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Allee's dynamics and bifurcation structures in von Bertalanffy's population size functions

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Abstract. The interest and the relevance of the study of the population dynamics and the extinction phenomenon are our main motivation to investigate the induction of Allee Effect in von Bertalanffy's population size functions. The adjustment or correction factor of rational type introduced allows us to analyze simultaneously strong and weak Allee's functions and functions with no Allee effect, whose classification is dependent on the stability of the fixed point $x = 0$. This classification is founded on the concepts of strong and weak Allee's effects to the *population growth rates* associated. The transition from strong Allee effect to no Allee effect, passing through the weak Allee effect, is verified with the evolution of the rarefaction critical density or Allee's limit. The existence of cusp points on a fold bifurcation curve is related to this phenomenon of transition on Allee's dynamics. Moreover, the "foliated" structure of the parameter plane considered is also explained, with respect to the evolution of the Allee limit. The bifurcation analysis is based on the configurations of fold and flip bifurcation curves. The chaotic semistability and the nonadmissibility bifurcation curves are proposed to this family of 1D maps, which allow us to define and characterize the corresponding Allee effect region.

1. Introduction and motivation

One of the most familiar growth equation used to describe the growth of marine populations, namely fishes, seabirds, marine mammals, invertebrates, reptiles and sea turtles is von Bertalanffy's equation, see for example [5], [8] and references therein. This equation remains one of the most popular flexible growth equations to model fish weight growth, since it was presented by von Bertalanffy for this aim in 1938, see [29] and [30]. An usual form of von Bertalanffy's growth equation is given by,

$$g(W_t) = \frac{dW_t}{dt} = \frac{V}{3} W_t^{\frac{2}{3}} \left(1 - \left(\frac{W_t}{W_\infty} \right)^{\frac{1}{3}} \right), \tag{1}$$

where W_t is the weight at age t, W_∞ is the asymptotic weight, $V > 0$ is von Bertalanffy's growth rate constant and t_0 is the theoretical age the chick would have at weight zero.

Due to various causes, the extinction of certain species of trees, plants and mammals is one of the most current and worrying problems. In fact, species extinction is usually a major focus of ecological and biological research. The Allee effect is an important dynamic phenomenon first described by Allee in 1931, see [3]. In this work, we use the concept of demographic Allee effect, which is manifested by a reduction in the *per capita* growth rate at low population sizes. The Allee effect is described as *strong* when there is a density threshold (where the *per capita* growth rate becomes null) below which the population decrease and go to extinction. On the other hand, the Allee effect is described as *weak* when population growth is slowed down for small densities but not to the point of becoming negative. The distinction between strong and weak Allee effects is important, most authors neglect the former and almost exclusively consider the strong Allee effect.

Generically, in the context of dynamical systems or difference equations, a function representing Allee effect must have three fixed points: an asymptotically stable zero fixed point, an unsable fixed point, called the threshold point, and a bigger positive fixed point, which is asymptotically stable. For more information on Allee effect see, for example, [7], [11], [13], [17], [18], [19], [22], [23] and references therein.

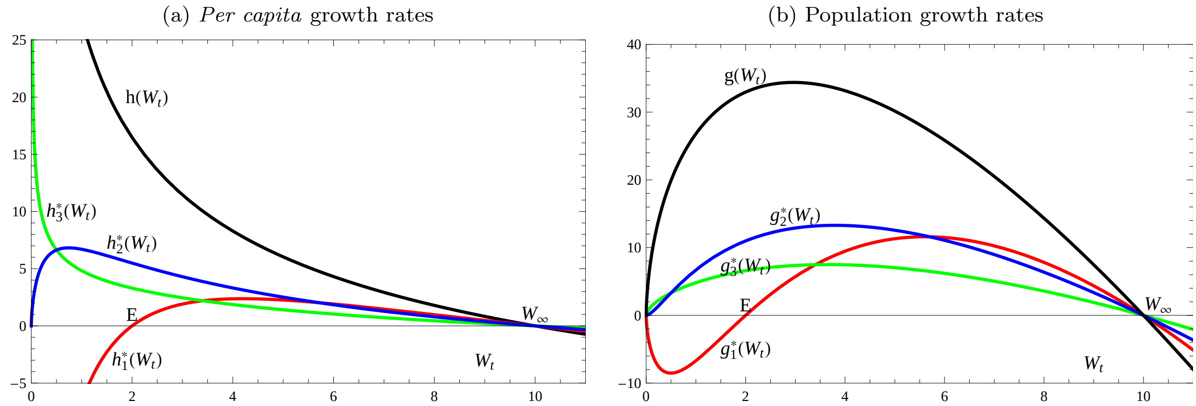

Figure 1. (a) *Per capita* growth rates defined by Eqs.(2) and (4): $h(W_t)$ at $V=150$ and $W_\infty=10$ (without correction); $h_1^*(W_t)$ at $V=150$, $W_\infty=10$, $E=2$ and $C=3$ (strong Allee effect); $h_2^*(W_t)$ at $V=75$, $W_\infty=10$, $E=0$ and $C=1$ (weak Allee effect); $h_3^*(W_t)$ at $V=45$, $W_\infty=10$, $E=-2$ and $C=4$ (no Allee effect); (b) population growth rates defined by Eqs.(1) and (3): $g(W_t)$ at $V=150$ and $W_\infty=10$ (without correction); $g_1^*(W_t)$ at $V=150$, $W_\infty=10$, $E=2$ and $C=3$ (strong Allee effect); $g_2^*(W_t)$ at $V=75$, $W_\infty=10$, $E=0$ and $C=1$ (weak Allee effect); $g_3^*(W_t)$ at $V=45$, $W_\infty=10$, $E=-2$ and $C=4$ (no Allee effect).

The von Bertalanffy growth model, Eq.(1), do not exhibit the Allee effect, because the *per capita* growth rate defined as

$$h\left(W_t\right)=\frac{g\left(W_t\right)}{W_t}=\frac{V}{3}\;W_t^{-\frac{1}{3}}\left(1-\left(\frac{W_t}{W_\infty}\right)^{\frac{1}{3}}\right), \tag{2}$$

decreases at low densities, see Fig.1(a). In this case, the *per capita* growth rates are higher than the population growth rate at the initial time, see Fig.1(b). However, this drawback was corrected using suitable corrections in the work presented by Rocha *et al.* in [18]. One of the corrected von Bertalanffy's models proposed is defined by the differential equation,

$$g^*\left(W_t\right)=\frac{dW_t}{dt}=\frac{V}{3}W_t^{\frac{2}{3}}\left(1-\left(\frac{W_t}{W_\infty}\right)^{\frac{1}{3}}\right)\frac{W_t-E}{W_t+C}, \tag{3}$$

with $|E|<W_\infty$ and $C>0$, where E is the rarefaction critical density (density threshold) or Allee's limit and W_∞ is the carrying capacity. The parameter C allows us to define and study more flexible models, with variable extinction rates, see [1], [2] and [4]. Note that the purpose of introducing Allee effect in the *population growth rates*, given by Eq.(1), is satisfied under a correction factor of rational type that defines Eq.(3). The corresponding *per capita* growth rates are defined as follows:

$$h^* (W_t) = \frac{g^* (W_t)}{W_t} = \frac{V}{3} W_t^{-\frac{1}{3}} \left(1 - \left(\frac{W_t}{W_\infty}\right)^{\frac{1}{3}}\right) \frac{W_t - E}{W_t + C}, \tag{4}$$

Fig.1 illustrates some examples of the parameter values for which there is strong Allee effect, weak Allee effect and no Allee effect corresponding to Eqs.(3) and (4). Similar approaches were made in [1], [2], [7], [17], [19], [25] and [26].

In Lotka's works on the logistic growth concept, presented in [14], the rate of population growth, $\frac{dW_t}{dt}$, at any moment t, is given by a function of the population size at that moment, W_t, namely, $\frac{dW_t}{dt} = g(W_t)$. This procedure leads to the Verhulst logistic equation, by expanding $g(W_t)$ as a Taylor series in a neighborhood of $W_t = 0$, see also [28]. On the other hand, many stochastic models of population growth can be approximated by a diffusion process (including a discrete time birth-death process), where the infinitesimal mean (defined also by $g(W_t)$) specifies the underlying deterministic trend, see, for example, [27]. The previous works are examples of different approaches in the use of the population size function $g(W_t)$ associated to the differential equation that defines the growth model under study. Similarly, considering population size functions, Rocha *et al.* presents probabilistic and dynamical studies in different logistic growth models, which can be seen in [18], [19], [20], [21], [22], [23], [24], [25] and [26].

1.1. Purpose of the paper and plan

In this paper, with the purpose of presenting a dynamical approach to generalized von Bertalanffy's models, given by Eq.(3), we consider the corresponding family of population size functions, defined as follows:

$$f(x; r, K, E, C) = r \; x^{\frac{2}{3}} \; \left(1 - x^{\frac{1}{3}}\right) \; \frac{Kx - E}{Kx + C},$$

which is studied in the next section. Note that this 4-parameter family of 1D maps is proportional to the right-hand side of the corrected von Bertalanffy's growth model, Eq.(3), with an adjustment or correction factor of rational type. For more details on the definition and interpretation of these parameters see, for example, [1], [2] and [4].

We remark that our approach to the Allee effect phenomenon is founded on the concepts of strong and weak Allee effects to the *population growth rates* (given by Eq.(3)) and not to the usual *per capita growth rates* (given by Eq.(2)). Our main in this work is not to study the difference equation or the discrete dynamical system associated to Eq.(3).

The plan of the work is as follows. In Section 2, we study von Bertalanffy's population size functions under a rational correction: new classes of Allee's functions and functions with no Allee effect. In particular, are also characterized subclasses of strong and weak Allee's functions and functions with no Allee effect, whose classification is dependent on the stability of the fixed point $x = 0$. The parameters E and C play a fundamental role in distinguishing these classes of functions. These families of unimodal and bimodal maps are proportional to the right hand side of the corrected von Bertalanffy's model, defined by Eq.(3). The dynamical behavior of these von Bertalanffy's population size functions is investigated, are also defined and characterized the unconditional extinction and essential extinction, Lemmas 1, 2 and 3. In Lemma 5, we discuss the dynamic behavior of the von Bertalanffy's population size functions with no Allee effect.

Section 3 is devoted to the study of the bifurcation structures of von Bertalanffy's population size functions, in the (C, r) parameters plane. This analysis is based on the configurations of fold and flip bifurcation curves. The concepts of chaotic semistability and nonadmissibility

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bifurcation curves to this family of maps are also considered, which allow us to define the corresponding Allee effect region. The transition from strong Allee effect to no Allee effect, passing through the weak Allee effect, is analyzed with the evolution of the Allee's limit E. To support our results, we present fold and flip bifurcations curves, numerical simulations of several bifurcation diagrams and the"foliated" structure of the (C, r) parameters plane considered. Section 4 concludes.

2. Von Bertalanffy's chaotic dynamics with strong and weak Allee effects and with no Allee effect

In the sequence of the work presented by Rocha *et al.* in [18], new one-dimensional discrete dynamical systems were established. In this section, we present a dynamical approach to von Bertalanffy's equation with the purpose of introducing Allee effect: von Bertalanffy's population size functions under a rational correction. We define new classes of von Bertalanffy's functions with different types of Allee effect, designated by strong and weak Allee's functions, which describe the growth of a population with an Allee effect and the existence of extinction. Dependent on the variation of the parameters, the von Bertalanffy's population size functions also includes another class of important functions: functions with no Allee effect. Generically, these functions are unimodal or bimodal maps for which the extinction is inevitable for too high or too low initial population densities. The transition from strong Allee effect to no Allee effect, passing through the weak Allee effect, involve the several parameters considered in the models, standing out the Allee's limit. The complex dynamical behavior of these von Bertalanffy's population size functions is investigated in detail.

2.1. Von Bertalanffy's population size functions of rational type

We consider a 4-parameter family of 1D maps, $f : [0, 1] \to]-\infty, 1]$, defined by

$$f\left(x; r, K, E, C\right) = r\ x^{\frac{2}{3}}\ \left(1 - x^{\frac{1}{3}}\right)\ \frac{Kx - E}{Kx + C}, \tag{5}$$

with $x = \frac{W_t}{W_\infty} = \frac{W_t}{K} \in [0, 1]$ the normalized weight, $r = \frac{V}{3} \times K^{\frac{2}{3}} > 0$ an intrinsic growth rate of the individual weight, the Allee's limit E and the carrying capacity K satisfying $|E| < K$ and $C > 0$. Note that in [25] and [26] the parameter C can be null or negative, that because the correction factors are of the polynomial type. See some examples of graphics of f in Figs.2, 3 and 4.

Let $A_r = f\left(A_r; r, K, E, C\right)$ be the first positive fixed point of f and $A_r^* = max\{f^{-1}\left(A_r; r, K, E, C\right)\}$. From Eq.(5), we have that $E/K < A_r$. The interval $[0, E/K]$ will be designated by extinction interval. The following properties are verified:

(A1) f is continuous on $[0, 1]$;

(A2) there exists $[A_r, A_r^*] \subset [0, 1] : f\left(x; r, K, E, C\right) \leq A_r$, $\forall x \in [0, 1] \setminus]A_r, A_r^*[$;

(A3) f has an unique critical point $c \in]A_r, A_r^*[$;

(A4) $f'(x; r, K, E, C) \neq 0$, $\forall x \in]A_r, A_r^*[\setminus \{c\}$, $f'(c; r, K, E, C) = 0$ and $f''(c; r, K, E, C) < 0$;

(A5) $f \in C^3\left(]A_r, A_r^*[\right)$;

(A6) the Schwarzian derivative of f given by

$$S\left(f(x; r, K, E, C)\right) = \frac{f'''(x; r, K, E, C)}{f'(x; r, K, E, C)} - \frac{3}{2}\left(\frac{f''(x; r, K, E, C)}{f'(x; r, K, E, C)}\right)^2$$

verifies the next conditions,

$$S\left(f(x; r, K, E, C)\right) < 0, \forall x \in]A_r, A_r^*[\setminus \{c\} \ \text{ and } \ S\left(f(c; r, K, E, C)\right) = -\infty.$$

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Properties (A1)-(A6) and more particularly the negative Schwarzian derivative ensures a "good" dynamic behavior of the models: continuity and monotonicity of topological entropy, order in the succession of bifurcations, the non-existence of wandering intervals and the existence of an upper limit to the number of stable orbits, see [15]. In the case of unimodal maps there is at most one stable orbit, and in the case of bimodal maps there are at most two stable orbits. In general, the growth models studied have negative Schwarzian derivative and the use of unimodal maps is also frequent, see for example [19], [20], [21], [22] and [28]. Depending on the parameters, to von Bertalanffy's population size functions f, we define a class of functions that describes the growth of a population with Allee effect.

Definition 1 *Consider* $0 \leq E < K$, $C > 0$, $r > 0$ *and* f *defined by Eq.(5). The von Bertalanffy's population size functions* $f : [0,1] \to]-\infty, 1]$ *attaining its maximum at* c *are Allee's functions, if there are numbers* A_r *and* B_r *such that:*

(i) $0 < A_r < c < B_r < 1$;

(ii) $\forall x \in [0,1] \setminus [A_r, B_r] \Rightarrow f(x; r, K, E, C) < x$;

(iii) $\forall x \in]A_r, B_r[\Rightarrow f(x; r, K, E, C) > x$.

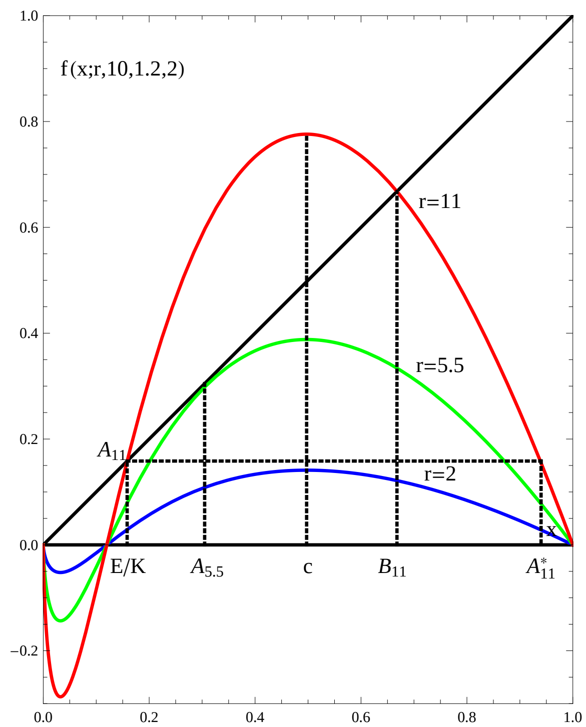

Figure 2. Von Bertalanffy's population size functions with strong Allee effect $f(x; r, K, E, C)$ at $K = 10$, $E = 1.2$, $C = 2$ and $r = 2$ (1 fixed point: 0), $r = 5.5$ (2 fixed points: 0 and $A_{5.5}$) and $r = 11$ (3 fixed points: $0, A_{11}$ and B_{11}). E/K is the Allee's threshold.

Note that,

$$\lim_{x \to 0^+} f'(x; r, K, E, C) = \begin{cases} -\infty, & \text{if } 0 < E < K \\ 0, & \text{if } E = 0 \\ +\infty, & \text{if } -K < E < 0 \end{cases}. \quad (6)$$

Consequently, this class of Allee's functions is divided into two subclasses of functions that verify the following properties. We remark that this classification is dependent on the evolution of the Allee's limit E.

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Property 1 *Under the conditions of Definition 1, it is established that:*

(i) If $0 < E < K$, the von Bertalanffy population size functions f are a family of strong Allee's functions, see Fig.2;

(ii) If $E = 0$, the von Bertalanffy population size functions f are a family of weak Allee's functions, see Fig.3.

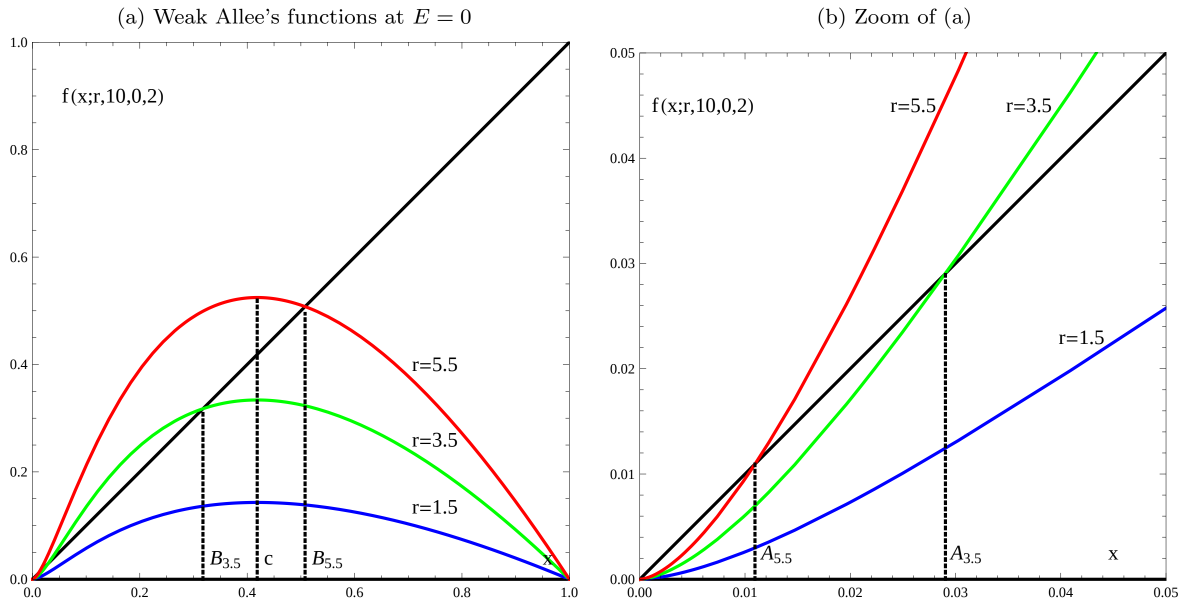

Figure 3. Von Bertalanffy's population size functions $f(x; r, K, E, C)$: (a) weak Allee's functions at $K = 10$, $E = 0$, $C = 2$ and $r = 3.5,\ 5.5$ (3 fixed points: $0, A_r$ and B_r), at the same parameters values and $r = 1.5$ (1 fixed point: 0); (b) Zoom of (a), where the fixed points $A_{3.5}$ and $A_{5.5}$ are well highlighted.

This classification is based on the concepts of strong and weak Allee effects to *population growth rates* associated, see Eq.(3). In this context, the Allee effect is described as *strong* when there is a density threshold, where the population growth rate becomes null and below which are negative. The Allee effect is described as *weak* when the populations do not exhibit any thresholds, the point $x = 0$ becomes stable and the population growth rate are positive. See also [19]. The von Bertalanffy's population size functions f also includes another class of important functions, for which does not exist Allee effect.

Definition 2 *Consider $-K < E < 0$, $C > 0$, $r > 0$ and f defined by Eq.(5). The von Bertalanffy's population size functions $f : [0, 1] \to [0, 1]$ attaining its maximum at c are functions with no Allee effect, if there are numbers O_r, A_r and B_r such that:*

(i) $0 < O_r < A_r < c < B_r < 1$;

(ii) $\forall x \in [0, 1] \setminus ([0, O_r] \cup [A_r, B_r]) \Rightarrow f(x; r, K, E, C) < x$;

(iii) $\forall x \in (]0, O_r[\cup]A_r, B_r[) \Rightarrow f(x; r, K, E, C) > x$.

The class of functions with no Allee effect verifies $\lim_{x \to 0^+} f'(x; r, K, E, C) > 1$, i.e., the fixed point $x = 0$ is always unstable. It is for this reason that this class of functions does not include extinction. Note that from Definition 1 it follows that, in the cases of strong and weak Allee's functions, we will have at maximum three fixed points $0, A_r$ and B_r, see Figs.2 and 3. Without

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loss of generality, we also consider the case where $x = 0$ is the only fixed point, see Figs.2, 3 and Lemma 1. On the other hand, from Definition 2, in the case of functions with no Allee effect, we will have at least two fixed points 0 and O_r, and at most four fixed points $0, O_r, A_r$ and B_r, depending on the evolution of the parameter r, see Figs.4 and 5. See also the definitions of Allee's functions, for example, at [13], [19], [22] and [28].

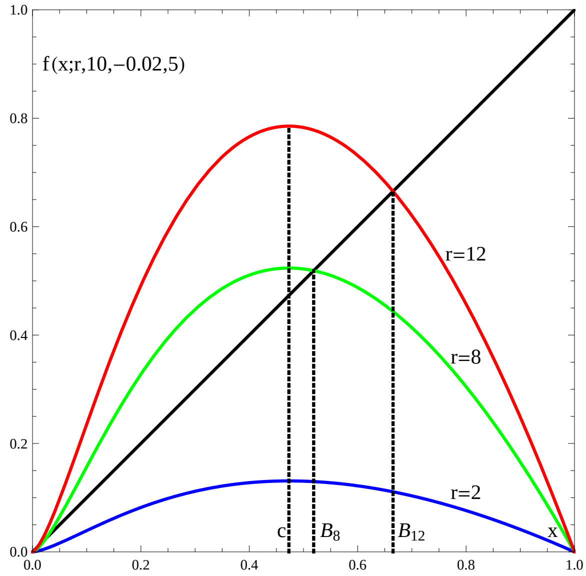

Figure 4. Von Bertalanffy's population size functions with no Allee effect $f(x; r, K, E, C)$ at $K = 10$, $E = -0.02$, $C = 5$ and $r = 2$ (2 fixed points: 0 and O_r), $r = 8$ and $r = 12$ (4 fixed points: $0, O_r, A_r$ and B_r). See Fig.5 for zoom of the fixed points O_r and A_r.

2.2. Unconditional extinction and essential extinction in von Bertalanffy's population size functions with strong and weak Allee effects

This section is devoted to the study of one of the most important phenomena in population dynamics: the extinction and the existence of Allee effect. In the next results, we study the cases where the extinction is inevitable in finite time, for all initial densities.

Lemma 1 *(Unconditional extinction) Let $f(x; r, K, E, C)$ be the von Bertalanffy population size functions, Eq.(5), with $0 \leq E < K$, $C > 0$, $r > 0$, satisfying (A1) and (A4). Consider that $f(x; r, K, E, C)$ has $x = 0$ as the unique fixed point.*

(i) If $0 < E < K$, then $\exists\ n_0 \in \mathbf{N} : n \geq n_0,\ f^n(x; r, K, E, C) < 0,\ \forall x \in \,]0, 1[\setminus \{E/K\}$;

(ii) If $E = 0$, then $\lim_{n\to\infty} f^n(x; r, K, E, C) = 0,\ \forall x \in [0, 1]$.

Proof 1 *Consider that $0 < E < K$, from Definition 1 and Property 1 (i) we have that strong Allee's functions are defined as $f : [0, 1] \to \,]-\infty, 1]$, see Fig.2 (blue graphic for $r = 2$). If $x = 0$ is the only fixed point of f and considering (A1), then $f(x; r, K, E, C) < x, \forall x \in [0, 1]$. So, we obtain $f(x; r, K, E, C) < 0,\ \forall x \in \,]0, E/K[$. However, when given $\forall x \in \,]E/K, 1[$, we have two different cases:*

(a) $\exists\ r > 0 : f(c; r, K, E, C) \leq E/K$. Considering conditions (A1) and (A4), we have that $f^2(x; r, K, E, C) < 0, \forall x \in \,]E/K, 1[$. In particular, $f^2(c; r, K, E, C) = 0$.

(b) $\exists\ r > 0 : f\,(c; r, K, E, C) > E/K$. *Consider*

$$Q_r = min\{f^{-1}\,(E/K; r, K, E, C)\} \quad and \quad Q_r^* = max\{f^{-1}\,(E/K; r, K, E, C)\},$$

such that $]Q_r, Q_r^*[\, \subset\,]E/K, 1[$. *Given conditions (A1) and (A4), we have that*

$$f^2\,(x; r, K, E, C) < 0, \forall x \in (]E/K, Q_r[\, \cup\,]Q_r^*, 1[)\,.$$

Otherwise, the sequence $f^n\,(x; r, K, E, C)$ *is decreasing and finite,* $\forall x \in\,]Q_r, Q_r^*[$*, where exist an order* $n_0 - 1$ *such that the sequence is bounded below by* E/K*. This implies that, for* $n \geq n_0$*, the iterates* $f^n\,(x; r, K, E, C) < 0,\ \forall x \in\,]E/K, 1[$*. So, the first claim is proved.*

Assume that $E = 0$*, whence from Definition 1 and Property 1* (ii) *weak Allee's functions are defined as* $f : [0, 1] \to [0, 1]$ *and* $x = 0$ *is the only fixed point of* $f\,(x; r, K, 0, C)$*, see Fig.3 (blue graphic for* $r = 1.5$*). By (A1), continuity of* f*, we have that* $f\,(x; r, K, 0, C) < x, \forall x \in\,]0, 1]$*. Considering that* $f^n\,(x; r, K, 0, C)$ *is a decreasing sequence that is bounded below by* $x = 0$*, then* $f^n\,(x; r, K, 0, C)$ *converges to some point* $a \in [0, 1]$*. By (A1), this point* a *is a fixed point. So, the claim* (ii) *is proved.* ■

Lemma 2 *(Unconditional extinction) Let* $f\,(x; r, K, E, C)$ *be the von Bertalanffy population size functions, Eq.(5), with* $0 \leq E < K$*,* $C > 0$*,* $r > 0$*, satisfying (A1)-(A4). Consider that* $f\,(x; r, K, E, C)$ *has exactly three fixed points.*

(i) If $0 < E < K$*, then*

$$f\,(x; r, K, E, C) < 0,\ \forall x \in\,]0, E/K[\quad and \quad f^2\,(x; r, K, E, C) < 0,\ \forall x \in\,]E/K, A_r[\, \cup\,]A_r^*, 1[\,;$$

(ii) If $E = 0$*, then* $\lim\limits_{n\to\infty} f^n\,(x; r, K, E, C) = 0,\ \forall x \in [0, 1] \setminus [A_r, A_r^*]\,.$

Proof 2 *Consider now that strong Allee's functions* f *has three fixed points, see Fig.2 (red graphic for* $r = 11$*). The first part of the claim* (i) *follows from Definition 1 and Property 1* (i)*. On the other hand, given (A3) and (A4), we conclude that* $E/K < A_r < c < A_r^*$*. Furthermore, by (A4) we have*

$$f'(x; r, K, E, C) > 0,\ \forall x \in\,]E/K, A_r[\quad and \quad f'(x; r, K, E, C) < 0,\ \forall x \in\,]A_r^*, 1[\,.$$

Considering $(A2)$ *it follows that,*

$$f\,(]E/K, A_{r_i}[\,; r, K, E, C) \subseteq\,]0, A_r[\quad and \quad f\,(]A_r^*, 1[\,; r, K, E, C) \subseteq\,]0, A_r[\,.$$

Therefore, we have that $f^2\,(x; r, K, E, C) < 0,\ \forall x \in\,]E/K, A_r[\, \cup\,]A_r^*, 1[$*. The second claim is proved.*

To prove (ii) *assertion, consider that weak Allee's functions* $f\,(x; r, K, 0, C)$ *has three fixed points, see Fig.3 (green graphic for* $r = 3.5$ *and red graphic for* $r = 5.5$*). Given (A3) and (A4), we have that* $A_r < c < A_r^*$*. Considering (A4) it follows that,*

$$f'(x; r, K, 0, C) > 0,\ \forall x \in\,]0, A_r[\quad and \quad f'(x; r, K, 0, C) < 0,\ \forall x \in\,]A_r^*, 1[\,.$$

Moreover, by (A2) we have that $f\,(x; r, K, 0, C) \leq x,\ \forall x \in [0, 1] \setminus [A_r, A_r^*]$*. Conclusively, since* $x = 0$ *is the only fixed point in the set* $[0, 1] \setminus [A_r, A_r^*]$*, we have that* $f^n\,(x; r, K, 0, C)$ *is a decreasing sequence that converges to* $x = 0$*,* $\forall x \in [0, 1] \setminus [A_r, A_r^*]$*. The desired result follows.* ■

See the unconditional extinction region in the (C, r) parameters plane at Fig.9. Other kind of extinction, also extremely important, is characterized by the essential extinction, which is defined when the maximum size growth of the population exceeds the critical density and the populations are almost surely doomed to extinction, see [28]. This populational occurrence is also characterized as a randomly choice of the initial densities leading to extinction with probability one. The following result characterizes this extinction in finite time, for a subset of all initial densities. We remark that, in [28] is proved a general result for similar models to Lebesgue almost every initial population densities, see also [19]. To simplify the writing of the next results, we consider c the positive critical point and A_r the first positive fixed point of f,

$$A_r^* = max\{f^{-1}(A_r; r, K, E, C)\}, P_r = min\{f^{-1}(A_r^*; r, K, E, C)\}, P_r^* = max\{f^{-1}(A_r^*; r, K, E, C)\},$$

such that $\tilde{I} =]P_r, P_r^*[\subset]A_r, A_r^*[$. We note that in [21], the point A_r is designated by Allee's point and this type of extinction is associated to Allee's effect region. In Section 3.2 we present an approach to the bifurcation structure associated to this phenomenon, in the (C, r) parameters plane, which is also designated by Allee's effect region.

Lemma 3 *(Essential extinction) Let $f(x; r, K, E, C)$ be the von Bertalanffy population size functions, Eq.(5), with $0 \leq E < K$, $C > 0$, $r > 0$, satisfying (A1)-(A4). Consider that $f(x; r, K, E, C)$ has exactly three fixed points and $f^2(c; r, K, E, C) < A_r$.*

(i) If $0 < E < K$, then $\exists\, n_0 \in \mathbf{N} : n \geq n_0$, $f^n(x; r, K, E, C) < 0$, $\forall x \in \tilde{I}$;

(ii) If $E = 0$, then $\lim_{n\to\infty} f^n(x; r, K, 0, C) = 0$, $\forall x \in \tilde{I}$.

Proof 3 *If $f(x; r, K, E, C)$ is a family of strong or weak Allee's functions, then by conditions (A1)-(A4), it follows that $A_r < P_r < c < P_r^* < A_r^*$, see Figs.4 and 5. Given (A4), we have*

$$f'(x; r, K, E, C) > 0, \ \forall x \in [A_r, c[\quad \textit{and} \quad f'(x; r, K, E, C) < 0, \ \forall x \in]c, 1].$$

Considering Definition 1 (iii), we verify that there exists $[P_r, P_r^*] \subset]A_r, B_r[$, such that,

$$f(x; r, K, E, C) > x, \ \forall x \in [P_r, P_r^*].$$

By hypothesis, we have that $f^2(c; r, K, E, C) < A_r \Leftrightarrow A_r^* < f(c; r, K, E, C) \leq 1$. Consequently, we obtain $A_r^* \leq f(x; r, K, E, C) \leq 1$, $\forall x \in [P_r, P_r^*]$. By (A4), monotonicity of f, we have $0 \leq f^2(x; r, K, E, C) \leq A_r$, $\forall x \in [P_r, P_r^*]$. If $f(x; r, K, E, C)$ is a family of strong Allee's functions, then by Lemma 2 (i) follows the desired result of (i). On the other hand, if $f(x; r, K, 0, C)$ is a family of weak Allee's functions, then by (ii) of Lemma 2 follows (ii). ■

See the essential extinction region in the (C, r) parameters plane at Fig.9. However, considering (i) and (ii) of Lemma 2, Lemma 3 and (4) of Theorem 1 proved in [28], we can state that:

Lemma 4 *Let $f(x; r, K, E, C)$ be the von Bertalanffy population size functions, Eq.(5), with $0 \leq E < K$, $C > 0$, $r > 0$, satisfying (A1)-(A4). Consider that $f(x; r, K, E, C)$ has exactly three fixed points and $f^2(c; r, K, E, C) < A_r$.*

(i) If $0 < E < K$, then $\exists\, n_0 \in \mathbf{N} : n \geq n_0$, $f^n(x; r, K, E, C) < 0$, for Lebesgue almost every $x \in]A_r, P_r[\cup]P_r^*, A_r^*[$. The basin of attraction of the extinction interval $[0, E/K]$ is given by

$$\Omega_{[0,E/K]} = [E/K, A_r[\bigcup]P_r, P_r^*[\bigcup]A_r^*, 1] \bigcup \left\{ \bigcup_{n\geq 1} f^{-n}\left(\tilde{I}; r, K, E, C\right) \right\};$$

(ii) If $E = 0$, then $\lim_{n\to\infty} f^n(x; r, K, 0, C) = 0$, for Lebesgue almost every $x \in]A_r, P_r[\cup]P_r^*, A_r^*[$. The origin's basin of attraction is given by

$$\Omega_0 = [0, A_r[\bigcup]P_r, P_r^*[\bigcup]A_r^*, 1] \bigcup \left\{ \bigcup_{n\geq 1} f^{-n}\left(\tilde{I}; r, K, 0, C\right) \right\}.$$

2.3. Chaotic dynamics of von Bertalanffy's population size functions with no Allee effect

In this section we study von Bertalanffy's population size functions with no Allee effect, which require special attention, because we no longer have three fixed points but four fixed points, at maximum, see Figs.4 and 5. Considering Definition 2 (i), it is verified that,

$$0 < O_r < A_r < c < B_r < 1, \tag{7}$$

where now O_r is the first positive fixed point, extremely near zero, see Fig.5 (b). In this case, the fixed point $x = 0$ becomes unstable. The next result characterizes the chaotic dynamics of this class of functions.

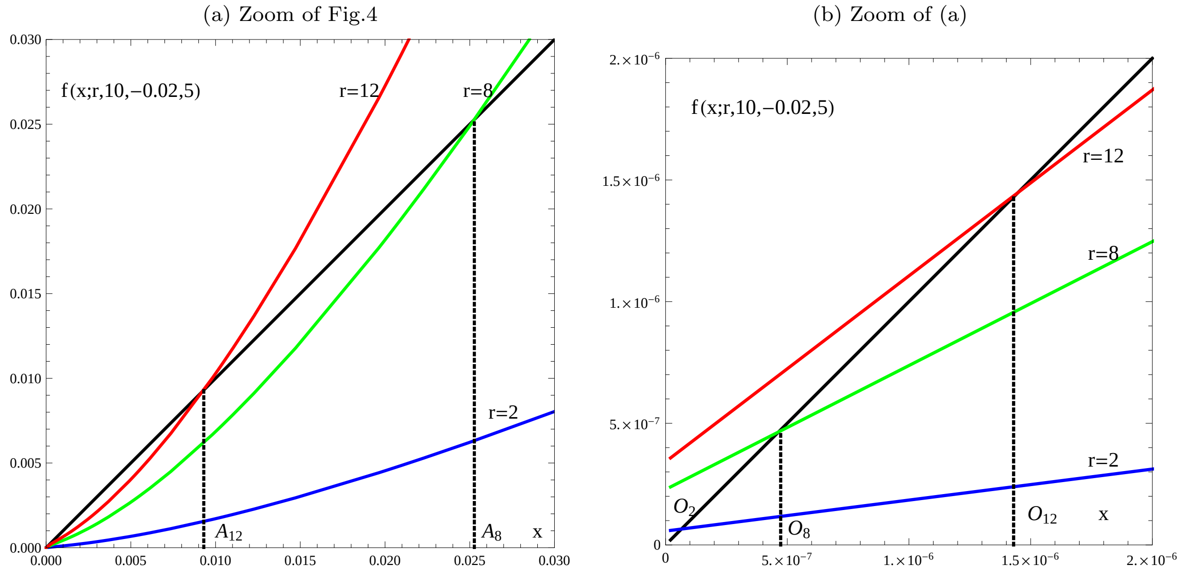

Figure 5. Von Bertalanffy's population size functions with no Allee effect $f(x; r, K, E, C)$ at $K = 10$, $E = -0.02$, $C = 5$ and $r = 2$, $r = 8$ and $r = 12$: (a) Zoom of Fig.4, where the fixed points A_8 and A_{12} are well highlighted; (b) Zoom of (a), where the fixed points O_2, O_8 and O_{12} are well highlighted.

Lemma 5 *Let $f(x; r, K, E, C)$ be the von Bertalanffy population size functions, Eq.(5), with $-K < E < 0$, $C > 0$, $r > 0$, satisfying (A1)-(A6).*

(i) If the functions $f(x; r, K, E, C)$ with no Allee effect have two fixed points 0 and O_r, then

$$\lim_{n\to\infty} f^n(x; r, K, E, C) = O_r, \ \forall x \in]0, 1[\,;$$

(ii) If the functions $f(x; r, K, E, C)$ with no Allee effect have four fixed points $0, O_r, A_r$ and B_r, then:

- $\lim_{n\to\infty} f^n(x; r, K, E, C) = O_r, \ \forall x \in]0, 1[\setminus [A_r, A_r^*]\,;$

- *there is a linearly stable fixed point* $B_r \in \,]A_r, A_r^*[$ *whose basin of attraction is* $]A_r, A_r^*[$ *or the interval* $[f^2(c; r, K, E, C), f(c; r, K, E, C)]$ *is forward invariant with basin of attraction* $]A_r, A_r^*[$.

Proof 4 *Given that* $\lim_{x\to 0^+} f'(x; r, K, E, C) > 1$, *by (A1)* f *is continuous, and that* $x = 0$ *and* $x = O_r$ *are two fixed points satisfying Eq.(7), we have*

$$f(x; r, K, E, C) > x, \ \forall x \in \,]0, O_r[\ \text{ and } f(x; r, K, E, C) < x, \ \forall x \in \,]O_r, 1].$$

Considering that $f^n(x; r, K, E, C)$ *is an increasing sequence,* $\forall x \in \,]0, O_r] \cup [O_r^*, 1[$, *with*

$$O_r^* = max\{f^{-1}(O_r; r, K, E, C)\},$$

that is bounded above by O_r, *then* $f^n(x; r, K, E, C)$ *converges to some point* $a \in]0, O_r]$. *By (A1), this point is the first positive fixed point* O_r, *because the fixed point* $x = 0$ *is unstable.*

On the other hand, we have $f(x; r, K, E, C) < x$, $\forall x \in \,]O_r, O_r^*[$. *By Eq.(7), follows that* $0 < O_r < c < 1$, *and by (A4), we have that*

$$f'(x; r, K, E, C) > 0, \ \forall x \in \,]O_r, c[\ \text{ and } f'(x; r, K, E, C) < 0, \ \forall x \in \,]c, A_r^*[.$$

So, $f^n(x; r, K, E, C)$ *is a decreasing sequence, that is bounded below by* O_r, *whence* $f^n(x; r, K, E, C)$ *converges to some point which belongs to* $[O_r, O_r^*[$. *By (A1), this point is the fixed point* O_r. *Thus, the fixed point* O_r *is stable. The claim* (i) *is proved.*

If $x \in \,]0, O_r] \cup [O_r^*, 1[$, *then by* (i) *we obtain the desired result. Moreover, if* $x \in [O_r, A_r[\cup]A_r^*, O_r^*]$, *then by Lemma 1* (iv) *and considering* $0 \equiv O_r$, *we obtain the desired outcome. So, the first equality of* (ii) *is proved. Now, if we are restricted to* $x \in [A_r, A_r^*]$, *then we have an unimodal map that satisfies (A1)-(A6) conditions,* $f^2(c; r, K, E, C) > A_r$ *and the Schwarzian derivative of* f *verifies*

$$S(f(x; r, K, E, C)) < 0, \forall x \in \,]A_r, A_r^*[\setminus \{c\} \ \text{ and } S(f(c; r, K, E, C)) = -\infty.$$

Depending on the evolution of the parameter r, *if* $|f'(B_r; r, K, E, C)| < 1$, *then there is a linearly stable fixed point* $B_r \in \,]A_r, A_r^*[$, *whose basin of attraction is* $]A_r, A_r^*[$. *Otherwise, it is verified that*

$$f\left([f^2(c; r, K, E, C), f(c; r, K, E, C)]\right) \subseteq [f^2(c; r, K, E, C), f(c; r, K, E, C)],$$

i.e., the interval $[f^2(c; r, K, E, C), f(c; r, K, E, C)]$ *is forward invariant with basin of attraction* $]A_r, A_r^*[$. *For more details of the proof of this claim see Theorem 1* (2) *at [28]. Therefore, the claim* (ii) *is proved.* ■

Note that the von Bertalanffy's population size functions of rational type have the same dynamical behavior concerning to the fixed point B_r.

Remark 1 *The fixed point* O_r *exists analytically, although extremely near zero, consider its valid existence is debatable from a practical (or physical) point of view. Indeed, it implies the non existence of Allee effect. Only the biological or ecological datas might clarify whether or not we should consider the existence of this fixed point* O_r. *For example, see [10] to observe the dimension of microbiological data. In Fig.5* (b), *the values are* $O_2 = 5.00277 \times 10^{-7}$, $O_8 = 0.0000311786...$ *and* $O_{12} = 0.000112243....$

3. Bifurcation structures of von Bertalanffy's population size functions

In this section we investigate in detail the bifurcation structures of von Bertalanffy's population size functions, in the (C, r) two-dimensional parameters space. We will make use of the classical fold and flip bifurcations. We study the behavior of such curves related with some cycles of order $n \in \mathbf{N}$. For more details about bifurcation theory see for example [9] and [16]. We introduce the notions of chaotic semistability curve and Allee's effect region. We also analyze the existence of cusp points and the "foliated" structure of the (C, r) parameters plane, dependent on the evolution of the parameter E.

3.1. Fold and flip bifurcations of von Bertalanffy's population size functions

Generically, to von Bertalanffy's population size functions $f(x; r, K, E, C)$, defined by Eq.(5), with $r > 0$, $|E| < K$ and $C > 0$, the fold and flip bifurcation curves relative to a cycle of order n are determined as follows. In the (C, r) parameters plane, if $x \in [0, 1[$ is a point of an order n cycle that satisfies the equations

$$f^n(x; r, K, E, C) = x \quad \text{and} \quad \frac{\partial f^n}{\partial x}(x; r, K, E, C) = 1 \tag{8}$$

then there exists a solution φ_n, such that the fold bifurcation curves relative to a cycle of order $n \in \mathbf{N}$ are given by $r(C) = \varphi_n(x; K, E, C)$, and are denoted by $\Lambda^j_{(n)_0}$, where j is the number of the curve, which differentiates cycles of the same order, see Figs.6, 7 and 8. On the other hand, if $x \in [0, 1[$ is such that,

$$f^n(x; r, K, E, C) = x \quad \text{and} \quad \frac{\partial f^n}{\partial x}(x; r, K, E, C) = -1 \tag{9}$$

then exists a solution ψ_n, such that the flip bifurcation curves relative to a cycle of order $n \in \mathbf{N}$ are given by $r(C) = \psi_n(x; K, E, C)$, and are denoted by Λ_n, see also Figs.6, 7 and 8.

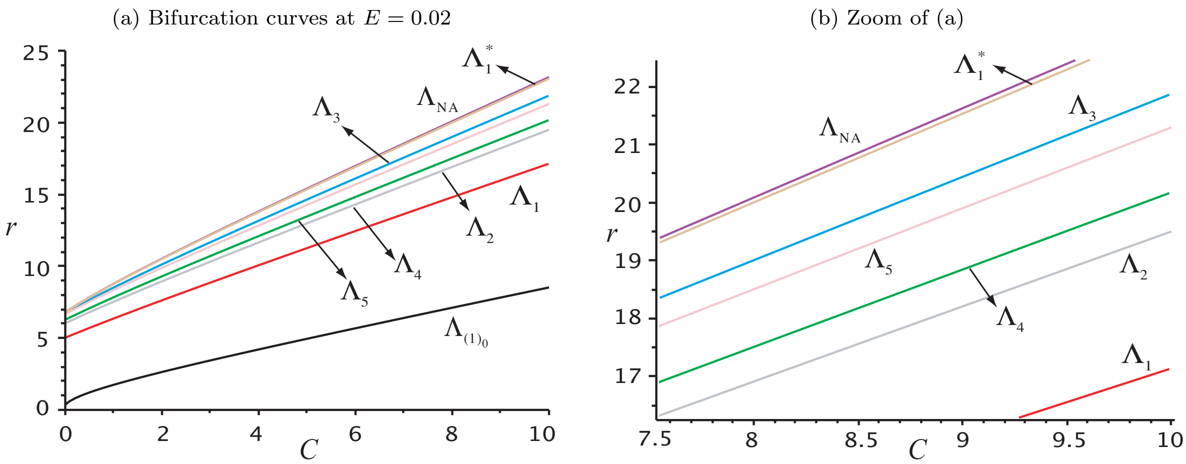

Figure 6. (a) Bifurcation curves of von Bertalanffy's population size functions with strong Allee effect $f^n(x; r, K, E, C)$, with $n = 1, 2, 3, 4, 5$, in the (C, r) parameters plane at $K = 10$ and $E = 0.02$. $\Lambda_{(1)_0}$ is the fold bifurcation curve; Λ_1, Λ_2, Λ_3, Λ_4 and Λ_5 are the flip bifurcation curves of the cycles of order $n = 1, 2, 3, 4, 5$, respectively; Λ_1^* is the chaotic semistability curve and Λ_{NA} is the bifurcation curve of nonadmissibility, in this case $\Lambda_1^* \neq \Lambda_{NA}$, according to Property 2. (b) Zoom of (a), where the region between the bifurcation curves Λ_1^* and Λ_{NA} is very well highlighted.

In particular, to von Bertalanffy's population size functions $f(x; r, K, E, C)$, defined by Eq.(5), with $r > 0$, $|E| < K$ and $C > 0$, the Eqs.(8) and (9) for $n = 1$ are given by the

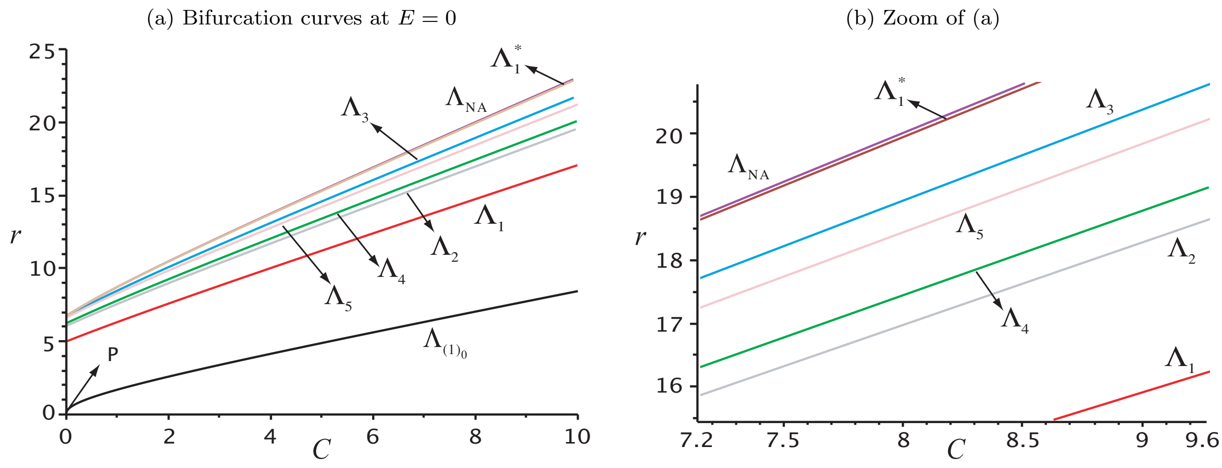

Figure 7. (a) Bifurcation curves of von Bertalanffy's population size functions with weak Allee effect $f^n(x; r, K, E, C)$, with $n = 1, 2, 3, 4, 5$, in the (C, r) parameters plane at $K = 10$ and $E = 0$. $\Lambda_{(1)_0}$ is the fold bifurcation curve; Λ_1, Λ_2, Λ_3, Λ_4 and Λ_5 are the flip bifurcation curves of the cycles of order $n = 1, 2, 3, 4, 5$, respectively; Λ_1^* is the chaotic semistability curve and Λ_{NA} is the bifurcation curve of nonadmissibility, in this case $\Lambda_1^* \neq \Lambda_{NA}$, according to Property 2. P is the cusp point, given by Eq.(15). (b) Zoom of (a), where the region between the bifurcation curves Λ_1^* and Λ_{NA} is very well highlighted.

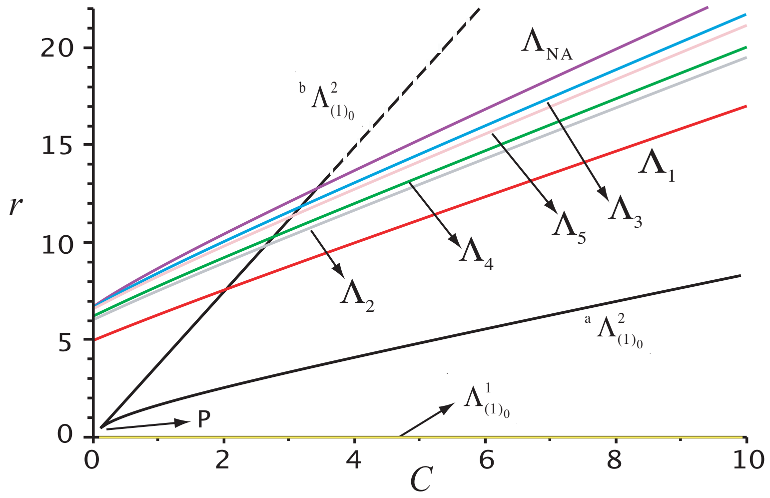

Figure 8. Bifurcation curves of von Bertalanffy's population size functions with no Allee effect $f^n(x; r, K, E, C)$, with $n = 1, 2, 3, 4, 5$, in the (C, r) parameters plane at $K = 10$ and $E = -0.02$. $\Lambda^1_{(1)_0}$, ${}^a\Lambda^2_{(1)_0}$ and ${}^b\Lambda^2_{(1)_0}$ are the fold bifurcation curves of different cycles of order $n = 1$; Λ_1, Λ_2, Λ_3, Λ_4 and Λ_5 are the flip bifurcation curves of the cycles of order $n = 1, 2, 3, 4, 5$, respectively; Λ_{NA} is the bifurcation curve of nonadmissibility, in this case not exists Λ_1^*, according to Property 2. P is the cusp point, given by Eq.(15).

following expressions,

$$\begin{cases} x = 0 \ \vee \ x = \left(r \left(1 - x^{\frac{1}{3}} \right) \ \frac{Kx - E}{Kx + C} \right)^3 \\ \varphi_1 (x; K, E, C) = \dfrac{3 \ x^{\frac{1}{3}} \ (Kx+C)^2}{3Kx \ \left(1 - x^{\frac{1}{3}}\right)(C+E) + (Kx-E)(Kx+C)\left(2 - 3x^{\frac{1}{3}}\right)} \\ \psi_1 (x; K, E, C) = -\varphi_1 (x; K, E, C) \end{cases} . \tag{10}$$

We remark that in the case where $0 < E < K$ and $C > 0$, we consider

$$f^n(x; r, K, E, C) = \begin{cases} f(x; r, K, E, C), & \text{if } x \in [0, E/K] \\ f\left(f^{n-1}(x; r, K, E, C)\right), & \text{if } x \in]E/K, 1] \end{cases}.$$

Note that in Fig.8, ${}^{a}\Lambda^{2}_{(1)_0}$ is the fold bifurcation curve of the fixed points A_r and B_r, and ${}^{b}\Lambda^{2}_{(1)_0}$ is the fold bifurcation curve of the fixed points O_r and A_r.

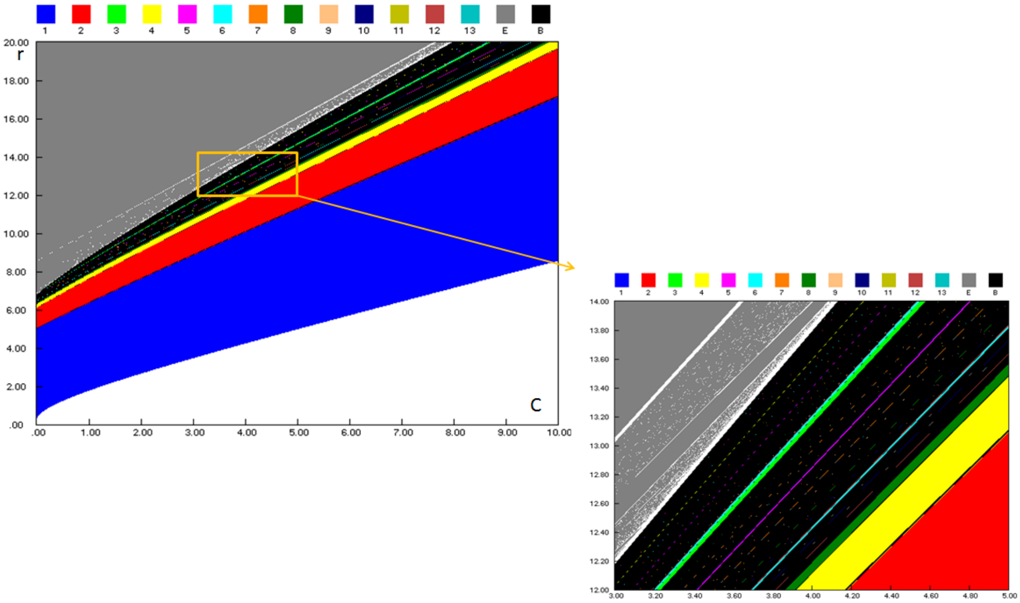

Figure 9. (on the left) Bifurcation diagram of von Bertalanffy's population size functions with strong Allee effect $f(x; r, K, E, C)$, at $K = 10$ and $E = 0.02$, in the (C, r) parameters plane. The white region is the unconditional extinction region, established in Lemmas 1 and 2. The blue region is the stability region of the fixed point A_r. The region between the blue, the gray and the white ones corresponds to period doubling region and chaotic region (existence region of cycles as shown on top of figure). The white region is the Allee effect region R_{AE} or essential extinction region, stated in Lemma 3 (i). The gray region is the no admissible region. (on the rights) Zoom of the bifurcation diagram where the Allee effect region is well highlighted.

Remark 2 *There follows a presentation of some fundamental properties of the fold and flip bifurcation curves, which allow a better understanding of the bifurcation curves and the bifurcation diagrams of von Bertalanffy's population size functions, Figs.6, 7, 8, 9 and 10, in the* (C, r) *parameters plane:*

(i) in the cases of strong and weak Allee's functions, the fold bifurcation curve $\Lambda_{(1)_0}$ *of the fixed points* A_r *and* B_r*, given by Eq.(10), is the bifurcation curve which defines the transition between the unconditional extinction region, stated in Lemmas 1 and 2, and the stability region of the fixed point* B_r*. See Figs.6, 7 and 9;*

(ii) in the case of functions with no Allee effect the fold bifurcation curves ${}^{a}\Lambda^{2}_{(1)_0}$ *and* ${}^{b}\Lambda^{2}_{(1)_0}$ *characterize the stability of the fixed points* O_r *and* B_r*, and the instability of the fixed points* 0 *and* A_r*. Reason why for this class of functions does not exist an unconditional extinction region, such as established in Lemma 5. See Figs.8 and 10;*

(iii) in all cases, the flip bifurcation curve Λ_1*, of the stable fixed point* B_r*, given by Eq.(10), is the bifurcation curve which defines the transition between the stability region and the period*

doubling region. The upper limit of the period doubling region is defined by the accumulation value of the flip bifurcation curves of the cycle of order 2^n, of the stable fixed points nonzero, see [16]. This bifurcation curve is denoted by Λ_∞, from Eq.(9) and considering $x \in [0,1[$ a fixed point, we have $\Lambda_\infty = \lim_{n\to\infty} \psi_{2^n}(x; K, E, C)$;

(iv) the chaotic region is bounded below by Λ_∞ and is upper bounded by the chaotic semistability curve Λ_1^* (in the cases of strong and weak Allee's functions) or is upper bounded by the curve of nonadmissibility Λ_{NA} (in the case of functions with no Allee effect), both of these bifurcation curves are studied in Section 3.2, see also Property 2. In this region are observed all fold and flip bifurcation curves of the cycle of order different than 2^n, Λ_k, with $k \neq 2^n$, identified and ordered in the "box-within-a-box" bifurcation structure, see [9] and [16]. See Figs.9 and 10.

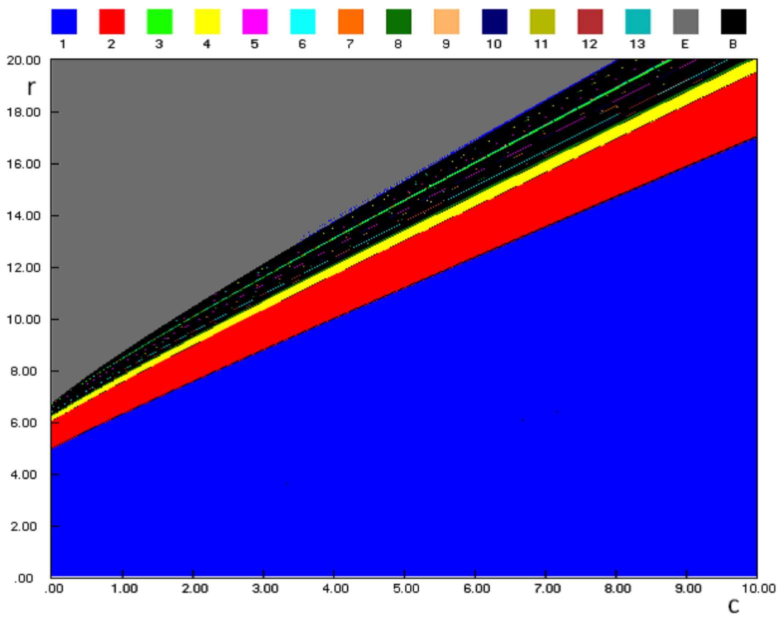

Figure 10. Bifurcation diagram of von Bertalanffy's population size functions with no Allee effect $f(x; r, K, E, C)$, at $K = 10$ and $E = -0.02$, in the (C, r) parameters plane. The blue region is the stability region of the fixed point O_r, stated in Lemma 5 (ii). The region between the blue and the gray ones corresponds to period doubling region and chaotic region (existence region of cycles as shown on top of figure), stated in Lemma 5 (ii). The gray region is the no admissible region.

3.2. Chaotic semistability and nonadmissibility bifurcation curves: Allee's effect region

However from Section 2.2, for all von Bertalanffy's population size functions with strong and weak Allee effects f, with $0 \le E < K$, $C > 0$ and initial states in $[A_r, A_r^*]$, given (A2) and (A3), it is verified that

$$c \in \,]A_r, A_r^*[\,: \; f(c; r, K, E, C) = A_r^* \Leftrightarrow \; f^2(c; r, K, E, C) = A_r, \tag{11}$$

i.e., the maximum size growth of the population is equal to the critical density. In this case, the populations can persist in a semistable chaotic interval. In the (C, r) parameters plane, the densities to which is verified the chaotic behavior previously described by Eq.(11) characterize the *chaotic semistability curve*.

Definition 3 *Let $f(x;r,K,E,C)$ be the von Bertalanffy population size functions, Eq.(5), with $0 \le E < K,\ C > 0,\ r > 0$, satisfying (A1)-(A4). The chaotic semistability curve, denoted by Λ_1^*, is defined by*

$$\Lambda_1^* = \left\{(C,r) \in \mathbf{R}^2 : f^2(c;r,K,E,C) = A_r\right\}, \tag{12}$$

where A_r is the first positive fixed point of f and c is the critical point defined in $(A3)$.

This bifurcation curve defines where the chaotic region finishes and begins another region with a different kind of dynamics, the *Allee effect region*, see Figs.6, 7, 8 and 9. The Allee effect region is characterized by an essential extinction, i.e., a populational occurrence where the maximum size growth of the population exceeds the critical density and the populations are almost surely doomed to extinction, verifying $f^2(c;r,K,E,C) < A_r$, see Lemma 3.

Definition 4 *Let $f(x;r,K,E,C)$ be the von Bertalanffy population size functions, Eq.(5), with $0 \le E < K,\ C > 0,\ r > 0$, satisfying (A1)-(A4). The Allee effect region, denoted by R_{AE}, is defined by*

$$R_{AE} = \left\{(C,r) \in \mathbf{R}^2 : A_r^* < f(c;r,K,E,C) < 1\right\},$$

where $A_r^* = max\{f^{-1}(A_r;r,K,E,C)\}$, A_r is the first positive fixed point of f and c is the critical point defined in (A3).

If f is a family of strong or weak Allee's functions, then the Allee effect region is limited superiorly by the fullshift curve or curve of nonadmissibility. Otherwise, if f is a family of functions with no Allee effect, then the chaotic region is also limited superiorly by the fullshift curve or curve of nonadmissibility, which is defined as follows.

Definition 5 *Let $f(x;r,K,E,C)$ be the von Bertalanffy population size functions, Eq.(5), with $|E| < K,\ C > 0,\ r > 0$, satisfying (A1)-(A4) and under the conditions of Definitions 1 and 2. The nonadmissibility curve, denoted by Λ_{NA}, is defined by*

$$\Lambda_{NA} = \left\{(C,r) \in \mathbf{R}^2 : f(c;r,K,E,C) = 1\right\}, \tag{13}$$

where c is the critical point defined in $(A3)$.

This bifurcation curve separates the Allee effect region or the chaotic region and the no admissible region, see Figs.6(b), 7(b), 9 and 10. In the no admissible region the graphic of any von Bertalanffy's population size function is no longer totally in the invariant set $[0,1]$. Almost all trajectories of f (besides a hyperbolic set of zero measure) leave the interval $[0,1]$ and either escape to infinity. The maps under these conditions are not good models for populations dynamics. For more details about the bifurcation structure of similar models to this one see for example [18], [19], [20], [21] and [22].

From the above definitions, we can claim the following properties:

Property 2 *Under the conditions of Definitions 3, 4 and 5, it is established that the existence of Allee's effect region R_{AE} implies that:*

(i) if $0 \le E < K$ and $C > 0$, then exists Λ_1^* and $\Lambda_1^* \neq \Lambda_{NA}$, see Figs.6, 7 and 9;

(ii) if $-K < E < 0$ and $C > 0$, then not exists Λ_1^*, see Figs.8 and 10.

Property 3 *Under the conditions of Definition 5, it is established that the existence of the nonadmissibility curve Λ_{NA} implies that:*

(i) if $0 \le E < K$ and $C > 0$, then Λ_{NA} is the bifurcation curve that separates the Allee effect region R_{AE} and the no admissible region, see Figs.6, 7 and 9;

(ii) if $-K < E < 0$ and $C > 0$, then Λ_{NA} is the bifurcation curve that separates the chaotic region and the no admissible region, see Figs.8 and 10.

In Figs.9 and 10, we present the bifurcation diagrams of von Bertalanffy's population size functions $f(x; r, K, E, C)$, with strong Allee effect ($E = 0.02$) and with no Allee effect ($E = -0.02$), at $K = 10$, in the (C, r) parameters plane. In these figures is quite clear the introduction of Allee effect in the models under study, the unconditional extinction region and the Allee effect region R_{AE} or essential extinction region are well highlighted. We remark that the bifurcation diagram of von Bertalanffy's population size functions with weak Allee effect ($E = 0$) is similar to the bifurcation diagram of von Bertalanffy's population size functions with strong Allee effect, such as the bifurcations curves presented in Figs.6 and 7.

3.3. Cusp points and the "foliated" structure of the (C, r) parameters plane, with respect to Allee's limit E

In [12] is given a necessary and sufficient condition for the existence of a cusp point on a fold bifurcation curve relative to a cycle of order n, in a parameter plane of a 1D map, see also [6]. Generically, to von Bertalanffy's population size functions $f(x; r, K, E, C)$, given by Eq.(5), with $|E| < K$, $C > 0$ and $r > 0$, that condition is given by the following expressions:

$$\begin{cases} H_1(x; r, K, E, C) \equiv f^n(x; r, K, E, C) - x = 0, \\ H_2(x; r, K, E, C) \equiv \frac{\partial f^n}{\partial x}(x; r, K, E, C) - 1 = 0, \\ D_r(x; r, K, E, C) = 0, \; D_C(x; r, K, E, C) = 0, \\ \begin{cases} D_x(x; r, K, E, C) \times \frac{\partial D_r}{\partial x}(x; r, K, E, C) \neq 0, \\ \text{or} \\ D_x(x; r, K, E, C) \times \frac{\partial D_C}{\partial x}(x; r, K, E, C) \neq 0, \end{cases} \end{cases} \tag{14}$$

where D_x, D_r and D_C are the minors of the Jacobian matrix $\frac{\partial(H_1, H_2)}{\partial(x, r, C)}$.

Note that, we use the parameters r and C to establish the minors, because we obtain explicit expressions for the existence of the cusp points on the fold bifurcation curves of the fixed point 0 and of the first positive fixed point A_r. For von Bertalanffy's population size functions $f(x; r, K, E, C)$, the necessary and sufficient condition given by Eq.(14) can be written in the following form:

$$\begin{cases} x = -\frac{2E}{K} \\ r = -\frac{2E\sqrt[3]{-\frac{2E}{K}}}{2\sqrt[3]{-\frac{2E}{K}} - 1}, \quad \forall -K < E \leq 0 \\ C = -\frac{2E\left(\sqrt[3]{-\frac{2E}{K}} - 2\right)}{2\sqrt[3]{-\frac{2E}{K}} - 1} \end{cases} . \tag{15}$$

Consequently, from Eq.(15) results the next property:

Property 4 *Let $f(x; r, K, E, C)$ be the von Bertalanffy population size functions, Eq.(5), with $|E| < K$, $C > 0$ and $r > 0$, satisfying (A1)-(A6). If $0 < E < K$, then von Bertalanffy's population size functions with strong Allee effect not admit cusp points, given by Eq.(15).*

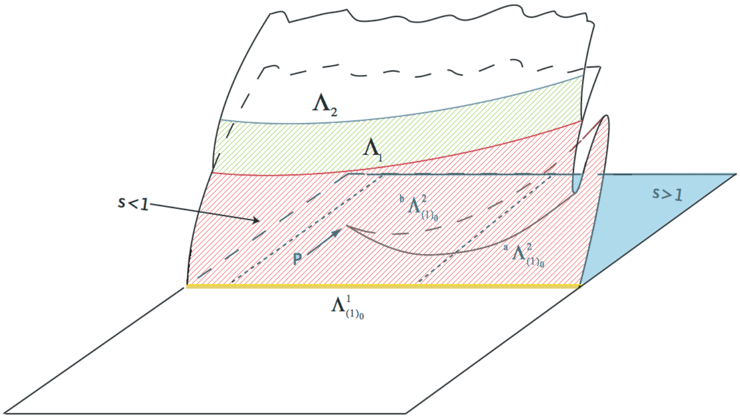

Figure 11. Foliated bifurcation structure of von Bertalanffy's population size functions $f(x; r, K, E, C)$, at $K = 10$ and $E = -0.02$. $\Lambda^1_{(1)_0}$, ${}^a\Lambda^2_{(1)_0}$ and ${}^b\Lambda^2_{(1)_0}$ are the fold bifurcation curves of different cycles of order $n = 1$; Λ_1 and Λ_2 are the flip bifurcation curves of the cycles of order $n = 1, 2$, respectively. P is the cusp point, given by Eq.(15) and $s = \frac{\partial f}{\partial x}(x; r, K, E, C)$.

Note that the existence of cusp points is dependent again of the Allee's limit E. In particular, for $E = -0.02$ and $K = 10$ we obtain $x = 0.004$, $r = 0.4651589...$ and $C = 0.1079095...$, see the cusp point P at Fig.8. In this case, for von Bertalanffy's population size functions with no Allee effect, the cusp point P corresponds to the intersection between the fold bifurcation curves ${}^a\Lambda^2_{(1)_0}$, of the fixed points A_r and B_r, and ${}^b\Lambda^2_{(1)_0}$, of the fixed points O_r and A_r, see also Fig.11. On the other hand, for $E = 0$ and $K = 10$ we obtain the fixed point $x = 0$ and $r = C = 0$, see the cusp point P at Figs.7(a) and 12. Similar phenomenon has been observed in [19] just to weak Allee effect and in [11] for a continuous time system.

In the (C, r) parameters plane, a given point is generally related to different $(n; j)$ cycles with diverse stability properties. For a better understanding of the bifurcation structures the (C, r) parameters plane must be considered as made up of sheets ("foliated" in the sense of Mira, see [16]), each one being associated with a given cycle $(n; j)$ or other singularities. A complete knowledge of the bifurcations organization implies the identification of the sheets "geometry" in a three-dimensional auxiliary qualitative space $(C, r, |z|)$. Here, the $|z|$ is a qualitative norm associated with the considered cycle. This identification amounts to seeing how to pass continuously from a sheet to another one following a continuous path in the (C, r) parameters plane, i.e., to knowing the possible communications between sheets, see for example [6].

In the simplest case, a fold bifurcation curve ($s = +1$, where $s = \frac{\partial f}{\partial x}(x; r, K, E, C)$) is the junction of two sheets, one related to an unstable (or repulsive) $(n; j)$ cycle, with $s > 1$, the other to a stable (or attractive) $(n; j)$ cycle, with $-1 < s < 1$. A flip bifurcation curve ($s = -1$) is the junction of three sheets, one associated with a $(n; j)$ cycle with $-1 < s < 1$, another with an unstable (or repulsive) $(n; j)$ cycle, with $s < -1$, the third being related to a cycle $(2n; j)$ with $-1 < s < 1$. In Figs.11 and 12, the blue-mixed domain corresponds to the existence of the fixed point $x = 0$, which is unstable in Fig.11 ($s > 1$) and is stable in Fig.12 ($s < 1$); the red-mixed domain corresponds to the existence of the stable fixed point A_r and the green-mixed domain corresponds to the existence of the stable order 2-cycles.

This analysis allows us to understand the transition from the weak Allee effect to no Allee effect with respect to Allee's limit E. Note that the parameter E is directly associated with the classification of von Bertalanffy's population size functions in functions with weak Allee effect and functions with no Allee effect, according to Property 1 and Definition 2, respectively. In [23]

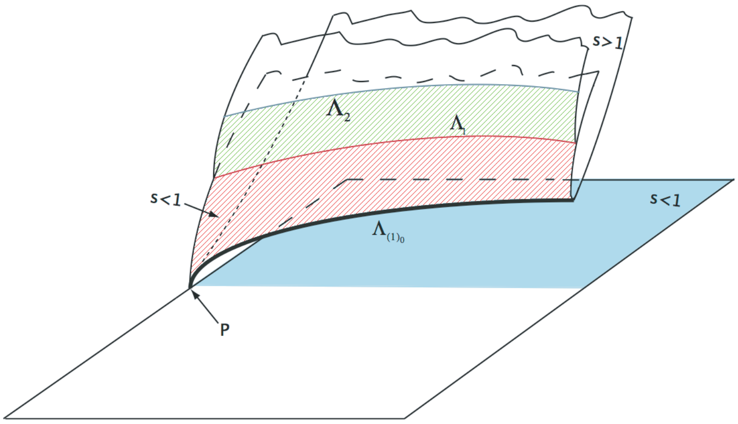

Figure 12. Foliated bifurcation structure of von Bertalanffy's population size functions $f(x; r, K, E, C)$, at $K = 10$ and $E = 0$. $\Lambda_{(1)_0}$ is the fold bifurcation curve; Λ_1 and Λ_2 are the flip bifurcation curves of the cycles of order $n = 1, 2$, respectively. P is the cusp point, given by Eq.(15) and $s = \frac{\partial f}{\partial x}(x; r, K, E, C)$.

is presented a similar analysis for Richards' growth models associated to a dovetail structure.

For the parameter values considered, we verify that when the parameter E tends toward 0, by negative values, the cusp point P tends to the cusp point $P = (0, 0)$ (evolution from Fig.11 to Fig.12). Simultaneously, the fold bifurcation curves ${}^a\Lambda^2_{(1)_0}$, of the fixed points A_r and B_r, and ${}^b\Lambda^2_{(1)_0}$, of the fixed points O_r and A_r, tend toward infinity at $E = 0$ and disappear. The bifurcation curve $\Lambda_{(1)_0}$ tends toward $(C, r) = (0, 0)$ and only remaining the cusp point $P = (0, 0)$. The cusp point $P = (0, 0)$ will make the connection between the sheet of the fixed point $x = 0$ and the sheets of the other fixed points. On the other hand, for $E > 0$ the foliated bifurcation structure is incomplete, because everything takes place in the parameters plane $C < 0$ and $r < 0$, but there is always three fixed points (0, A_r and B_r) and the foliated bifurcation structure is similar to the case $E = 0$, see Fig.12.

4. Conclusion and discussion

The distinction between strong and weak Allee effects is a very important research subject, although most authors neglect the last one and almost exclusively consider the strong Allee effect, as if it were the only one. In this sense, the main goal of this work has been the definition and investigation of new von Bertalanffy's population size functions with strong and weak Allee effects, or no Allee effect, from the point of view of stability analysis and bifurcation theory.

The introduction of strong Allee effect in von Bertalanffy's growth models was made in [18], through the ordinary differential equations defined. However, in the context of the population dynamic discrete models with the growth of the population given by a family of von Bertalanffy's population size functions, the unexpected appearance of weak Allee effect in these models is something new, obtained from the correction factor of rational type considered. In this framework, Definitions 1 and 2, and Property 1 are a significant contribution that establishes the transition from strong Allee effect to no Allee effect, involving the crossing through the weak Allee effect, for von Bertalanffy's population size functions proposed, depending on the evolution of the rarefaction critical density or Allee's limit E. Again, we highlight that we use the concepts of strong and weak Allee's effects to *population growth rates*, because we consider the population size functions associated with Eq.(3). The extinction phenomenon was analyzed in Lemmas 1, 2 and 3, providing a detailed characterization of the unconditional extinction and the essential extinction.

Another central point of our investigation was the study of bifurcation structures of von Bertalanffy's population size functions, in the (C, r) two-dimensional parameters space, dependent on the evolution of the Allee limit E. These analysis is founded on the configurations of classical fold and flip bifurcation curves. The definitions of chaotic semistability and nonadmissibility bifurcation curves were crucial in the description of Allee's effect region. In this sense, Definitions 3, 4 and 5, and Properties 2 and 3 complete the characterization and the discussion of the bifurcation structures to this family of 1D maps. Moreover, this discussion is complemented with the analysis of the existence of cusp points on a fold bifurcation curve, given by Property 4, and the corresponding "foliated" structure of the (C, r) parameters plane, with respect to Allee's limit E. A qualitative representation of the evolution of the bifurcation structure at $(C, r, |z|)$ parameters space is obtained, where the Allee limit E varies, see Figs.11 and 12. This study highlights the complex structure of von Bertalanffy's population size functions under a rational correction, as regards the number of fixed points and their dynamics.

One of the major discussions of this work is the use of the correction factor of rational type and the parameter C be positive. In [25] and [26] the parameter C can be null or negative, that because the correction factors are of the polynomial type. In the study of the bifurcation structures, in the (C, r) parameters plane, the consideration of different correction factors is evident: while in [25] and [26] are defined and characterized big bang bifurcations of the "box-within-a-box" fractal type, in the present work this special kind of bifurcation is not identified. Certainly, our work is an original contribution to the study of population dynamics and extinction phenomenon.

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