$$V'(a_t) = \beta V'(a_{t+1}) \frac{\partial a_{t+1}}{\partial a_t} \Rightarrow V'(a_t) = \beta(1 + r) V'(a_{t+1}) \quad (52)$$

The transversality condition, as presented in previous occasions, holds as well. Combining (51) and (52), and advancing one time period, the difference equation for consumption comes

$$c_{t+1} = [b + \beta(1 + r)] c_t - b\beta(1 + r) c_{t-1} \quad (53)$$

Under the exponential discounting bias,

$$V(a_t) = \max_c \left\{ u(c_t, c_{t-1}) + \beta^{1-\theta_0} [V(a_{t+1}) - (1 - \beta^{\theta_1}) u(c_{t+1}, c_t)] \right\} \quad (54)$$

The optimality conditions required for finding the consumption dynamic equation are, in the current case,

$$u'(c_t, c_{t-1}) + \beta^{1-\theta_0} \left[V'(a_{t+1}) \frac{\partial a_{t+1}}{\partial c_t} - (1 - \beta^{\theta_1}) u'(c_{t+1}, c_t) \frac{dc_{t+1}}{dc_t} \right] = 0 \Rightarrow$$

$$V'(a_{t+1}) = \frac{1}{\beta^{1-\theta_0}} \frac{1}{c_t - bc_{t-1}} - (1 - \beta^{\theta_1}) [b + \beta(1 + r)] \frac{1}{c_{t+1} - bc_t} \tag{55}$$

and

$$V'(a_t) = \beta V'(a_{t+1}) \frac{\partial a_{t+1}}{\partial a_t} \Rightarrow V'(a_t) = \beta^{1-\theta_0} (1 + r) V'(a_{t+1}) \tag{56}$$

Note that, in (55), $\frac{dc_{t+1}}{dc_t} = b + \beta(1 + r)$ given the expression for the evolution of consumption in (53). Taking together (55) and (56) and proceeding as in the previous settings, one obtains the consumption difference equation in the proposition.

Appendix F: stability of the steady-state points in the Model with habit persistence

The study of stability requires, in this case, addressing the local properties of system

$$\begin{cases} \psi_{t+1} = b + \frac{\psi_1^* \psi_2^*}{\frac{\psi_1^* + \psi_2^*}{1 - b/\psi_t} - \frac{\psi_t}{1 - b/\phi_t}} \\ \phi_{t+1} = \psi_t \end{cases} \tag{57}$$

Values ψ_1^* and ψ_2^* are the steady-state values, as presented in section 5; variable ϕ_t is the lag variable associated with ψ_t . The linearization of the equations in (57), with respect to each steady-state value yields:

- For ψ_1^* :

$$\begin{bmatrix} \psi_{t+1} - \psi_1^* \\ \phi_{t+1} - \psi_1^* \end{bmatrix} = \begin{bmatrix} \frac{b}{\psi_1^*} + \frac{\psi_1^*}{\psi_2^*} - \frac{b}{\psi_2^*} \\ 1 \quad 0 \end{bmatrix} \cdot \begin{bmatrix} \psi_t - \psi_1^* \\ \phi_t - \psi_1^* \end{bmatrix} \tag{58}$$

- For ψ_2^* :

$$\begin{bmatrix} \psi_{t+1} - \psi_2^* \\ \phi_{t+1} - \psi_2^* \end{bmatrix} = \begin{bmatrix} \frac{b}{\psi_2^*} + \frac{\psi_2^*}{\psi_1^*} - \frac{b}{\psi_1^*} \\ 1 \quad 0 \end{bmatrix} \cdot \begin{bmatrix} \psi_t - \psi_2^* \\ \phi_t - \psi_2^* \end{bmatrix} \tag{59}$$

The eigenvalues of the Jacobian matrices for the cases ψ_1^* and ψ_2^* are, respectively, $\lambda_1|_{\psi_1^*} = \frac{b}{\psi_1^*}$; $\lambda_2|_{\psi_1^*} = \frac{\psi_1^*}{\psi_2^*}$; $\lambda_1|_{\psi_2^*} = \frac{b}{\psi_2^*}$; $\lambda_2|_{\psi_2^*} = \frac{\psi_2^*}{\psi_1^*}$. The presented eigenvalues are all positive.

Equilibrium point ψ_1^* is stable if:

- $\lambda_1|_{\psi_1^*} < 1 \Rightarrow b < \beta^{1-\theta_0} (1 + r)$ and

$$\bullet \lambda_2|_{\psi_1^*} < 1 \Rightarrow b > \frac{1-\beta(1-\beta^{\theta_1})}{1-\beta^{\theta_1}}(1+r).$$

Equilibrium point ψ_2^* is stable if:

$$\bullet \lambda_1|_{\psi_2^*} < 1 \Rightarrow b < \frac{\beta(1+r)}{1-\beta^{1-\theta_0}(1-\beta^{\theta_1})} \text{ and}$$

$$\bullet \lambda_2|_{\psi_2^*} < 1 \Rightarrow b < \frac{1-\beta(1-\beta^{\theta_1})}{1-\beta^{\theta_1}}(1+r).$$

Saddle-path stability holds when only one of the eigenvalues, for each of the above pairs, falls inside the unit circle and instability prevails when both eigenvalues of the Jacobian matrix fall outside the unit circle. Results point for a crucial role of the habit persistence parameter b ; it is the value of this parameter that will determine the stability profile.

Note that $\lambda_2|_{\psi_1^*} = 1/\lambda_2|_{\psi_2^*}$. Observe, as well, that the condition allowing for $\lambda_2|_{\psi_2^*} < 1$ is identical to (17), i.e., it is the one that will determine which of the two points ψ_1^* and ψ_2^* is preferable from an utility maximization point of view. As a result, the desired steady-state will have one of its dimensions necessarily unstable, while the other steady-state is such that the same dimension is stable. We reemphasize, though, that this instability of the steady-state the agent wants to achieve is not an obstacle for the success of the agent's optimal strategy because consumption is a control variable.

Recover the example that has taken the following array of parameters: $\beta = 0.97$, $r = 0.05$, $\theta_0 = 0.95$ and $\theta_1 = 50$. In this case, independently of the value of $b \in [0, 1)$, $\lambda_1|_{\psi_1^*} < 1$ and $\lambda_1|_{\psi_2^*} < 1$. Relatively to the other eigenvalue, it is true that:

- If $b < 0.3243$, then $\lambda_1|_{\psi_1^*} > 1$ and $\lambda_1|_{\psi_2^*} < 1$;
- If $b > 0.3243$, then $\lambda_1|_{\psi_1^*} < 1$ and $\lambda_1|_{\psi_2^*} > 1$;

In this case, a low level of habit persistence implies that ψ_1^* is saddle-path stable and ψ_2^* is a stable equilibrium point. This result is reversed once the habit persistence parameter acquires a value larger than 0.3243. Condition $b < 0.3243$ also implies $\psi_1^* > \psi_2^*$, meaning that the unstable dimension is associated with the steady-state point that the household prefers to achieve.

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