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Weakly spectrally complete pair of matrices
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Let $A$ and $B$ be $n \times n$ matrices over an algebraically closed field $F$. Let $c_1, \ldots, c_n$ be elements of $F$ such that $\det(AB) = c_1 \ldots c_n$ and $\# \{i \in \{1, \ldots, n\} : c_i \neq 0\} \leq \min\{\text{rank}(A), \text{rank}(B)\}$. We give necessary and sufficient condition for the existence of matrices $A'$ and $B'$ similar to $A$ and $B$, respectively, such that $A'B'$ has eigenvalues $c_1, \ldots, c_n$.

**Keywords:** eigenvalues; invariant polynomials; factorization of matrices

**AMS Subject Classifications:** 15A18; 15A23

Let $F$ be an algebraically closed field and $A, B \in F^{n \times n}$, where $n \geq 2$.

In this paper, we study the possible eigenvalues of the product $A'B'$, where $A', B' \in F^{n \times n}$ are matrices similar to $A, B$, respectively. If $c_1, \ldots, c_n \in F$ are the eigenvalues of $A'B'$ then there are two conditions that the eigenvalues must satisfy:

\[
\det(AB) = c_1 \ldots c_n, \quad (1)
\]
\[
\# \{i \in \{1, \ldots, n\} : c_i \neq 0\} \leq \min\{\text{rank}(A), \text{rank}(B)\}. \quad (2)
\]

The pair $(A, B)$ is **spectrally complete**, if for every sequence $c_1, \ldots, c_n \in F$ such that (1) is satisfied, there exist matrices $A', B' \in F^{n \times n}$ similar to $A, B$, respectively, such that $A'B'$ has eigenvalues $c_1, \ldots, c_n$.

A complete description of the spectrally complete pair of matrices was given in [1], and previously, was given in [2] for the nonsingular case. The concept of spectral completeness was introduced in [3] in order to study the possible eigenvalues of the sum of matrices.

The pair $(A, B)$ is said to be **weakly spectrally complete** if, for every sequence $c_1, \ldots, c_n \in F$ such that (1) and (2) are satisfied, there exist matrices $A', B'$ similar to $A, B$, respectively, such that $A'B'$ has eigenvalues $c_1, \ldots, c_n$.

Note that there exist $A', B'$ similar to $A, B$, respectively, such that $A'B'$ has eigenvalues $c_1, \ldots, c_n$ if and only if there exists $A''$ similar to $A$ such that $A''B$ has eigenvalues $c_1, \ldots, c_n$ if and only if there exists $B''$ similar to $B$ such that $AB''$ has eigenvalues $c_1, \ldots, c_n$.

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Given a monic polynomial \( f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \), we denote by \( C(f) \) the companion matrix of \( f \):

\[
C(f) = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
-1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-a_{n-1} & -a_{n-2} & \cdots & \cdots & 0 \\
-a_0 & -a_1 & \cdots & \cdots & 1
\end{bmatrix} \in \mathbb{F}^{n \times n}.
\]

We denote by \( i(A) \) the number of nonconstant invariant polynomials of \( A \). We make the convention that the invariant polynomials are always monic. If \( \alpha_1, \ldots, \alpha_n \) are the invariant polynomials of \( A \), then \( A \) is similar to \( C(\alpha_{n-i(A)+1}) \oplus \cdots \oplus C(\alpha_n) \).

We say that \( \lambda \in \mathbb{F} \) is a primary eigenvalue of \( A \) if \( \lambda \) is an eigenvalue of \( \alpha_{n-i(A)+1} \). Note that if \( \lambda \) is a primary eigenvalue of \( A \), then \( \text{rank}(A - \lambda I_n) = n - i(A) \).

If \( C = [c_{i,j}] \in \mathbb{F}^{n \times n} \) is a matrix such that \( c_{i,j} = 0 \) if \( j > i + 1 \), we denote by \( \chi(C) \) the number of indices \( i \in [1, \ldots, n-1] \) such that \( c_{i,i+1} \neq 0 \). We have \( i(C) \leq n - \chi(C) \).

The next theorem is our main theorem:

**Theorem 1** Let \( \alpha_1, \ldots, \alpha_n \) and \( \beta_1, \ldots, \beta_n \) be the invariant polynomials of \( A \) and \( B \), respectively. The pair \((A, B)\) is weakly spectrally complete if and only if the following are satisfied:

1. If \( i(A) + i(B) > n \) and \( \alpha_{n-i(A)+1}(x) = x - \lambda, \) with \( \lambda \in \mathbb{F} \{0\} \), then
   \[
   \beta_1(x) \cdots \beta_{i(A)}(x) = x^{i(A)+i(B)-n};
   \]
2. If \( i(A) + i(B) > n \) and \( \beta_{n-i(B)+1}(x) = x - \mu, \) with \( \mu \in \mathbb{F} \{0\} \), then
   \[
   \alpha_1(x) \cdots \alpha_{i(B)}(x) = x^{i(A)+i(B)-n};
   \]
3. At least one of the following conditions holds:
   - \( n = 2 \),
   - \( \deg(\alpha_n) \neq 2 \),
   - \( \deg(\beta_n) \neq 2 \),
   - \( i(A) \leq i(B) \) and 0 is a primary eigenvalue of \( B \),
   - \( i(B) \leq i(A) \) and 0 is a primary eigenvalue of \( A \).

**Lemma 2** If the pair \( (A, B) \) is weakly spectrally complete, then (1.1) is satisfied.

**Proof** Suppose that \( (A, B) \) is weakly spectrally complete, \( i(A) + i(B) > n \) and \( \alpha_{n-i(A)+1}(x) = x - \lambda, \) with \( \lambda \in \mathbb{F} \{0\} \). If \( A \) and \( B \) are nonsingular then for every sequence \( c_1, \ldots, c_n \in \mathbb{F} \) such that \( \det(AB) = c_1 \cdots c_n \), there exist matrices \( A', B' \in \mathbb{F}^{n \times n} \) similar to \( A, B \), respectively, such that \( A'B' \) has eigenvalues \( c_1, \ldots, c_n \) and then the pair \( (A, B) \) is spectrally complete. By Theorem 1 of [2], we have \( i(A) + i(B) \leq n \), which is impossible. Then one of the matrices \( A, B \) is singular and there exists a matrix \( B' \in \mathbb{F}^{n \times n} \) similar to \( B \) such that \( AB' \) has all its eigenvalues equal to 0. Let \( \gamma_1(x), \ldots, \gamma_n(x) \) be the invariant polynomials of \( AB' \). Then

\[
\gamma_1(x) \cdots \gamma_n(x) = x^n.
\]
We have \( AB' = \lambda B' + (A - \lambda I_n)B' \). If \( \beta_1(x)|\ldots|\beta_n(x) \) are the invariant polynomials of \( B \) then \( \beta_1(\lambda^{-1}x)|\ldots|\beta_n(\lambda^{-1}x) \) are the invariant polynomials of \( \lambda B' \). As \( \lambda \) is a primary eigenvalue of \( A \), we have \( \text{rank}(A - \lambda I_n)B' \leq n - i(A) \), and by [4, Theorem 2], we conclude that
\[
\beta_j(\lambda^{-1}x)|\gamma_j+n-i(A)(x), \quad j \in \{1, \ldots, i(A)\}. \tag{4}
\]
Using (3) and (4), the invariant polynomials \( \beta_{n-i(B)+1}(x), \ldots, \beta_{i(A)}(x) \) must be powers of \( x \) and \( \text{rank}(B) = n - i(B) < i(A) \), \leq \text{rank}(A).

Let \( c_1 = \cdots = c_{n-i(B)} = 1 \) and \( c_{i(B)} = \cdots = c_n = 0 \). There exists a matrix \( B'' \in F^{n \times n} \) similar to \( B \) such that \( AB'' \) has eigenvalues \( c_1, \ldots, c_n \). Let \( \delta_1(x)|\ldots|\delta_n(x) \) be the invariant polynomials of \( AB'' \). As in the previous argument, we have
\[
\beta_j(\lambda^{-1}x)|\delta_{j+n-i(A)}(x), \quad j \in \{1, \ldots, i(A)\}.
\]

Note that
\[
\delta_1(x) \ldots \delta_n(x) = x^{i(B)}(x - 1)^{n-i(B)} \tag{5}
\]
and \( \text{rank}(AB'') \leq \text{rank}(B'') = n - i(B) \), so \( \delta_{n-i(B)+1}(0) = \cdots = \delta_n(0) = 0 \). Then
\[
\delta_k(x) = x(x - 1)^{l_k}, \quad k \in \{n - i(B) + 1, \ldots, n\},
\]
for some \( l_k \in \mathbb{N}_0 \). Therefore,
\[
\beta_{n-i(B)+1}(x) = \cdots = \beta_{i(A)}(x) = x
\]
and
\[
\beta_1(x) \ldots \beta_{i(A)}(x) = x^{i(A)+i(B)-n}.
\]

\[\text{Lemma 3}\]

If the pair \((A, B)\) is weakly spectrally complete then (1.3) is satisfied.

**Proof** Suppose that the pair \((A, B)\) is weakly spectrally complete and \( n \neq 2 \), \( \deg(\alpha_n) = \deg(\beta_n) = 2 \). Then \( A \) and \( B \) are similar to matrices of the form
\[
A' = \begin{bmatrix} \lambda I_{i(A)} & * \\ 0 & \nu I_{n-i(A)} \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} \mu I_{i(B)} & * \\ 0 & \epsilon I_{n-i(B)} \end{bmatrix},
\]
respectively, where \( \lambda, \nu \) are the roots of \( \alpha_n \) and \( \mu, \epsilon \) are the roots of \( \beta_n \).

Suppose that \( i(A) \leq i(B) \) as the complementary case is analogous. We shall say that a sequence \( c_1, \ldots, c_n \) of elements of \( F \) are admissible if there exist matrices \( A', B' \) similar to \( A, B \), respectively, such that \( A'B' \) has eigenvalues \( c_1, \ldots, c_n \).

Let \( c_1, \ldots, c_n \in F \) be any admissible sequence. Using the arguments presented in the proof of Theorem 1 of [2], we deduce that there exists a permutation \( \pi \) of \( \{1, \ldots, n\} \) such that
\[
c_{\pi(2i-1)}c_{\pi(2i)} = \lambda\nu\mu\epsilon, \quad 1 \leq i \leq n - i(B) \tag{6}
\]
\[
c_{\pi(j)} = \lambda\mu, \quad 2(n - i(B)) < j \leq n + i(A) - i(B) \tag{7}
\]
\[
c_{\pi(j)} = \nu\mu, \quad n + i(A) - i(B) < j \leq n. \tag{8}
\]

If \( A \) and \( B \) are nonsingular, we can find a sequence \( c_1, \ldots, c_n \in F \) such that \( \det(AB) = c_1 \ldots c_n \) but the equalities (6)–(8) are not satisfied.
Suppose that at least one of the matrices \( A, B \) is singular. As the pair \((A, B)\) is weakly spectrally complete, the sequence of \( n \) zeros is admissible and should satisfy the equalities (6)–(8). Then \( \lambda = \nu = 0 \) or \( \mu = 0 \). If \( \lambda = \nu = 0 \), then the sequence of \( n \) zeros is the only admissible sequence, which contradicts the assumption that the pair \((A, B)\) is weakly spectrally complete, \( A \neq 0 \) and \( B \neq 0 \). Therefore, \( \mu = 0 \) and 0 is a primary eigenvalue of \( B \).

Using the definition of weakly spectrally complete pair, Lemma 11 of [5] can be stated as follows:

**Lemma 4** If one of the matrices \( A, B \) is singular and the other is nonderogatory, then the pair \( (A, B) \) is weakly spectrally complete.

**Lemma 5** [1, Lemma 4] If \( \min(\text{rank}(A), \text{rank}(B)) \geq n - 1 \), one of the matrices \( A, B \) is nonderogatory and the other is nonscalar, then the pair \( (A, B) \) is spectrally complete.

According to the two previous Lemmas, we have:

**Lemma 6** If one of the matrices \( A, B \) is nonderogatory and the other is nonscalar, then the pair \( (A, B) \) is weakly spectrally complete.

**Lemma 7** If \( i(A) + i(B) \leq n \) and, either \( n = 2 \) or at least one of the polynomials \( \alpha_n, \beta_n \) has degree different from 2, then \( (A, B) \) is weakly spectrally complete.

**Proof** This proof is by induction on \( n \). If \( \min(\text{rank}(A), \text{rank}(B)) \geq n - 1 \), then, according to [1, Theorem 1], the pair \( (A, B) \) is spectrally complete and then is weakly spectrally complete.

Suppose that \( \min(\text{rank}(A), \text{rank}(B)) < n - 1 \). Suppose, without loss of generality [2, Lemma 1], that \( \text{rank}(A) \leq \text{rank}(B) \). If \( B \) is nonderogatory the result follows from Lemma 4. In particular, Lemma 4 covers the case \( n \leq 3 \).

Suppose that \( n \geq 4 \) and \( B \) is derogatory. Let \( c_1, \ldots, c_n \) be elements of \( F \) such that \( \det(AB) = c_1 \ldots c_n \) and \# \{ \( i \in \{1, \ldots, n\} : c_i \neq 0 \} \leq \min(\text{rank}(A), \text{rank}(B)) \) in order to prove that there exist matrices \( A', B' \in F^{n \times n} \) similar to \( A, B \), respectively, such that \( A'B' \) has eigenvalues \( c_1, \ldots, c_n \). Suppose, without loss of generality, that \( c_{n-1} = c_n = 0 \). If there exists \( i \in \{1, \ldots, n - 2\} : c_i \neq 0 \), suppose, without loss of generality, that \( c_1 \neq 0 \).

**Case 1.** Suppose that \( c_1 \neq 0 \). The matrix \( A \) is similar to the direct sum of the companion matrices of its nonconstant invariant polynomials \( K = C(\alpha_n) \oplus \cdots \oplus C(\alpha_{n-i(A)+1}) \).

- If \( \deg(\alpha_n) \geq 3 \), then, according to [1, Lemma 5], \( K \) is similar to a matrix of the form

\[
K' = \begin{bmatrix}
* & * & 1 \\
* & K_0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

where \( K_0 \in F^{(n-2) \times (n-2)} \) is a direct sum of companion matrices, \( \chi(K_0) = \chi(K) - 1 \) and \( \det(K_0) = 0 \). Moreover, if \( i(A) \leq n - 3 \) (i.e. \( \chi(K) \geq 3 \)), then \( K_0 \) has been
chosen so that at least one of the companion matrices appearing in $K_0$ is of size $u \times u$, with $u \geq 3$ and then the minimum polynomial of $K_0$ has degree greater than 2;

- If $\deg(\alpha_n) = 2$, then $C(\alpha_n)$ is similar to a matrix of the form

$$\begin{bmatrix} * & 1 \\ 0 & 0 \end{bmatrix}$$

and $K$ is similar to a matrix of the form

$$K' = \begin{bmatrix} * & 0 & 1 \\ 0 & K_0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $K_0 = C(\alpha_{n-i(A)+1}) \oplus \cdots \oplus C(\alpha_{n-1})$. Note that $\det K_0 = 0$ and $\chi(K_0) = \chi(K) - 1$.

Analogously, the matrix $B$ is similar to the direct sum of the companion matrices of its nonconstant invariant polynomials $L = C(\beta_n) \oplus \cdots \oplus C(\beta_{n-i(B)+1})$.

- If $\deg(\beta_n) \geq 3$, then, according to a variant of [1, Lemma 5] or a variant of [2, Lemma 4], $L$ is similar to a matrix of the form

$$L' = \begin{bmatrix} 0 & 0 & * \\ 0 & L_0 & * \\ c_1 & * & * \end{bmatrix},$$

where $L_0 \in F^{(n-2)\times(n-2)}$ is a direct sum of companion matrices, $\det(K_0) = \det(L_2 \oplus \cdots \oplus L_r)$, and $\chi(L_0) = \chi(L) - 1$. Moreover, if $i(B) \leq n - 3$ (i.e. $\chi(L) \geq 3$), then $L_0$ has been chosen so that at least one of the companion matrices appearing in $L_0$ is of size $u \times u$, with $u \geq 3$ and then the minimum polynomial of $L_0$ has degree greater than 2;

- If $\deg(\beta_n) = 2$, then $C(\beta_n)$ is similar to a matrix of the form

$$\begin{bmatrix} 0 & * \\ c_1 & * \end{bmatrix}$$

and $L$ is similar to a matrix of the form

$$L' = \begin{bmatrix} 0 & 0 & * \\ 0 & L_0 & 0 \\ c_1 & 0 & * \end{bmatrix},$$

where $L_0 = C(\beta_{n-i(B)-1}) \oplus \cdots \oplus C(\beta_{n-1})$. Note that $\chi(L_0) = \chi(L) - 1$.

We have $\det(K_0L_0) = 0 = c_2 \cdots c_{n-1}, \# \{i \in \{2, \ldots, n-1\} : c_i \neq 0\} \leq \min\{\rank(A), \rank(B)\} - 1 = \min\{\rank(K_0), \rank(L_0)\}$ and $i(K_0) + i(L_0) \leq (n - 2 - \chi(K_0)) + (n - 2 - \chi(L_0)) = 2n - \chi(K) - \chi(L) - 2 = i(A) + i(B) - 2 \leq n - 2$. Now, we shall prove that either $n = 4$ or at least one of the minimum polynomial of the matrices $K_0, L_0$ has degree greater than 2.
We have $\alpha_n \geq 3$ and $i(A) \leq n - 3$, then the minimum polynomial of the matrix $K_0$ has degree greater than 2;

- If deg$(\alpha_n) \geq 3$ and $i(B) \leq n - 3$, then the minimum polynomial of the matrix $L_0$ has degree greater than 2;

- If deg$(\alpha_n) = 2$ and $i(B) > n - 3$, then $(n/2) + (n - 2) \leq i(A) + i(B) \leq n$ and therefore $n = 4$;

- If deg$(\beta_n) = 2$ and $i(A) > n - 3$, then with similar arguments to the previous case, we conclude that $n = 4$.

By the induction assumption, there exist nonsingular matrices $X_0, Y_0 \in F^{(n-2)\times(n-2)}$ such that $X_0 K_0 X_0^{-1} Y_0 L_0 Y_0^{-1}$ has eigenvalues $c_2, \ldots, c_{n-1}$. Let $X = [1] \oplus X_0 \oplus [1]$ and $Y = [1] \oplus Y_0 \oplus [1]$. The matrix $X^{-1}K'XY^{-1} L'Y$ has eigenvalues $c_1, \ldots, c_n$.

Case 2. Suppose that $c_1 = 0$. Then $c_1 = \cdots = c_n = 0$. Let $p = \min\{j \in \{n - i(A) + 1, \ldots, n - 1\} : \alpha_j(0) = 0\}$. Let $\alpha'_p = \alpha_p(x)/x$ and $\alpha'_j = \alpha_{j+1}$, for every $j \in \{1, \ldots, n - 1\}$ and $j \neq p - 1$.

The matrix $A'$ is similar to a matrix of the form

$$A' = \begin{bmatrix} A_0 \ast & \ast & \ast \\ \ast & 0 & \ast \\ \ast & \ast & 0 \end{bmatrix},$$

where $A_0$ has invariant polynomials $\alpha'_1 \ldots \alpha'_{n-1}$ and det$(A_0) = 0$.

Subcase 2.1 Suppose that $i(A) + i(B) < n$ or deg$(\beta_{n-i(B)+1}) = 1$. Let $\mu$ be a primary eigenvalue of $B$. Let $\beta'_{n-i(B)+1}(x) = \beta_{n-i(B)+1}(x)/(x - \mu)$. The matrix $B'$ is similar to a matrix of the form

$$B' = \begin{bmatrix} B_0 \ast & \ast & \ast \\ \ast & 0 & \ast \\ \ast & \ast & 0 \end{bmatrix},$$

where

$$B_0 = C(\beta'_{n-i(B)+1}) + C(\beta_{n-i(B)+2}) + \cdots + C(\beta_n), \text{ if deg}(\beta_{n-i(B)+1}) \geq 2,$$

$$B_0 = C(\beta_{n-i(B)+2}) + \cdots + C(\beta_n), \text{ if deg}(\beta_{n-i(B)+1}) = 1.$$ We have $i(A_0) + i(B_0) \leq n - 1$ and at least one of the minimum polynomials of $A_0, B_0$ has degree greater than 2. According to the induction assumption, $(A_0, B_0)$ is spectrally complete and it is easy to conclude that $(A, B)$ is also weakly spectrally complete.

Subcase 2.2 Suppose that $i(A) + i(B) = n$ and deg$(\beta_{n-i(B)+1}) \geq 2$. Let $d = \deg(\beta_{n-i(B)+1})$. Analogously to the subcase 2.2.2 of the proof of Theorem 1 of [1], we conclude that

$$\#\{j \in \{1, \ldots, n\} : \deg(\alpha_j) = 1\} \geq d - 1.$$ Then $\alpha_{n-i(A)+d-1}(x) = \cdots = \alpha_{n-i(A)+d-1}(x) = x - \lambda$, where $\lambda$ is a primary eigenvalue of $A$. If $\lambda = 0$, then $p = n - i(A) + 1$ and $i(A_0) = i(A) - 1$. Let $B'$ be the matrix similar to $B$ as in the previous subcase. We have $i(A_0) + i(B_0) = n - 1$ and $\alpha_1, \beta_n$ are the minimum polynomials of $A', B'$. According to the induction assumption, there exist $X_0, Y_0 \in F^{(n-1)\times(n-1)}$ such that $X_0 A_0 X_0^{-1} Y_0 B_0 Y_0^{-1}$ has eigenvalues $c_1, \ldots, c_{n-1}$. The matrix $(X_0 \oplus [1]) A'(X_0 \oplus [1])^{-1} (Y_0 \oplus [1]) B'(Y_0 \oplus [1])^{-1}$ has eigenvalues $c_1, \ldots, c_n$.

Suppose that $\lambda \neq 0$. Let $\alpha''_{p-d}(x) = \alpha_p(x)/x$ and $\alpha'_j = \alpha'_{j+d}$, for every $j \in \{1, \ldots, n - d\}$ and $j \neq p - d$. The matrix $A'$ is permutation similar to a matrix of the form

$$\begin{bmatrix} D \ast & \ast \\ \ast & 0 \end{bmatrix},$$
where \( D = I_{d-1} \oplus [0] \) and \( K_0 \in F^{(n-d) \times (n-d)} \) has invariant polynomials \( \alpha_i'' \). The matrix \( B \) is similar to

\[
C(\beta_{n-i(B)+1}) \oplus L_0, \quad \text{where} \quad L_0 = C(\beta_{n-i(B)+2}) \oplus \cdots \oplus C(\beta_n).
\]

We have \( i(K_0) + i(L_0) = (i(A) - d + 1) + (i(B) - 1) = n - d \) and \( \alpha_n, \beta_n \) are the minimum polynomials of \( A', B' \). Then, we conclude that \( (K_0, L_0) \) is weakly spectrally complete. By Lemma 4 the pair \( (D, C(\beta_{n-i(B)+1})) \) is also weakly spectrally complete. It is easy to complete the proof.

**Lemma 8** If \((1.1)\) and \((1.2)\) are satisfied and at least one of the polynomials \( \alpha_n, \beta_n \) has degree different from 2, then the pair \( (A, B) \) is weakly spectrally complete.

**Proof** By induction on \( n \). The proof has already been done when \( i(A) + i(B) \leq n \).

Suppose that \( i(A) + i(B) > n \). Suppose, without loss of generality [2, Lemma 1], that \( i(A) \geq i(B) \). Then \( \deg(\alpha_{n-i(A)+1}) = 1 \). Let \( p = \#\{j \in \{1, \ldots, n\} : \deg(\alpha_j) = 1\} \) and \( d = \deg(\beta_{n-i(B)+1}) \). In order to obtain a contradiction, assume that \( p < d \). Then

\[
i(A) \leq p + \frac{n-p}{2}, \quad i(B) \leq \frac{n}{d} \leq \frac{n}{p+1}.
\]

From

\[
n+1 \leq i(A) + i(B) \leq p + \frac{n-p}{2} + \frac{n}{p+1},
\]

it follows that \( 0 \leq h(p) \), where \( h(p) = p^2 - (n+1)p + n - 2 \), which is impossible because \( h(1) \) and \( h(n) \) are negative numbers. Therefore \( p \geq d \). Let \( \lambda \) be the primary eigenvalue of \( A \). The matrices \( A, B \) are, respectively, similar to the matrices

\[
A' = \lambda I_d \oplus K_0, \quad \text{where} \quad K_0 = C(\alpha_{n-i(A)+d+1}) \oplus \cdots \oplus C(\alpha_n),
\]

\[
B' = C(\beta_{n-i(B)+1}) \oplus L_0, \quad \text{where} \quad L_0 = C(\beta_{n-i(B)+2}) \oplus \cdots \oplus C(\beta_n).
\]

Let \( \alpha_1' \ldots |\alpha_{n-d}' \) and \( \beta_1' \ldots |\beta_{n-d}' \) be the invariant polynomials of the matrices \( K_0 \) and \( L_0 \), respectively. Note that \( i(K_0) = i(A) - d \) and \( i(L_0) = i(B) - 1 \).

**Case 1.** Suppose that \( \lambda = 0 \). Then \( \text{rank}(A) = n - i(A) \). If \( p = n \), then \( A = 0 \) and the result is trivial.

Suppose that \( p < n \). If \( d = 1 \) and \( C(\beta_{n-i(B)+1}) \) is singular, then \( \text{rank}(L_0) = \text{rank}(B) = n - i(B) \geq n - i(A) = \text{rank}(A) = \text{rank}(K_0) \). If \( d > 1 \) or \( C(\beta_{n-i(B)+1}) \) is nonsingular, then \( \text{rank}(L_0) \geq i(L_0) = i(B) - 1 \geq n - i(A) = \text{rank}(A) = \text{rank}(K_0) \) and \( \text{rank}(B) \geq i(B) > n - i(A) = \text{rank}(A) \).

Let \( c_1, \ldots, c_n \in F \) be such that \( \#\{i \in \{1, \ldots, n\} : c_i \neq 0\} \leq \min\{\text{rank}(A), \text{rank}(B)\} = \text{rank}(A) = n - i(A) \). Suppose without loss of generality, that \( c_1 = \cdots = c_{i(A)} = 0 \).

If \( \beta_{(n-d)-i(L_0)+1}(x) = x - \mu, \) with \( \mu \in F \setminus \{0\} \), then, as \( \beta_{(n-d)-i(L_0)+1}(x) = \beta_{n-i(B)+2}(x) \), we have \( \beta_{n-i(B)+1}(x) = \beta_{n-i(B)+2}(x) = x - \mu \). By \((1.2)\), we have

\[
\alpha_1(x) \ldots \alpha_{i(B)}(x) = x^{i(A)+i(B)-n}
\]

and then

\[
\alpha_1'(x) \ldots \alpha_{i(L_0)}'(x) = \frac{\alpha_1(x) \ldots \alpha_{i(B)}(x)}{x} = x^{i(A)+i(B)-n-1} = x^{i(K_0)+i(L_0)-(n-1)}.
\]
Note that $\text{rank}(L_0) \geq \text{rank}(K_0) = n - i(A)$ and at least one of the polynomials $\alpha'_{n-d} = \alpha_n$ and $\beta'_{n-d} = \beta_n$ has degree different from 2. According to the induction assumption, there exist $X_0, Y_0 \in F^{(n-d) \times (n-d)}$ such that $X_0^{-1}K_0X_0Y_0^{-1}L_0Y_0$ has eigenvalues $c_{d+1}, \ldots, c_n$. Consider the matrices $X = I_d \oplus X_0$ and $Y = I_d \oplus Y_0$. The matrix $X^{-1}A'XY^{-1}B'Y$ has eigenvalues $c_1, \ldots, c_n$.

Case 2. Suppose that $\lambda \neq 0$. By (1.1), we have

$$\beta_1(x) \ldots \beta_{i(A)}(x) = x^{i(A) + i(B) - 1}$$

which implies that

$$\beta_{n-i(B)+1}(x) = \cdots = \beta_{i(A)}(x) = x.$$ 

Note that $d = 1$ and $\text{rank}(B) = n - i(B) < i(A) \leq \text{rank}(A)$. Let $c_1, \ldots, c_n \in F$ be such that $\# \{i \in \{1, \ldots, n\} : c_i \neq 0\} \leq \min\{\text{rank}(A), \text{rank}(B)\} = \text{rank}(B) = n - i(B)$. Suppose without loss of generality, that $c_1 = \cdots = c_{i(B)} = 0$.

If $\deg(\alpha_{n-i(A)+2}) = 1$, then

$$\beta_1'(x) \ldots \beta_{i(K_0)}'(x) = \frac{\beta_1(x) \ldots \beta_{i(A)}(x)}{x} = x^{i(A) + i(B) - n - 1} = x^{i(K_0) + i(L_0) - (n-1)}.$$ 

Note that $\text{rank}(L_0) = \text{rank}(B) < \text{rank}(A) = \text{rank}(K_0) + 1$ and at least one of the polynomials $\alpha'_{n-1} = \alpha_n$ and $\beta'_{n-1} = \beta_n$ has degree different from 2. According to the induction assumption, there exist $X_0, Y_0 \in F^{(n-1) \times (n-1)}$ such that $X_0^{-1}K_0X_0Y_0^{-1}L_0Y_0$ has eigenvalues $c_2, \ldots, c_n$. Consider the matrices $X = [1] \oplus X_0$ and $Y = [1] \oplus Y_0$. The matrix $X^{-1}A'XY^{-1}B'Y$ has eigenvalues $c_1, \ldots, c_n$. 

**Lemma 9** If $n = 2 = \deg(\alpha_2) = \deg(\beta_2)$, then the pair $(A, B)$ is weakly spectrally complete.

**Proof** Follows from Lemma 6.

**Lemma 10** If $\deg(\alpha_n) = \deg(\beta_n) = 2$, $i(A) \leq i(B)$ and 0 is a primary eigenvalue of $B$, then the pair $(A, B)$ is weakly spectrally complete.

**Proof** Let $\lambda, \nu$ be the roots of $\alpha_n$ and $\lambda$ a primary eigenvalue of $A$. Let $0, \epsilon$ be the roots of $\beta_n$. The matrix $A$ is similar to

$$A' = \lambda I_{2i(A) - n} \oplus \bigoplus_{i=1}^{n-i(A)} C,$$

where $C = \begin{bmatrix} \lambda & 1 \\ 0 & \nu \end{bmatrix}$,

and $B$ is similar to

$$B' = 0_{2i(B) - n} \oplus \bigoplus_{i=1}^{n-i(B)} D,$$

where $D = \begin{bmatrix} 0 & 1 \\ 0 & \epsilon \end{bmatrix}$.

Note that $\text{rank}(B) = n - i(B) \leq n - i(A) \leq \text{rank}(A)$. Let $c_1, \ldots, c_n \in F$ be such that $\# \{i \in \{1, \ldots, n\} : c_i \neq 0\} \leq \min\{\text{rank}(A), \text{rank}(B)\}$. Suppose, without loss of generality, that $c_{n-i(B)+1} = \cdots = c_n = 0$. According to the previous lemma, for every $j \in \{1, \ldots, n - i(B)\}$, there exists $D_j \in F^{2 \times 2}$ similar to $D$ such that $CD_j$ has eigenvalues
$c_j > 0$. Then, $B'$ is similar to $B'' = 0_{2i(B)−n} \oplus D_1 \oplus \cdots \oplus D_{n−i(B)}$ and $A'B''$ has eigenvalues $c_1, \ldots, c_n$. □

**Proof of Theorem 1** The necessity follows from Lemmas 2 and 3. The sufficiency follows from Lemmas 8–10. □

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**References**