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# Von Bertalanffy's dynamics under a polynomial correction: Allee effect and big bang bifurcation

J. Leonel Rocha<sup>1</sup>, A.K. Taha<sup>2</sup> and D. Fournier-Prunaret<sup>3</sup>

<sup>1</sup> ISEL-Instituto Superior de Engenharia de Lisboa, ADM, Rua Conselheiro Emídio Navarro 1, 1959-007 Lisboa, Portugal

<sup>2</sup> INSA, University of Toulouse, 135 Avenue du Rangueil, 31077 Toulouse, France

<sup>3</sup> LAAS-CNRS, INSA, University of Toulouse, 7 Avenue du Colonel Roche, 31077 Toulouse, France

E-mail: jrocha@adm.isel.pt

**Abstract.** In this work we consider new one-dimensional populational discrete dynamical systems in which the growth of the population is described by a family of von Bertalanffy's functions, as a dynamical approach to von Bertalanffy's growth equation. The purpose of introducing Allee effect in those models is satisfied under a correction factor of polynomial type. We study classes of von Bertalanffy's functions with different types of Allee effect: strong and weak Allee's functions. Dependent on the variation of four parameters, von Bertalanffy's functions also includes another class of important functions: functions with no Allee effect. The complex bifurcation structures of these von Bertalanffy's functions is investigated in detail. We verified that this family of functions has particular bifurcation structures: the big bang bifurcation of the so-called "box-within-a-box" type. The big bang bifurcation is associated to the asymptotic weight or carrying capacity. This work is a contribution to the study of the big bang bifurcation analysis for continuous maps and their relationship with explosion birth and extinction phenomena.

## 1. Introduction and motivation

Von Bertalanffy's equation remains one of the most popular flexible growth equations to describe the growth of marine populations, in particular fishes, seabirds, marine mammals, invertebrates, reptiles and sea turtles. The growth equation introduced by von Bertalanffy to model fish weight growth is given by the differential equation

$$\frac{dW_t}{dt} = \frac{V}{3} W_t^{\frac{2}{3}} \left( 1 - \left( \frac{W_t}{W_\infty} \right)^{\frac{1}{3}} \right), \quad (1)$$

where  $W_t$  is the weight at age  $t$ ,  $W_\infty$  is the asymptotic weight or carrying capacity,  $V > 0$  is von Bertalanffy's growth rate constant and  $t_0$  is the theoretical age the chick would have at weight zero, see [5].

Currently the extinction of certain marine species is one of the most worrying problems in ecological and biological research. The Allee effect is an important dynamic phenomenon first described by Allee in 1931, [1]. A demographic Allee effect occurs when individual fitness, at low densities, increases with population density. However, the von Bertalanffy growth model, Eq.(1), do not exhibit the Allee effect, because the associated *per capita* growth rate decreases



at low densities. This drawback was corrected using suitable corrections in the work presented by Rocha *et al* in [8]. One of the corrected von Bertalanffy's model proposed is defined by the differential equation,

$$\frac{dW_t}{dt} = \frac{V}{3} W_t^{\frac{2}{3}} \left( 1 - \left( \frac{W_t}{W_\infty} \right)^{\frac{1}{3}} \right) \frac{W_t - E}{W_\infty + C}, \quad (2)$$

with  $|E| < W_\infty$  and  $C > 0$ , where  $E$  is the rarefaction critical density (density threshold) or Allee limit and  $W_\infty$  is the carrying capacity. Note that the adjustment or correction factor is of polynomial type.

## 2. von Bertalanffy's functions of polynomial type

In the present work, we study a dynamical approach to von Bertalanffy's growth models: a new class of functions describing the existence or not of strong and weak Allee effect is investigated. Consider the family of functions  $f : [0, 1] \rightarrow ]-\infty, 1]$ , defined by

$$f(x; r, K, E, C) = r x^{\frac{2}{3}} \left( 1 - x^{\frac{1}{3}} \right) \frac{Kx - E}{K + C}, \quad (3)$$

with  $x = \frac{W_t}{W_\infty} = \frac{W_t}{K} \in [0, 1]$  the normalized weight,  $r = \frac{V}{3} \times K^{\frac{2}{3}} > 0$  an intrinsic growth rate of the individual weight, the Allee limit  $E \in \mathbf{R}$  and the carrying capacity  $K > 0$  satisfying  $|E| < K$  and  $C > -K$ . This family of functions will be designated by von Bertalanffy's functions of polynomial type, see some examples in Figs.1, 2 and 3. Generically, these functions are unimodal or bimodal maps for which extinction is inevitable for too high or too low initial population densities. Furthermore, this family of maps is proportional to the right hand side of the von Bertalanffy's growth model, Eq.(2).

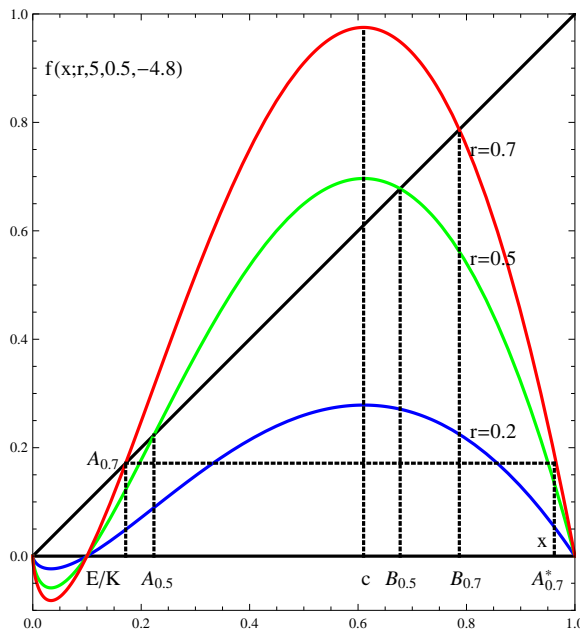
Let  $A_r = f(A_r; r, K, E, C)$  be a fixed point of  $f$  and  $A_r^* = \max\{f^{-1}(A_r; r, K, E, C)\}$ . From Eq.(3), we have that  $E/K < A_r$ . The interval  $[0, E/K]$  will be designated by extinction interval and  $E/K$  is the Allee threshold. The following properties are satisfied:

- (A1)  $f$  is continuous on  $[0, 1]$ ;
- (A2) there exists  $[A_r, A_r^*] \subset [0, 1] : f(x; r, K, E, C) \leq A_r, \forall x \in [0, 1] \setminus ]A_r, A_r^*];$
- (A3)  $f$  has an unique critical point  $c \in ]A_r, A_r^*];$
- (A4)  $f \in C^3([A_r, A_r^*]);$
- (A5)  $f'(x; r, K, E, C) \neq 0, \forall x \in ]A_r, A_r^*] \setminus \{c\}, f'(c; r, K, E, C) = 0$  and  $f''(c; r, K, E, C) < 0;$
- (A6) the Schwarzian derivative  $S(f(x; r, K, E, C)) = \frac{f'''(x; r, K, E, C)}{f'(x; r, K, E, C)} - \frac{3}{2} \left( \frac{f''(x; r, K, E, C)}{f'(x; r, K, E, C)} \right)^2$  verifies the following properties:
  - (i)  $S(f(x; r, K, E, C)) < 0, \forall x \in ]A_r, A_r^*] \setminus \{c\}$  and  $S(f(c; r, K, E, C)) = -\infty;$
  - (ii)  $S(f(x \rightarrow 0^+; r, K, E, C)) > 0$ , with  $|E| < K \wedge E \neq 0;$
  - (iii)  $S(f(x \rightarrow 0^+; r, K, E, C)) < 0$ , with  $E = 0.$

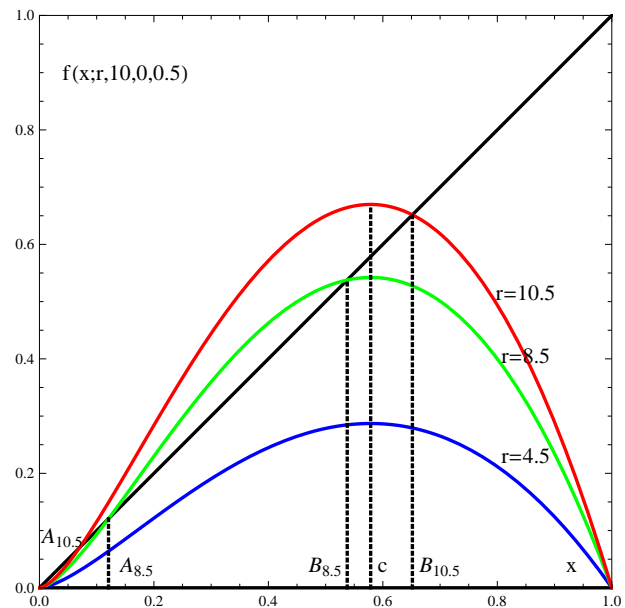
To von Bertalanffy's functions, Eq.(3), we define a class of functions that describes the growth of a population with Allee effect.

**Definition 1:** Consider  $0 \leq E < K, C > -K, r > 0$  and  $f(x; r, K, E, C)$  defined by Eq.(3). von Bertalanffy's functions  $f : [0, 1] \rightarrow ]-\infty, 1]$  attaining its maximum at  $c$  are Allee's functions, if there are real numbers  $A_r$  and  $B_r$  such that:

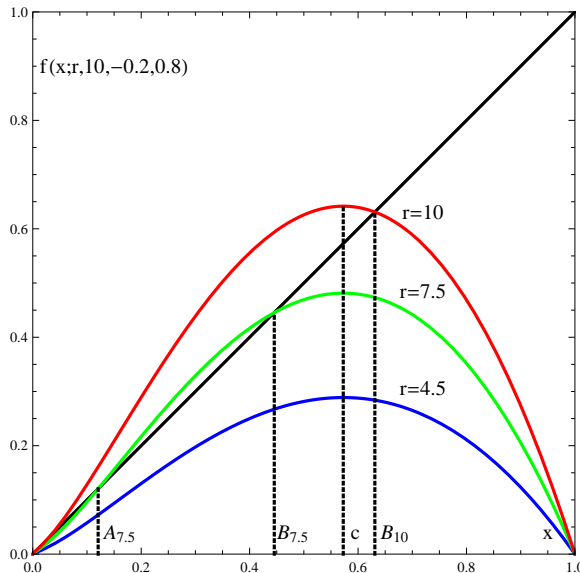
- (i)  $0 < A_r < c < B_r < 1;$
- (ii)  $\forall x \in [0, 1] \setminus [A_r, B_r] \Rightarrow f(x; r, K, E, C) < x;$
- (iii)  $\forall x \in ]A_r, B_r[ \Rightarrow f(x; r, K, E, C) > x.$



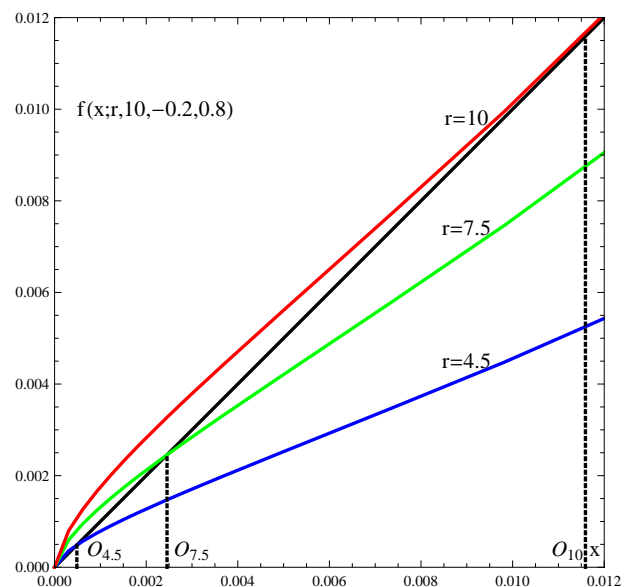
**Figure 1.** von Bertalanffy's functions with strong Allee effect  $f(x; r, K, E, C)$  at  $K = 5$ ,  $E = 0.5$ ,  $C = -4.8$  and  $r = 0.2$  (1 fix. pt.: 0),  $r = 0.5$  (3 fix. pts.: 0,  $A_{0.5}$  and  $B_{0.5}$ ) and  $r = 0.7$  (3 fix. pts.: 0,  $A_{0.7}$  and  $B_{0.7}$ ).



**Figure 2.** von Bertalanffy's functions with weak Allee effect  $f(x; r, K, E, C)$  at  $K = 10$ ,  $E = 0$ ,  $C = 0.5$  and  $r = 4.5$  (1 fix. pt.: 0),  $r = 8.5$  (3 fix. pts.: 0,  $A_{8.5}$  and  $B_{8.5}$ ) and  $r = 10.5$  (3 fix. pts.: 0,  $A_{10.5}$  and  $B_{10.5}$ ).



**Figure 3.** von Bertalanffy's functions with no Allee effect  $f(x; r, K, E, C)$  at  $K = 10$ ,  $E = -0.2$ ,  $C = 0.8$  and  $r = 4.5$  (2 fix. pts.: 0 and  $O_r$ ),  $r = 7.5$  and  $r = 10$  (4 fix. pts.: 0,  $O_r$ ,  $A_r$  and  $B_r$ ).



**Figure 4.** von Bertalanffy's functions with no Allee effect  $f(x; r, K, E, C)$  at  $K = 10$ ,  $E = -0.2$ ,  $C = 0.8$  and  $r = 4.5$  (2 fix. pts.: 0 and  $O_r$ ),  $r = 7.5$  and  $r = 10$  (4 fix. pts.: 0,  $O_r$ ,  $A_r$  and  $B_r$ ).

This class of Allee's functions is divided into two subclasses of functions that verify the following properties.

**Property 1:** Under the conditions of Def.1 it is established that:

- (i) If  $0 < E < K$  and  $C > -K$ , the von Bertalanffy functions  $f : [0, 1] \rightarrow ]-\infty, 1]$  are a family

of strong Allee's functions, see Fig.1;

- (ii) If  $E = 0$  and  $C > -K$ , the von Bertalanffy functions  $f : [0, 1] \rightarrow [0, 1]$  are a family of weak Allee's functions, see Fig.2.

This classification is based on the concepts of strong and weak Allee's effects to *population growth rates*. The Allee effect is described as *strong* when there is a density threshold, where the population growth rate becomes null and below which are negative. The Allee effect is described as *weak* when the populations do not exhibit any thresholds, the point  $x = 0$  becomes stable and the population growth rate are positive. von Bertalanffy's functions  $f$  also includes another class of important functions, for which the existence of Allee effect has no evidence.

**Definition 2:** Consider  $-K < E < 0$ ,  $C > -K$ ,  $r > 0$  and  $f(x; r, K, E, C)$  defined by Eq.(3). von Bertalanffy's functions  $f : [0, 1] \rightarrow [0, 1]$  attaining its maximum at  $c$  are functions with no Allee effect, if there are real numbers  $O_r, A_r$  and  $B_r$  such that:

- (i)  $0 < O_r < A_r < c < B_r < 1$ ;
- (ii)  $\forall x \in (]O_r, A_r[ \cup ]B_r, 1]) \Rightarrow f(x; r, K, E, C) < x$ ;
- (iii)  $\forall x \in (]0, O_r[ \cup ]A_r, B_r]) \Rightarrow f(x; r, K, E, C) > x$ .

From Def.2, the fixed point  $x = 0$  is always unstable. It is for this reason that this class of functions with no Allee effect does not contemplate the phenomenon of extinction and any thresholds. Note that from Def.1 and Prop.1 it follows that, in the cases of strong and weak Allee's functions, we will have at maximum three fixed points  $0, A_r$  and  $B_r$ , see Figs.1 and 2. On the other hand, from Def.2, in the case of functions with no Allee effect, we will have at least two fixed points  $0$  and  $O_r$ , and at most four fixed points  $0, O_r, A_r$  and  $B_r$ , depending on the evolution of the parameter  $r$ , see Fig.3, for details of the fixed point  $O_r$  see Fig.4.

**Remark 1:** Note that the transition from strong Allee effect ( $0 < E < K$  and  $C > -K$ ) to no Allee effect ( $-K < E < 0$  and  $C > -K$ ), passing through the weak Allee effect ( $E = 0$  and  $C > -K$ ), involve the several parameters considered in the definition of the new models, standing out the Allee limit  $E$  and the parameter  $C$ . See in Table 1 the summary of the most important properties for von Bertalanffy's functions of polynomial type.

**Remark 2:** Note that for von Bertalanffy's functions  $f$  when  $r$  varies monotonically in the interval  $]0, \tilde{r}[$ , where  $\tilde{r}$  is such that  $f(c; \tilde{r}, K, E, C) = 1$ , there exists a fixed point  $B_r$  such that its multiplier ( $s = \frac{\partial f}{\partial x}(x; r, K, E, C)$  calculated at the fixed point) decreases monotonically from  $+1$ , and a fixed point with  $s$  which increases monotonically from  $+1$ .

### 3. Bifurcation structures in von Bertalanffy's functions: big bang bifurcation

In this work we investigate in detail the bifurcation structures of von Bertalanffy's functions given by Eq.(3), in the  $(C, r)$  two-dimensional parameter space.

#### 3.1. Fold and flip bifurcation curves

We will make use of the classical fold and flip bifurcations. We study the behavior of such curves related with some cycles of order  $n \in \mathbf{N}$ , see Figs.5, 7 and 9. In a general way, to von Bertalanffy's functions  $f$ , the fold and flip bifurcation curves relative to a cycle of order  $n$  are determined as follows. If  $x \in [0, 1]$  is a point of an order  $n$  cycle that satisfies the equations

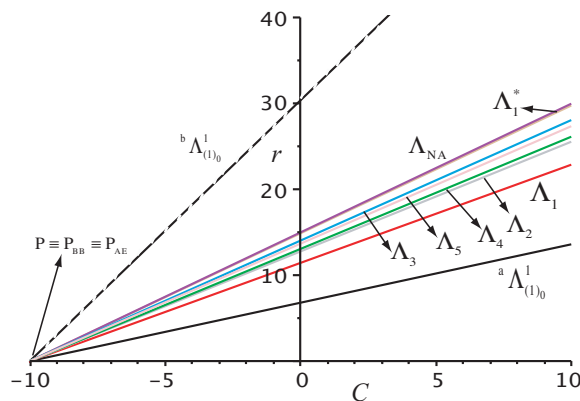
$$f^n(x; r, K, E, C) = x \quad \text{and} \quad \frac{\partial f^n}{\partial x}(x; r, K, E, C) = 1 \quad (4)$$

then there exists a solution  $\varphi_n$ , such that the fold bifurcation curves relative to a cycle of order  $n \in \mathbf{N}$  are given by  $r(C) = \varphi_n(x; K, E, C)$ , and are denoted by  $\Lambda_{(n)_0}^j$ , where  $j$  is the number of the curve, which differentiates cycles of the same order.

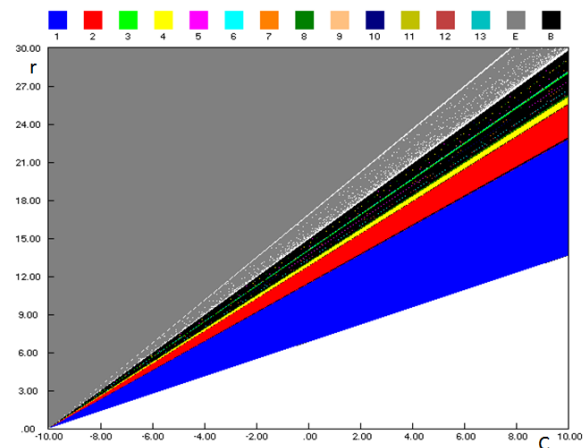
On the other hand, if  $x \in [0, 1[$  is such that,

$$f^n(x; r, K, E, C) = x \quad \text{and} \quad \frac{\partial f^n}{\partial x}(x; r, K, E, C) = -1 \quad (5)$$

then exists a solution  $\psi_n$ , such that the flip bifurcation curves relative to a cycle of order  $n \in \mathbb{N}$  are given by  $r(C) = \psi_n(x; K, E, C)$ , and are denoted by  $\Lambda_n^j$ , where  $j$  is the number of the curve, which differentiates cycles of the same order.



**Figure 5.** Bifurcations curves of  $f^n(x; r, K, E, C)$  with strong Allee effect, for  $n = 1, 2, 3, 4, 5$ , in the  $(C, r)$  parameter plane at  $K = 10$  and  $E = 0.02$ .  ${}^a\Lambda_{(1)_0}^1$  and  ${}^b\Lambda_{(1)_0}^1$  are the fold bifurcation curves of different cycles of order  $n = 1$ ,  $\Lambda_1$ ,  $\Lambda_2$ ,  $\Lambda_3$ ,  $\Lambda_4$  and  $\Lambda_5$  are the flip bifurcation curves of the cycles of order  $n = 1, 2, 3, 4, 5$ , respectively;  $\Lambda_1^*$  is the chaotic semistability curve and  $\Lambda_{NA}$  is the bifurcation curve of non admissibility, in this case  $\Lambda_1^* \neq \Lambda_{NA}$ .  $P_{BB}$  is the big bang bifurcation point and  $P_{AE}$  is the Allee bifurcation point.



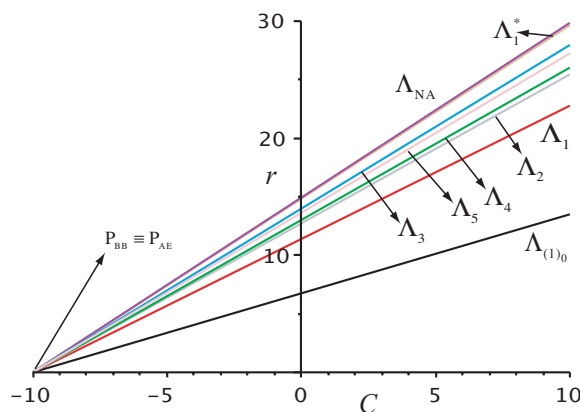
**Figure 6.** Bifurcation diagram of  $f(x; r, K, E, C)$  with strong Allee effect, at  $K = 10$  and  $E = 0.02$ , in the  $(C, r)$  parameter plane. The white region is the unconditional extinction region, the blue region is the stability region of the fixed point  $A_r$ . The region between the blue and the gray ones corresponds to period doubling region and chaotic region (existence region of cycles as shown on top of figure). The gray region is the no admissible region.

**Remark 3:** There follows some fundamental properties of the fold and flip bifurcation curves, which allow a better understanding of the bifurcation diagrams of von Bertalanffy's functions, at the  $(C, r)$  parameter plane:

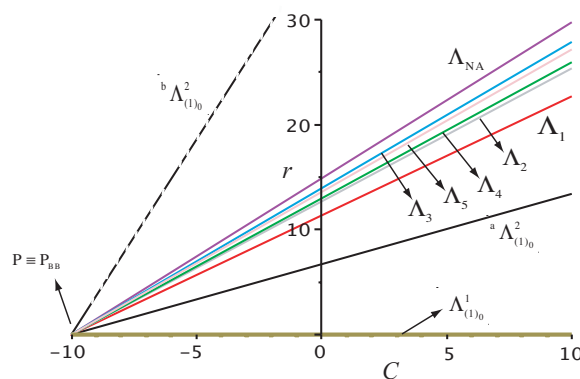
- (i) in the cases of strong and weak Allee's functions, the fold bifurcation curve  $\Lambda_{(1)_0}$  of the fixed points  $A_r$  and  $B_r$ , for  $n = 1$ , is the bifurcation curve which defines the transition between the unconditional extinction region and the stability region of the fixed point  $B_r$ ;
- (ii) in the case of functions with no Allee effect the fold bifurcation curves  ${}^a\Lambda_{(1)_0}^2$  and  ${}^b\Lambda_{(1)_0}^2$  characterize the stability of the fixed points  $O_r$  and  $B_r$ , and the instability of the fixed points 0 and  $A_r$ . Reason why for this class of functions does not exist an unconditional extinction region;
- (iii) in all cases, the flip bifurcation curve  $\Lambda_1$ , of the stable fixed point  $B_r$ , for  $n = 1$ , is the bifurcation curve which defines the transition between the stability region and the period doubling region. The upper limit of the period doubling region is defined by the accumulation value of the flip bifurcation curves of the cycle of order  $2^n$ , of the stable fixed points nonzero. This bifurcation curve is denoted by  $\Lambda_\infty$ , considering  $x \in [0, 1[$  a fixed point, we have,

$$\Lambda_\infty = \lim_{n \rightarrow \infty} \psi_{2^n}(x; K, E, C);$$

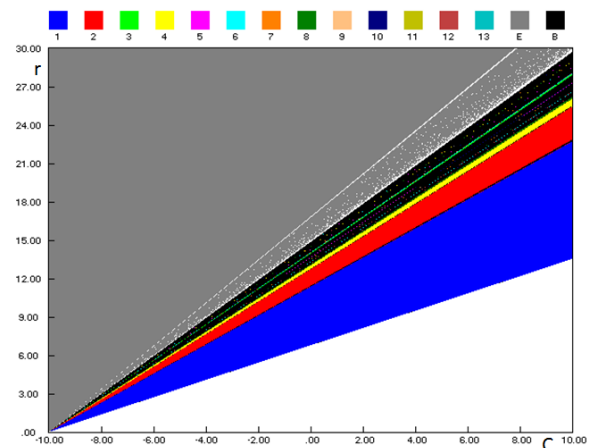
- (iv) the chaotic region is bounded below by  $\Lambda_\infty$  and is upper bounded by the chaotic semistability curve, which is then studied. In this region are observed all fold and flip bifurcation curves of the cycle of order different than  $2^n$ ,  $\Lambda_k$ , with  $k \neq 2^n$ , identified and ordered in the "box-within-a-box" bifurcation structure.



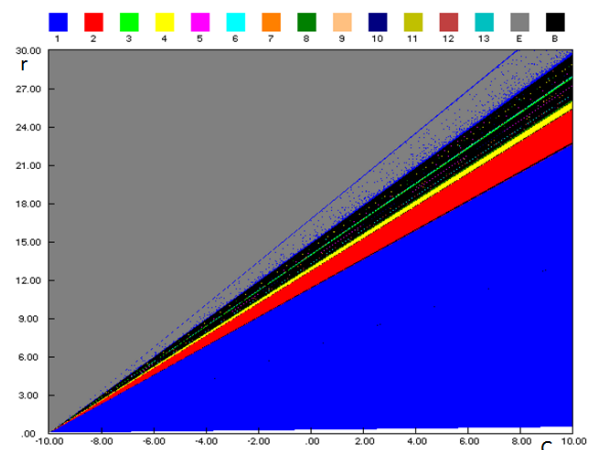
**Figure 7.** Bifurcations curves of  $f^n(x; r, K, E, C)$  with weak Allee effect, for  $n = 1, 2, 3, 4, 5$ , in the  $(C, r)$  parameter plane at  $K = 10$  and  $E = 0$ .  $\Lambda_{(1)0}$  is the fold bifurcation curve;  $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$  and  $\Lambda_5$  are the flip bifurcation curves of the cycles of order  $n = 1, 2, 3, 4, 5$ , respectively;  $\Lambda_1^*$  is the chaotic semistability curve and  $\Lambda_{NA}$  is the bifurcation curve of non admissibility, in this case  $\Lambda_1^* \neq \Lambda_{NA}$ .  $P_{BB}$  is the big bang bifurcation point and  $P_{AE}$  is the Allee bifurcation point.



**Figure 9.** Bifurcations curves of  $f^n(x; r, K, E, C)$  with no Allee effect, for  $n = 1, 2, 3, 4, 5$ , in the  $(C, r)$  parameter plane at  $K = 10$  and  $E = -0.02$ .  $\Lambda_{(1)0}^1, \Lambda_{(1)0}^2$  and  $\Lambda_{(1)0}^3$  are the fold bifurcation curves of different cycles of order  $n = 1$ ;  $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$  and  $\Lambda_5$  are the flip bifurcation curves of the cycles of order  $n = 1, 2, 3, 4, 5$ , respectively;  $\Lambda_{NA}$  is the bifurcation curve of non admissibility.  $P_{BB}$  is the big bang bifurcation point.



**Figure 8.** Bifurcation diagram of  $f(x; r, K, E, C)$  with weak Allee effect, at  $K = 10$  and  $E = 0$ , in the  $(C, r)$  parameter plane. The white region is the unconditional extinction region, the blue region is the stability region of the fixed point  $A_r$ . The region between the blue and the gray ones corresponds to period doubling region and chaotic region (existence region of cycles as shown on top of figure). The gray region is the no admissible region.



**Figure 10.** Bifurcation diagram of  $f(x; r, K, E, C)$  with no Allee effect, at  $K = 10$  and  $E = -0.02$ , in the  $(C, r)$  parameter plane. The blue region is the stability region of the fixed point  $O_r$ . The region between the blue and the gray ones corresponds to period doubling region and chaotic region (existence region of cycles as shown on top of figure). The gray region is the no admissible region.

### 3.2. Allee's effect region: chaotic semistability curve and Allee's bifurcation point

The notions of *chaotic semistability curve*, *Allee's effect region* and *Allee's bifurcation point* are fundamental for a detailed characterization of the bifurcation structures of von Bertalanffy's functions.

**Definition 3:** Let  $f(x; r, K, E, C)$  be von Bertalanffy's functions, Eq.(3), with  $r > 0$ ,  $C > -K$  and  $0 \leq E < K$ , satisfying (A1) – (A5). The chaotic semistability curve, denoted by

$\Lambda_1^*$ , is defined by

$$\Lambda_1^* = \left\{ (C, r) \in \mathbf{R}^2 : f^2(c; r, K, E, C) = A_r \right\}, \quad (6)$$

where  $A_r$  is the first positive fixed point of  $f$  and  $c$  is the critical point defined in (A3).

This curve is a bifurcation curve for von Bertalanffy's functions with strong and weak Allee effects, see Figs. 5 and 7, because it defines where the chaotic region finishes and begins another region with a different kind of dynamics: the Allee effect region. The Allee effect region is characterized by an essential extinction, i.e., a populational occurrence where the maximum size growth of the population exceeds the critical density and the populations are almost surely doomed to extinction, verifying  $f^2(c; r, K, E, C) < A_r$ .

**Definition 4:** Let  $f(x; r, K, E, C)$  be von Bertalanffy's functions, Eq.(3), with  $r > 0$ ,  $C > -K$  and  $0 \leq E < K$ , satisfying (A1) – (A5). The Allee effect region, denoted by  $R_{AE}$ , is defined by

$$R_{AE} = \left\{ (C, r) \in \mathbf{R}^2 : A_r^* < f(c; r, K, E, C) < 1 \right\},$$

where  $A_r^* = \max\{f^{-1}(A_r; r, K, E, C)\}$  and  $c$  is the critical point defined in (A3).

To von Bertalanffy's functions with strong and weak Allee effects, the Allee effect region is limited superiorly by the fullshift curve or curve of non admissibility, see Figs. 6 and 8. However, to von Bertalanffy's functions with no Allee effect this curve limit superiorly the chaotic region, see Figs.9 and 10.

**Definition 5:** Let  $f(x; r, K, E, C)$  be von Bertalanffy's functions, Eq.(3), with  $r > 0$ ,  $|E| < K$  and  $C > -K$ , satisfying (A1) – (A5) and under the conditions of Definitions 1 and 2. The non admissibility curve, denoted by  $\Lambda_{NA}$ , is defined by

$$\Lambda_{NA} = \left\{ (C, r) \in \mathbf{R}^2 : f(c; r, K, E, C) = 1 \right\}, \quad (7)$$

where  $c$  is the critical point defined in (A3).

The above definitions allow us to write the following properties.

**Property 2:** Under the conditions of Definitions 3, 4 and 5 it is established that the existence of Allee's effect region  $R_{AE}$  implies that:

- (i) if  $C > -K$  and  $0 \leq E < K$ , then  $\Lambda_1^*$  exists and  $\Lambda_1^* \neq \Lambda_{NA}$ ;
- (ii) if  $C > -K$  and  $-K < E < 0$ , then  $\Lambda_1^*$  not exists.

**Property 3:** Under the conditions of Definition 5 it is established that the existence of the non admissibility curve  $\Lambda_{NA}$  implies that:

- (i) if  $C > -K$  and  $0 \leq E < K$ , then  $\Lambda_{NA}$  is the bifurcation curve that separates the Allee effect region  $R_{AE}$  and the no admissible region;
- (ii) if  $C > -K$  and  $-K < E < 0$ , then  $\Lambda_{NA}$  is the bifurcation curve that separates the chaotic region and the no admissible region.

In the cases of strong and weak Allee's functions the bifurcation curves  $\Lambda_1^*$  and  $\Lambda_{NA}$  intersect in a bifurcation point, where it begins the Allee effect region  $R_{AE}$ .

**Definition 6:** Let  $f(x; r, K, E, C)$  be von Bertalanffy's functions, Eq.(3), with  $r > 0$ ,  $C > -K$  and  $0 \leq E < K$ , satisfying (A1) – (A5). The Allee bifurcation point, denoted by  $P_{AE}$ , is defined by

$$P_{AE} = \left\{ (C, r) \in \mathbf{R}^2 : f(c; r, K, E, C) = A_r^* \wedge f(c; r, K, E, C) = 1 \right\},$$

where  $A_r^* = \max\{f^{-1}(A_r; r, K, E, C)\}$  and  $c$  is the critical point defined in (A3).

In the  $(C, r)$  parameter plane, the Allee bifurcation points are characterized by the symmetric of the carrying capacity  $K$  and by a null intrinsic growth rate  $r$ , as follows in the next result.



**Proposition 1:** Let  $f(x; r, K, E, C)$  be von Bertalanffy's functions, Eq.(3), with  $r > 0$ ,  $C > -K$  and  $0 \leq E < K$ , satisfying (A1) – (A5). The Allee bifurcation point is given by

$$P_{AE} \equiv (C \rightarrow -K^+, r \rightarrow 0^+).$$

*Proof* Consider the intersection of the chaotic semistability curve  $\Lambda_1^*$  with the non admissibility curve  $\Lambda_{NA}$ , given by the following equations:

$$f(A_r; r, K, E, C) = A_r, \quad f^2(c; r, K, E, C) = A_r \quad \text{and} \quad f(c; r, K, E, C) = 1, \quad (8)$$

Consider conditions (A1) – (A5), we verify that the only solution of Eq.(8) is given by  $f(1; r, K, E, C) = A_r$ . From Eq.(3), we have  $f(1; r, K, E, C) = 0 \implies A_r \equiv 0$ . So, we can be in a limit case of bifurcation of the fixed point  $x = 0$ , where two fixed points of  $f$  becomes three fixed points. However, by hypothesis this case is not possible, because we are working with von Bertalanffy's functions with strong and weak Allee effects. From Definition 1 and Property 1, we will have at least one fixed point  $x = 0$ , and at most three fixed points  $0, A_r$  and  $B_r$ , depending on the evolution of the parameters. Thus, we are in the case of an only fixed point  $x = 0$ . Consequently, the implicit expression of the fixed points, given by

$$x = \left( r \left( 1 - x^{\frac{1}{3}} \right) \frac{Kx - E}{K + C} \right)^3$$

has no meaning, i.e.,  $C \rightarrow -K^+$ . Accordingly, so that Definition 1 and Property 1 make sense it is necessary that  $r \rightarrow 0^+$ . Therefore,  $P_{AE} \equiv (C \rightarrow -K^+, r \rightarrow 0^+)$  is the Allee bifurcation point at the  $(C, r)$  parameter plane.

### 3.3. Big bang bifurcation of von Bertalanffy's functions

New kinds of bifurcation points are studied to this family of functions, in the  $(C, r)$  parameter plane. These points are called *big bang bifurcation points* and are associated with particular bifurcation structures. Typically, big bang bifurcations occur in the context of piecewise-smooth discontinuous dynamics, whenever two fixed points cross simultaneously the boundary and become virtual. In [2] it was proposed that these specific bifurcations be designated as big bang bifurcations. For more details about big bang bifurcations for discontinuous piecewise maps see for example [3], [4], [6] and references therein. In [9] and [10] are presented the first studies on big bang bifurcation analysis for continuous maps.

As previously indicated in Sec.2, von Bertalanffy's functions  $f$  satisfy the properties (A1) – (A6) and Remark 2. In particular, it is also verified that  $f$  are continuous maps with respect to the parameter  $r > 0$ . However, as noted in (ii) of (A6), the Schwarzian derivative of  $f$  is not negative throughout the interval  $[0, 1]$ , for some parameter values. This disturbance alters some of the classical properties in the behavior of bifurcations, see for example [9] and [10]. Note that the occurrence (ii) of (A6) it is verified in a neighborhood of the fixed point  $x = 0$ . On the other hand, from Definition 6 and Proposition 1 the bifurcation curves, chaotic semistability curve  $\Lambda_1^*$  and non admissibility curve  $\Lambda_{NA}$ , intersect in the Allee bifurcation point  $P_{AE}$  (in a "limit" case of  $f$ , with only the fixed point  $x = 0$ ). Nevertheless, from Eqs.(4) and (5), if we consider  $n = 1$  and  $C \rightarrow -K^+$  becomes

$$x = 0, \quad \varphi_1(0; K, E, C \rightarrow -K^+) = 0 \quad \text{and} \quad \psi_1(0; K, E, C \rightarrow -K^+) = 0 \quad (9)$$

This means that the fold bifurcation curve  $\Lambda_{(1)0}$  and the flip bifurcation curve  $\Lambda_1$  also intersect in the Allee bifurcation point  $P_{AE} \equiv (C \rightarrow -K^+, r \rightarrow 0^+)$ . These special characteristics lead us to predict some changes in bifurcation structures of  $f$ , restricted to the parameter region

$R = \{(C, r) \in \mathbf{R}^2 : C \rightarrow -K^+, r > 0\}$ , with  $C > -K$  and  $|E| < K$ . Thus, the sufficient conditions required by Mira in [7] for the existence of “box-within-a-box” fractal bifurcations structure are not satisfied for von Bertalanffy’s functions, restricted to  $R$ .

**Remark 4:** An interval of existence of an attractive limit set at a finite distance can be defined for  $f$ :

- (i) if  $0 \leq E < K$  and  $C > -K$ , then this set is defined by

$$\bar{\Omega}_1 = [r_{(1)0}, r_1^*], \quad (10)$$

where  $r_{(1)0} = \varphi_1(x; K, E, C)$  and  $r_1^* = \zeta(C)$  is given by Eq.(6), see Figs.5 and 7;

- (ii) if  $-K < E < 0$  and  $C > -K$ , then this set is defined by

$$\Omega_1 = [r_{(1)0}, r_{NA}], \quad (11)$$

where  $r_{(1)0} = \varphi_1(x; K, E, C)$  and  $r_{NA} = \xi(C)$  is given by Eq.(7), see Fig.9.

The sets  $\bar{\Omega}_1$  and  $\Omega_1$  are called boxes, inside which occur all the possible bifurcations of  $f$ , for more details see for example [7].

**Property 4:** Let  $f(x; r, K, E, C)$  be von Bertalanffy’s functions, Eq.(3), with  $r > 0$  and satisfying (A1) – (A6). If  $0 \leq E < K$  and  $C > -K$ , then  $\bar{\Omega}_1 \subset \Omega_1$ .

From the above considerations, Eqs.(9), (10) and (11), the bifurcation analysis made from fold and flip bifurcation curves, Figs.5, 7 and 9, and numerical simulations of the bifurcation diagrams, Figs.6,8 and 10, allow us to formalize the following conjecture:

**Conjecture 1:** Let  $f(x; r, K, E, C)$  be von Bertalanffy’s functions, Eq.(3), with  $r > 0$ ,  $C > -K$  and  $|E| < K$ , satisfying (A1) – (A6),  $\Lambda_{(n)0}$  be the fold bifurcation curves, given by Eq.(4), and  $\Lambda_n$  be the flip bifurcation curves, given by Eq.(5), with all bifurcation curves relative to a cycle of order  $n \in \mathbf{N}$ . In the limit case, when  $C \rightarrow -K^+$ , it is verified that the fold bifurcation curves  $\Lambda_{(n)0}$ , for  $n \geq 1$ , and the flip bifurcation curves  $\Lambda_n$ , for  $n \geq 1$ , intersect on the bifurcation point  $P_{BB} = (C \rightarrow -K^+, r \rightarrow 0^+)$ , following the “box-within-a-box” fractal bifurcations structure in the box  $\Omega_1$ , Eq.(11).

In this context we expose one of the main definition of this work.

**Definition 7:** In conditions of Conjecture 1, suppose that there is a fold or a flip bifurcation curve of a cycle of order  $k \in \mathbf{N}$  that intersect the non admissibility curve  $\Lambda_{NA}$ , then the coordinates of all the cycles of order  $j > k$  converge to the coordinates of the cycle of order  $k$ . The coordinates of the cycle of order  $k$  define a big bang bifurcation point for von Bertalanffy’s functions  $f$  with the “box-within-a-box” bifurcation structure.

**Proposition 2:** If Conjecture 1 is satisfied, then the point  $P_{BB}$  is a big bang bifurcation point for  $f$ . It is verified that:

- (i) if  $0 \leq E < K$  and  $C > -K$ , then the fold bifurcation curves  $\Lambda_{(n)0}$ , for  $n \geq 1$ , the flip bifurcation curves  $\Lambda_n$ , for  $n \geq 1$  and the chaotic semi-stability curve  $\Lambda_1^*$  are issuing from  $P_{BB}$ , following the “box-within-a-box” fractal bifurcations structure in the box  $\bar{\Omega}_1$ ;
- (ii) if  $-K < E < 0$  and  $C > -K$ , then the fold bifurcation curves  $\Lambda_{(n)0}$ , for  $n \geq 1$ , the flip bifurcation curves  $\Lambda_n$ , for  $n \geq 1$  and the non admissibility curve  $\Lambda_{NA}$  are issuing from  $P_{BB}$ , following the “box-within-a-box” fractal bifurcations structure in the box  $\Omega_1$ .

*Proof* Consider the limit case of  $f(x; r, K, E, C)$ , when  $C \rightarrow -K^+$ , i.e., it is verified that  $f(x; r \rightarrow 0^+, K, E, C \rightarrow -K^+) \rightarrow 0$ , for all initial states in  $[0, 1]$ . From Eq.(9),  $x = 0$  is the only fixed point of  $f$  with  $r > 0$ ,  $C > -K$  and  $|E| < K$ . In this limit case, the dynamics cannot escape from  $[0, 1]$  (besides a hyperbolic set of zero measure). These conditions are sufficient to state the existence of the big bang bifurcation point  $P_{BB} = (C \rightarrow -K^+, r \rightarrow 0^+)$ . Note that the Allee bifurcation points  $P_{AE}$  is coincident with the big bang bifurcation point  $P_{BB}$ . Therefore, from Conjecture 1, Definitions 6 and 7, Proposition 1 and Property 4 the desired results follows.

#### 4. Conclusion

The main goal of this work has been the definition and investigation of new ones population dynamic discrete models with the growth of the population given by a family of von Bertalanffy's functions with a polynomial correction, from the point of view of stability analysis and bifurcation theory. The Allee bifurcation points and the big bang bifurcation points are associated to the asymptotic weight or carrying capacity of these models. The definitions of chaotic semistability and non admissibility bifurcation curves were crucial in the description of Allee's effect region and Allee's bifurcation points. Table 1 summarizes the results obtained from the stability analysis and the bifurcation theory approaches.

**Table 1.** Summary of the properties for von Bertalanffy's functions of polynomial type

	Strong Allee's functions	Weak Allee's functions	Functions with no Allee effect
Possible parameter vals.	$0 < E < K, C > -K$ $K > 0, r > 0$	$E = 0, C > -K$ $K > 0, r > 0$	$-K < E < 0, C > -K$ $K > 0, r > 0$
fixed points	(0) or $(0, A_r, B_r)$	(0) or $(0, A_r, B_r)$	$(0, O_r)$ or $(0, O_r, A_r, B_r)$
Stability $x = 0$	unstable, negative val.	always stable	always unstable
Unconditional extinction reg.	upper bounded by $\Lambda_{(1)0}$ of fix. pts. $A_r$ and $B_r$	upper bounded by $\Lambda_{(1)0}$ of fix. pts. $A_r$ and $B_r$	not exists this extinction reg.
$R_{AE}$	$\Lambda_1^* < R_{AE} < \Lambda_{NA}$	$\Lambda_1^* < R_{AE} < \Lambda_{NA}$	not exists $R_{AE}$
$P_{AE}$	$P_{AE} \equiv (-K^+, 0^+)$	$P_{AE} \equiv (-K^+, 0^+)$	not exists $P_{AE}$
$P_{BB}$	$P_{BB} \equiv (-K^+, 0^+)$	$P_{BB} \equiv (-K^+, 0^+)$	$P_{BB} \equiv (-K^+, 0^+)$

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