Journal of Mathematics Education
November 2017, Vol. 10, No. 2, pp. 67-82

https://doi.org/10.26711/007577152790013

'Day Number': A Promoter Routine of Flexibility and Conceptual Understanding

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This article reports part of the research developed by a Project focused on flexible calculation. In this article, we discuss different perspectives of flexibility and adaptive thinking in literature. We also discuss the idea of proceptual thinking and how this idea is important in our perspective of adaptive thinking. The article analyzes a situation named 'Day number' developed by a first grade classroom and its teacher. It is a daily activity at the beginning of the school day. It consists of looking for the date number and thinking about different ways of writing it using the four arithmetic operations. The analyzed activity occurred on March 19, so the challenge was to write the number 19 in several ways. The data show the students' enthusiasm and their efforts to find different ways of writing the number. Some used large numbers and division, which they were just starting to learn. The students presented symbolic expressions of 19, decomposing and recomposing it in a flexible manner.

Keywords: Flexibility of calculation, proceptual thinking, number symbolic representations

This article reports part of the research developed by a project focused on flexible calculation developed by teachers of the Higher Education Schools of Lisbon, Setúbal and Portalegre in Portugal. The objective of the Project is to build knowledge about the development of quantitative reasoning and calculation flexibility of students from 6 to 12 years. The Project focuses on the following critical aspects: (i) the relationship between the development of quantitative reasoning and the development of calculation flexibility; and (ii) teachers' practices oriented towards the implementation of hypothetical learning trajectories in this domain. The research questions for this article are: (1) How do students develop flexible calculation?; (2) How does the calculation flexibility and the process of concept formation relate to one another?

We present a classroom routine called "Day number" developed in a class of students in their first year of schooling (6 years old), and analyze how

calculation flexibility (Threlfall, 2009) is visible in the way students developed this activity. We also connect the flexibility in mental calculation with the process of concept formation (Sfard, 1991).

Mental calculation is assumed as a calculation made by students using global numbers, instead of their digits, through the application of properties of operations and establishing numerical relations involving the use of varied personal strategies (Buys, 2001). This type of calculation is closely associated with the development of number sense to the extent that number sense refers to the ability to flexibly use numbers and operations to develop strategies for manipulating numbers and operations (McIntosh, Reys, & Reys, 1992). Several authors emphasize the importance of daily, interactive oral sessions focused on mental calculation at the beginning of class as a way to develop this type of calculation in students (Brown, Millet, & Askew, 2008; Fosnot & Dolk, 2002).

Theoretical framework

The relevance of adaptive expertise and flexibility has been a topic under strong debate in mathematics education since the reform movement in 1980's. The ability to use knowledge flexibly by appropriately applying what is learned in one situation to another is considered fundamental to developing mathematical proficiency (NCTM, 2000).

There are several perspectives of flexibility. Star and Newton (2009) define it as knowing multiple solution pathways as well as having the capacity to choose the most appropriate for a given problem. Other authors see the flexibility as the use of efficient strategies (Heirdsfield & Cooper, 2004). Baroody and Rosu (2006) reject the strategy-choice model proposed by Thorndike (1992), relating it to a passive storage view in which fluency is best achieved through repeated practice and each basic combination is stored discretely in a factual memory network. In addition, Threlfall (2009) rejects those perspectives advocating that the strategy is not selected but emerges from a process that is not fully conscious or rational. Threfall further states that the process involves a connection between what the student notices about the specific features of the numbers presented in the proposed problem and what he knows about numbers and their relationships. This process (named as *zeroing-in*) involves the noticing and exploratory partial calculations occurring simultaneously until the solution to the calculation is reached. Baroody and Rosu (2006) refer to that flexibility in calculation as related to the discovering of patterns and relations as children develop number sense, thus building a network of relationships. Rechtsteiner-Merz and Rathgeb-Schnierer (2015) clarify that flexible mental calculation involves both flexibility (the ability to switch between different tools of solution) and adaptivity (the ability to select the most appropriate strategy). For them,

"adaptivity in flexible mental calculation is related to recognition of problem characteristics, number patterns and numerical relationships" (p. 2).

The idea of flexibility appears to be associated with mental calculation and solving arithmetic problems. There are different ways (usually referred to as strategies) of mentally solving an arithmetic problem. Strategic flexibility in mental calculation refers to how the problem is affected by the circumstances when it is solved (Threlfall, 2009). These circumstances can be related to specific characteristics of the tasks, to individual characteristics, or to contextual variables.

Baroody and Rosu (2006) stress the importance of constructing an explicit knowledge of big ideas, defined as "overarching concepts applicable to many topics and applications" (p. 6), for constructing number sense. According to them, the big idea of composition/decomposition is relevant in order for students to be able to invent reasoning strategies such us the doubles, plus one or make a ten. According to Fosnot and Dolk (2001), it is crucial for students to be able to establish relationships between the basic facts in order to facilitate memorization. These authors present several strategies that, when adopted by students, are important scaffolding in the development of these automatisms (relative to the memorization of the basic facts) and, consequently, of the mental calculation. Among these, we highlight the idea of double and almost double, developing from the calculation chains. For example, starting with a double of five (5 + 5), then 5 + 6 or 5 + 4 are almost doubles. This relationship can also be used in the reverse direction (11- 6 = 5), since 11 = 10 + 1 and 5 is half of 10, thus establishing the double/half relationship. Hartnett (2007) includes the doubles and halves strategy as a mental calculation strategy for both multiplication and division. The author also considers counting strategy as a way of thinking multiplication from the addition of equal parcels. In this way, students develop a network of relationships.

Developing the flexibility of calculation involves developing a relational and flexible understanding so that, for example, when students revert the addition in order to obtain subtraction they do not need to look at subtraction as a new process; and therefore, they compress mathematical ideas, making them simpler (Gray & Tall, 1994).

We assume, as Sfard (1991) did, that processes and mathematical objects are two sides of the same coin. Thus, the number can be conceived both structurally (as an object) and operationally (as a process). These conceptions are complementary. According to Sfard (1991), in the process of concept formation, operational conceptions precede the structural ones. Taking the example of natural numbers, counting is the operational process that leads to the structural conception as a property of a set, both historically and psychologically (Gray & Tall, 1994; Sfard, 1991). Thus, starting from the conjecture of the operational origin of mathematical objects, Sfard (1991)

proposes a schema with three hierarchical stages in concept formation: (1) *interiorization*; (2) *condensation*; and (3) *reification*.

At the interiorization stage, the student becomes skilled in performing the processes that are operations performed on lower-level mathematical objects. A process was interiorized if it could be carried out through mental representations, without the necessity of being performed to be analyzed.

In the condensation stage, the student can think of a given process as a whole, compressing the lengthy sequences of operations into more manageable units. This stage allows the student to combine this process with other processes and make comparisons and generalizations. Sfard (1991) also states that the evolution of the student in this stage leads to a growing ease in alternating between different representations of the concept. The condensation stage resonates with Gray and Tall (1994) when they indicate that students compress mathematical ideas, making them simpler. "The condensation phase lasts as long as a new entity remains tightly connected to a certain process" (Sfard, 1991, p. 19).

Passage to the third stage is sudden and coincides with the solidification of a process into an object, into a static structure. The qualitative change for reification occurs when the student is able to see something familiar in a very new light.

> Various representations of the concept become semantically unified by this abstract, purely imaginary construct. The new entity is soon detached from the process which produced it and begins to draw its meaning from the fact of its being a member of a certain category. (Sfard, 1991, p. 20)

This category justifies the existence of the new object and can be investigated both with respect to its general properties and to the various relations between its representatives. The new object A, originated from processes in concrete objects, reified as concept A, will be subject to a new evolution, according to those three stages, functioning as input to the processes, in the internalization stage, until reification of object B occurs as concept B, and so on. Therefore, "the stage of reification is the point where an interiorization of higher-level concepts (...) begins" (Sfard, 1991, p. 20).

According to Tall (2013), it is essential to consider the cognitive combination of process and concept, proposing the *procept* construct as an amalgam of three components: (1) a *process*; (2) a mathematical *object* produced by the process; and (3) a *symbol* representative of either process or object (the same notation represents the duality of process and concept). Furthermore, Tall (2013) refers to three types of abstraction: (1) *operational abstraction* focused on actions that became operations (actions on objects becoming thematised objects of thought); (2) *structural abstraction* focused on the structure of objects (properties of objects becoming thematised objects of thought); and (3) *formal abstraction* focused on definitions formulated linguistically (deducing from definitions to prove other properties to construct formal objects of thought). Operational abstraction and structural abstraction

resonate with operational and structural conceptions formulated by Sfard (1991) and “build from perceptual ideas that become conceptualized as mathematical concepts” (Tall, 2013, p. 13).

Proceptual thinking, as the combination of conceptual and procedural thinking, includes the flexible ways that the symbolism can be manipulated as process or object. In other words, a procedural action or mental object that, at a higher level, can be manipulated, decomposed or recomposed. Moreover, for Tall (2013), “the symbols themselves may be seen not only as operations to be performed but also compressed into mental number concepts that can be manipulated in the mind” (p. 4). In the *compression* of knowledge, immediate conceptual links replace lengthy operations.

Thus, proceptual thinking implies the flexibility of viewing symbolism as a simultaneous representation of the process and the object. For example, 10 + 6 is both the process of adding two numbers and the mathematical object corresponding to sum 16. In this example, the number 16, upon being reified, becomes a mental object whose manipulation, decomposition or recomposition made in a flexible way, allows students to consider the multiple representations of 16 as representations of the same object, unified in their meaning as a number.

Multiple representations for numbers constitute an important dimension of “knowledge of, and facility with, numbers” (McIntosh et al., 1992, p. 5). Cusi and Malara (2007) distinguish canonical representations of natural numbers from non-canonical representations. Taking the previous example, "16" is the canonical representation of its cardinality, and other representations, such as "10 + 6", "2 x 8", "2^4", "32/2", are non-canonical. According to the authors, although canonical representation is more popular, it is more opaque, indicating little about the number itself. On the contrary, each non-canonical representation adds information about the number deepening the knowledge of the number and facilitating the identification of numerical relationships. For example, "10 + 6" underlines the structure of 10; "2 x 8" indicates that it is a multiple of 2 and 8; "2^4" being a base power 2 also indicates that it is a multiple of 2; "32/2" indicates that it is half of 32, and therefore its divisor. Baroody and Rosu (2006) relate the non-canonical representations to the big idea of composition/decomposition when the students understand the part-whole number relations and the multiple representations for a number.

Therefore, adaptive thinking involves the development of a flexible and relational understanding enabling the students to compress mathematical ideas into more flexible, simpler forms (Tall, 2013). In this perspective, what matters is the ability to produce new known facts from old ones, acting as an autonomous knowledge generator instead of the ability to efficiently produce answers from a memory network.

In our project that focused on flexible calculation, we adopt an integrative approach to develop flexibility in mental calculation (Brocardo,

2014). We emphasize the relationship between conceptual and procedural knowledge. In this approach, we integrate the views of Tall (2013) on proceptual thinking, Sfard (1991) on concept formation, and Threlfall (2009) on zeroing-in behind calculation-strategy-flexibility.

Methodology

This study followed a qualitative approach within an interpretative paradigm. It aimed to describe and interpret an educational phenomenon (Erickson, 1986). In the article we analyze a situation developed in a first grade classroom named 'Day number' that is a daily activity at the beginning of the school day.

The 'Day number' routine was conducted orally by the teacher interacting with all students for about half an hour. The challenge was to present different symbolic representations of the number of the day. In this case, the number 19, as the date was March 19. The hundred-square, which often worked as a calculation aid, was affixed on the left side of the blackboard. The teacher recorded the different representations of 19 proposed by the students on the blackboard and asked for explanations. When a student knew a different representation, he/she raised his/her hand and waited for his/her turn.

This teacher valued the development of the mental calculation and focused on the work of the students in establishing numerical relationships, making connections between the various arithmetic operations, through an interactive dialogue with focused questioning. Although she privileged addition and subtraction, she had introduced multiplication and division in an intuitive and comprehensive way in connection with the addition and subtraction. The teacher was an expert primary teacher with an active participation in projects and in-service education courses related to mathematics teaching and learning in the first years of schooling. This class was an average class from a public school in Portugal with 25 students (6-7 years old) — 13 girls and 12 boys. The teacher practice made the difference concerning mathematics teaching.

The students’ names used in the study are fictitious for ethical reasons. Data collection was made by the authors of this article through participant observation. This activity was videotaped and later transcribed.

The data were placed into categories for analysis. The categories were constructed from the theoretical framework and were focused on the students' resolution processes: applied relationships (how they related the numbers and the operations); properties of operations; and the stages of concept formation — interiorization, condensation, and reification (Sfard, 1991).

Results

Students were organized in small groups and each pair had a necklace with 30 beads, alternating in color in groups of five, and a spring to mark the 19, but not all the pairs have done the marking. One of the students counted by fives - 5, 10, 15 - moving the beads and marking the 19 with the spring, by displacing four more (Figure 1). However, she does not verbalize any expression.

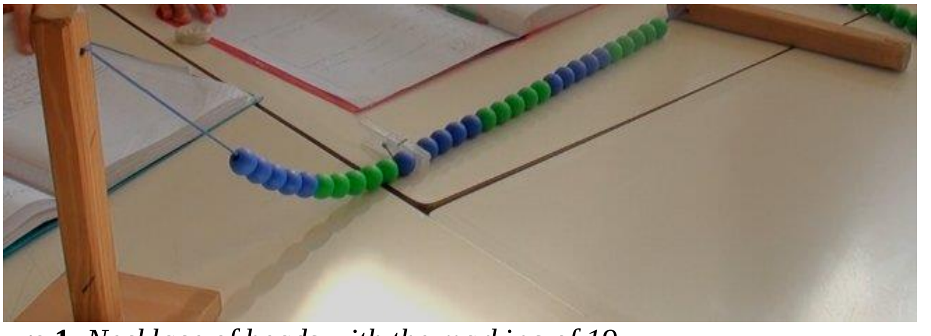

Figure 1. *Necklace of beads with the marking of 19.*

Although available, necklaces were not used as resources by most students. Only three students used them. Throughout the activity, the teacher occasionally used the necklace to materialize the explanation made by some of the students or to support their reasoning.

The teacher began the dialogue with the students and went on doing the respective records on the blackboard. Usually the teacher addressed a specific student who had his/her hand raised. Students followed the oral discussion around the various records written on the board and did not record them in their notebooks.

Teacher: 19 is next to what number, Maria?
Maria: 20.
Teacher: Before or after 20, António?
Marta: 10+10-1.
Teacher: How much is 10+10?
Students: 20.
Teacher: Renato, is 19 double or almost double?
Renato: Almost double.
Teacher: What is the double following 19?
Renato: 20.
Teacher: So we get the 10 + 10. And what is before?
Renato: 18.
Students: 9 + 9 + 1.
Teacher: Lara, 19 is even or odd?
Lara: It's odd.

The mathematical idea of double and almost double of a number, coupled with the notion of even and odd numbers, had already been worked on previously, and the teacher mobilizes this idea in her inquiry to students to express 19, relating it to neighboring numbers, 20 and 18. The questioning of teacher aims to clarify the students' thinking broadening their understanding of the situation. The 20 is verbalized first by Maria, perhaps because it is a multiple of 10. Thus, the expressions that frame 19 between 18 and 20 appear first, and these numbers were decomposed into their halves: 10 + 10-1; 9 + 9 + 1. Students continued to express their solutions:

Ilda: 5+5+5+4.
Teacher: Can you change this, Ilda?
Ilda: 3x5+4.

Ilda verbalized an expression that reveals the structure of groups of five. Incited by the teacher, she transformed the expression, already revealing a sense of multiplication associated with the addition of equal parcels.

Students continued to verbalize their solutions:
Dario: 38:2.
Teacher: Why?
Dario: Because 19+19 is 38.
Teacher: How did you get 38?
Dario: Because 18+18 is 36.
Prof[a]: Calm! ... So we went to 18 plus 18, which is something, you know, right?
Dario: Yes.
Teacher: How did you get 19+19?
Dario: It´s +2.
Teacher: Where does this 2 comes from?
Dario: From 18 to 19 plus 1; from 18 to 19 plus 1.
Teacher: Another way to do the 19 + 19, without going to 18 + 18?
Maria: We put together ten of the first 19 and then the other ten.
The teacher registers 10+9+10+9.
Teacher: And now?
Maria: 10+10 is 20. 9+9 is 18. (*The teacher registers 20+18, equating the expressions*)

Dario expressed 19 as half of 38 (38: 2). He shows, therefore, the understanding of the relationship between double and half. The teacher challenged students to use another strategy. She continued inquiring students, drawing attention to the use of the table (hundred square – Figure 2) and looking for more students to participate.

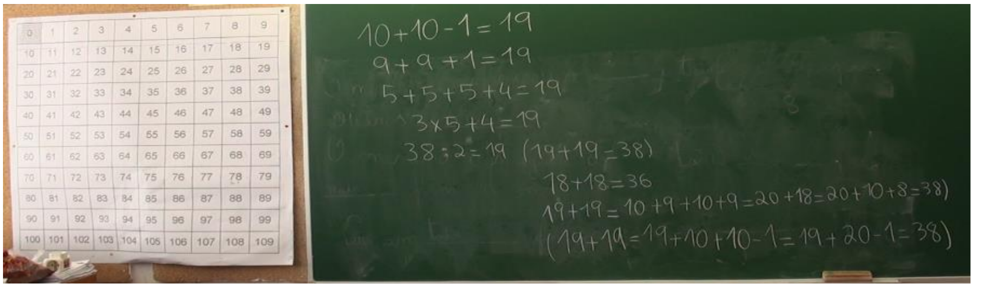

Figure 2. *The hundred square and the records of expressions representing 19.*

Teacher: Look at the table [hundred square]. How do you do quickly 19+19, Dario?
Dario: 19+10=29, coming down, 29+10=39, 39-1=38.
Dario (*looking at the table, sitting in his place*): 19+10=29, coming down, 29+10=39, 39-1=38; plus 9, because it does diagonally.
Teacher: Why is diagonally plus 9?
Maria: Because if we come down, is plus 10, diagonally, is plus 9.
Teacher: So, walking diagonally is the same as walking 10 down ...
Maria: And then backwards.
Teacher: Backwards, what? How do you say that?
Maria: Plus 10 minus 1.

Here the teacher took the opportunity to review the relationships between the numbers in the table and recorded them on the blackboard - the last notations in Figure 2. She asked the students to find new ways of calculating 19 + 19. Maria used the ten decomposition, and Dario, following the teacher's suggestion to use the table, read the table in a column and then diagonally. The teacher continues to suggest the use of the table (the hundred square) as a resource to expedite the calculation:

Teacher: Faster? 19 is close to 20. Why do not I go to 20 right on the first 19? How much is 20+20?
Students: 40.
Teacher: So how do I do this (*notes 19+19 = 20+20*)*?* How many others did I put here? (*points with one hand to the 19th and the other to the 20th*).
Students: Plus one.
Teacher: And here? (*points with one hand to the other 19 and with the other to the second 20*)
Students: Plus one.
Teacher: I put one more here and one more here (*pointing with one hand to the 19 and with the other to 20, and again with the second 19 and 20*). What do I have to do?
Students: Minus 2.
Teacher: I have to take 2. (*The teacher completes writing 19+19=20+20-2*) How much?

Students: 38.
Teacher: I agree. (...) So, coming back to 38:2, what is 38 relating to 19?
Students: It is the double.
Teacher: It is the double. (...) When I divide 38 into two equal parts, what am I doing?
Students: I'm finding the half.

The teacher suggested a new use of the table focusing the attention of the students in the 20 + 20 and compensating, then minus 2. She then asked what is the fastest way to calculate 19 + 19. She emphasized the use of double of 20. At the end, the teacher returned to the expression proposed by Dario, 38: 2, systematizing the double/half relation.

The teacher continued the dialogue by recording the student's expressions on the board. For example, 3x6+1=19 and 2x9+1=19, stated by João. Each new expression generated another, as if chained. Although they have a similar structure, these two expressions were proposed with some temporal detachment. The expression 2x9+1 was proposed immediately after a student had proposed 18+1. It appeared that João was inspired by the last expression, and looked for numbers to represent 18 as a product. A student started to do 4+4+, but hesitated; the teacher asked:

Teacher: Is 19 even or odd?
Students: Odd.
Teacher: What does happen when I count by fours? Do I get an even or odd number? Let's count.
Students: 4, 8, 12, 16, 20.
One student: So, 19 is odd, that is 4+4+4+4+3.
Teacher: Do I arrive at 19?
Students: No.
Teacher: Why?
Student: Because 19 is odd and 4 is even.

The teacher addressed something students already knew and already had taken for granted – the distinction between even and odd numbers. After more proposals coming from several students, an incident appeared:

Renato: 100:100...
Teacher: 100:100 how much?
Renato: Zero.
Teacher: If you have 100 marbles divided by 100 boys, how many marbles does each boy get?
Renato: 1.
Teacher: So how much is 100 divided by 100?
Renato: One. 100:100+18=19. (*the teacher records on the blackboard*)

Renato seems to transpose the difference between 100 and 100 to the quotient between 100 and 100. The teacher then referred to the context of the marbles already explored in previous tasks in order to give meaning to the proposed expression. Quickly, Renato rectified zero for one.

The expression proposed by Renato generated similar ones that will later be proposed by other students in the final moments of this routine: 10:10+18, 1000:1000+18, 2000:2000+18 and 4000:4000+18 (verbalized but not recorded). They used large numbers to express a generalization: 1 is the result of dividing any number by itself. It should be noted that these students were beginning to study multiplication, and the division had only appeared informally. Despite this, they could apply appropriately the two operations and start to make generalizations, giving evidence of being in the condensation stage. Moreover, with the expression of this generalization, they achieve their aim of obtaining a high number of expressions representative of the 19.

Dario: 51:3=19. (*the teacher records on the blackboard*)
Teacher: Where did you get 51?
Dario: 19+19+19 is 51.
Teacher: Why? Let's see... (*writes on the blackboard 19+19+19*). What did we see here that was easy to do?
Student: We can go to 20.
Dario and Maria: 20+20+20...
Teacher: And now, what should I do?
Dario: Minus 3.
The teacher records 19+19+19=20+20+20-3.
Teacher: How much is 20+20, Dario?
Dario: 40.
Teacher: Plus 20?
Dario: Hann...
Teacher: How much is 2+2 and +2?
Dario: Six.
Teacher: How much is 20+20? Plus 20?
Students: 60.
Teacher: Now minus 3. The 60-3?
Student: 56.
Maria: 57!

Dario demonstrated some difficulty in operating with numbers close to 60. This did not happen with smaller numbers. The teacher guided the students to read the table, from 60 to 57, and recorded 57 on the blackboard, ahead of the equal sign: 19+19+19=20+20+20-3=57.

Teacher: (*coming back 51:3*) Then, is it ok?
Students: No.

The teacher changed what was recorded on the blackboard to 57:3 (Figure 3).

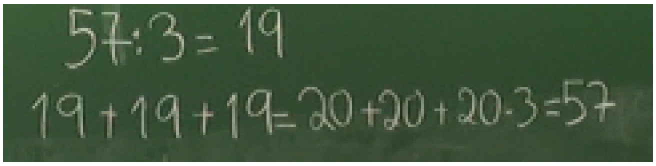

Figure 3. *19 as the third part of 57.*

She continued to elicit expressions from each of the students, taking care to give everyone the opportunity to express themselves.

João: 15+2+2=19 or 15+4=19.

Teacher: Miguel?

Miguel: 4+5+3+3+4=19. (*Miguel moves the beads accompanied by the teacher who also points to the groups of beads in the necklace*)

João decomposed 19 to 15+2+2, seeming to compress the iteration of fives into a more manageable unit, 15. He also recomposed 2+2 in 4, showing flexibility in moving the equivalence of different non-canonical expressions. Miguel, without taking advantage of the necklace structured into groups of 5 beads of alternating colors, proposed decomposition into groups of 4, 5 and 3, and had to be supported by the teacher with the necklace (Figure 4).

Figure 4. *Miguel trying to structure the 19 with the necklace and with the teacher support.*

João and Miguel appeared to be in different stages of number formation: João appeared to be in the condensation stage while Miguel gave evidence of being in the interiorization stage.

The students continued to propose new expressions until the teacher finished the routine: "We have to finish". It was only at the end of the routine that the students recorded some of the expressions representative of 19 in their daily notebooks. Each student had the discretion of selecting what he or she wanted to copy. The list of all representative expressions of 19 proposed by the students, as well as the recording of some of their underlying reasoning is shown in Figure 5.

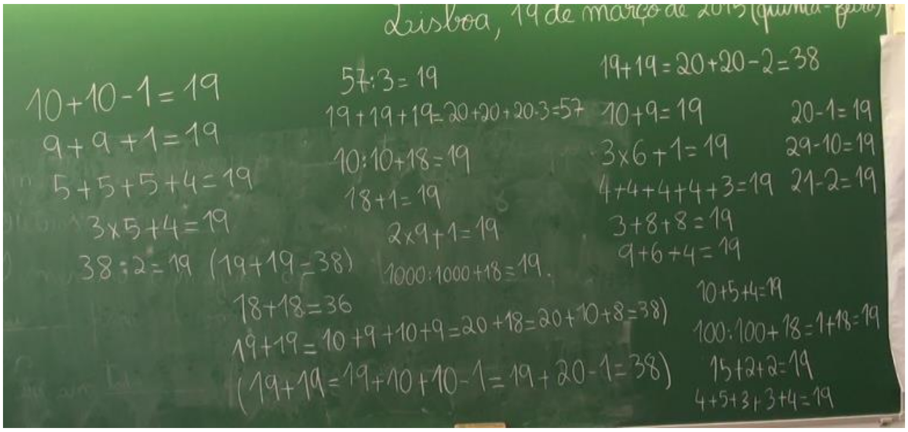

Figure 5. *Proposed non-canonical representations of 19.*

In the great diversity of representations of 19, students mobilized the four arithmetic operations, the relation of double and half, and applied generalizations; thereby, giving evidence that they had already developed important foundations for flexible mental calculation.

Conclusion

The majority of students thought about numbers as mathematical objects without connection to life contexts. Here the context of date was just a motif for the work of generating multiple symbolic expressions of the number of the day (Baroody & Rosu, 2006; Cusi & Malara, 2007). The manipulative material provided by the teacher, in order to guarantee its access by all the students, was not used by the majority of them. Thus, students who did not need to use the concrete material seem to have already passed the interiorization stage (Sfard, 1991) for the magnitude of the number in question, 19. They were able to deal with different representations of the number and reason mentally without the support of concrete material. The provided material was, however, an important resource for a few of students as a support for decomposition of the number 19. At the stage of interiorization, these students needed the manipulative material to become skillful in performing decompositions of a number.

The different expressions represented the number as an object and also represented the process of manipulating several numbers using the four arithmetic operations in order to get the given number (Gray & Tall, 1994; Sfard, 1991; Tall, 2013). The students used the operations in a related way. They represented the number using multiplication and division, which they were just starting to learn, from the additive structure. Some of the students demonstrate that they are in the condensation stage by their ability to do generalizations (as a number divided by itself is one). The procept 19 includes a collection of other representations that are obtained through different processes although representing the same object. Each symbolic expression

represents both a process (the involved operations) and a concept (the result of the operations) (Tall, 2013). The students seem to easily grasp the idea of double (Baroody & Rosu, 2006; Fosnot & Dolk, 2001), using it as a thinkable object in a flexible way to derive new relationships.

The fact that the teacher asked for new ways of making a certain calculation and encouraged the establishment of multiple numerical relations promoted the development of flexibility of thinking and calculation in students. Although they were 1st grade students, they already show great flexibility in the decomposition and recomposition of 19 (Threlfall, 2009).

This routine is an open task and may contemplate different learning rhythms, giving space to students to present different number representations, according to their level of conceptual development. The role of the teacher was fundamental to maintain the involvement and interest of all students. Furthermore, this study may be extended to larger numbers involving students from higher grades and with the same research goal focused on developing flexible calculation.

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