Additive adaptive thinking in 1st and 2nd grades pupils

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This paper is part of the Project "Adaptive thinking and flexible computation: Critical issues". It discusses what is meant by adaptive thinking and presents the results of individual interviews with four pupils. The main goal of the study is to understand pupils' reasoning when solving numerical tasks involving additive situations, and identify features associated with adaptive thinking. The results show that, in the case of first grade pupils, the semantic aspects of the problem are involved in its resolution and the pupils' performance appears to be related to the development of number sense. The 2nd grade pupils seem to see the quantitative difference as an invariant numerical relationship.

Keywords: Adaptive thinking, numerical relationships, quantitative difference.

INTRODUCTION

This paper is part of the Project "Adaptive thinking and flexible computation: Critical issues" being developed by the Schools of Education of Lisbon, Setúbal and Portalegre that has two main focus: (a) to characterize the development of pupils' numerical thinking and flexibility in mental calculation from 5 to 12 years, and (b) to describe teachers' practices that facilitate that development.

In this paper we will discuss different perspectives on adaptive thinking and flexible calculation with regard to addition and subtraction in the literature, and present preliminary findings obtained by conducting individual clinical interviews with four pupils (two from first grade and two from second grade). We intend to understand pupils' reasoning when solving numerical tasks involving additive situations, and identify aspects related to adaptive thinking. Besides this goal, the interviews were also conducted to test tasks to later implement a teaching experiment.

THEORETHICAL FRAMEWORK

In the last decade, flexible calculation has been considered an ability that all pupils should develop in elementary school (Anghileri, 2001). Being proficient in mathematics implies the ability to use the knowledge in a flexible way and apply in an appropriate way what is learned in a situation to another (NCTM, 2000).

The flexible idea appears associated to mental calculation and arithmetic problem solving. There are different ways of solving an arithmetic problem mentally, usually mentioned as strategies. Strategic flexibility in mental calculation refers to the way as the problem is affected by circumstances to be solved (Threlfall, 2009). These circumstances may be related with specific features of the tasks or with individual characteristics or contextual variables. Threlfall (2009) refers to the mechanism behind calculation-strategy-flexibility as *zeroing-in*, still referring that it is not a fully conscious and rational process, involving partial exploratory calculations arising from noticing specific features of the numbers involved and their respective relationships. "The calculation-strategy is not selected and applied, it is arrived to" (Threlfall, 2009, p. 548). In a different perspective, Star and Newton (2009) define flexibility as knowing multiple solutions as well as the capacity and tendency to choose the most appropriated for a given problem and a particular objective of problem solving. These authors also stated that flexibility exists at a continuum; when the pupils gain flexibility they may first show a greater knowledge of multiple strategies, then particular preferences, and finally, the appropriate use of the preferred strategy. The 'appropriate' term refers to the more effective strategy: one which requires the least number of intermediate calculation steps to arrive at the result. Other authors (Baroody & Rosu, 2006; Rathgeb-Schnierer & Green, 2013) reported that flexibility in calculation is related to the fact that children discover

patterns and relations, as they develop number sense, thus building a network of relationships. For example, pupils who recognize the commutative property of addition, given the need to calculate 3 + 9, know they can do 9 + 3. The way this property is mobilized, revealing or not the contextual aspects of the tasks, can vary depending on the age of the children. In this regard, several authors (De Corte & Verschaffel, 1987; Greer, 2012) report that the semantic aspects of the tasks influence how young children solve them. Pupils who understand the various compositions of a number in different parts (for example, 1 + 7, 2 + 6, 3 + 5, and 4 + 4 = 8...) and decompositions (e.g., 8 = 1 + 7, 2 + 6, 3 + 5, 4 + 4) are more likely to develop ways of thinking as "doubles +1" (e.g., 7 + 8 = 7 + 7 + 1 = 14 + 1) or making a "ten" (9 + 7 = 9 + 1 + 6 = 10+6). As the network of relationships is being built, children acquire the flexibility to use these relationships in concrete situations of calculation, which depends on their knowledge of numbers and operations (Rathgeb-Schnierer & Green, 2013).

In our perspective, adaptive thinking refers to a thinking that can be flexibly adapted to new as well as familiar tasks. Its focus is not on calculation-strategy, but on quantitative reasoning. Children can mechanically use learned strategies without considering the context or the numbers involved in the task (Brocardo, 2014) and in this sense, they can compute accurately without flexibility. The flexible calculation and the additive quantitative reasoning are two dimensions that are interrelated to each other. Because the quantitative reasoning focuses on the description and modeling of situations and comparative relationships involved (Thompson, 1993), it ultimately underlies the development of flexible calculation as a calculation that mobilizes numerical relationships, in an intelligent and adaptive way to situations and numbers themselves. So the adaptive thinking involves the development of a flexible and relational understanding enabling the pupils to produce new known facts from old ones.

The quantitative reasoning involves reasoning about relationships between quantities. It "is the analysis of a situation into a quantitative structure — a network of quantities and quantitative relationships" (Thompson, 1993, p. 165). What matters are the relationships between quantities and not the numbers and number relations. In this regard, this kind of reasoning approaches the algebraic reasoning. To clarify the distinction between quantity and number, Thompson (1993) connects the idea of measure to the notion of quantity, although this is not only applicable to continuous measurable quantities, and the reasoning does not depend on their measures. It is important to develop research in children's abilities to deal with complexity in situations. A relationally complex situation involves at least six quantities and three quantitative operations. Comparing two quantities to find the excess of one relative to the other is a quantitative operation. The result of the quantitative operation of comparing two quantities additively is the excess found, that is to say, the *quantitative difference*. The author stresses also the distinction between the concepts of numerical difference, as the result of subtracting, and quantitative difference. On the one hand, a quantitative difference is not always evaluated by subtraction and on the other hand, subtraction can be used to compute quantities that are not quantitative differences. The results of the teaching experiment held with 5th-grade children referred in Thompson (1993) show that these children (i) did not distinguish the quantitative and the arithmetical operations, and (ii) had trouble with two aspects of the concept of quantitative difference, namely the difference as an additive comparison of quantities and when they conceived the quantitative difference as an invariant numerical relationship as they assumed the relative change as an absolute amount and needed to know absolute values before they could make comparisons.

The additive comparison is closely linked to inverse reasoning, involving the mobilization of reversible thought. According to Greer (2012), the inversion is of central importance to the arithmetic of natural numbers and the four basic operations involving these numbers, with important implications in relation to flexible computation. Regarding comparison problems, the author draws attention to the fact that the inverse relationship relates the difference between A and B to the complementary difference between B and A, which is "quite a different conception" (p. 434). So, although this author refers to the inverse relationship between addition and subtraction and the quantitative reasoning involves additive quantitative operations that are distinct of these arithmetic operations, we can consider that the inversion is an intrinsically topic underlying the quantitative reasoning.

METHODOLOGY

This study follows a qualitative approach within an interpretive paradigm. It aims describing and inter-

preting an educational phenomenon (Erickson, 1986). Data collection for this paper was done through clinical interviews (Hunting, 1997). It is a technique that is directed by the researcher and seeks a description of the ways of thinking of respondents.

Individual interviews were conducted in January 2014 by the authors of this paper, both members of the research team of the Project. The four pupils were attending for the first time the respective grades and were selected by their teachers. The selection criterions were: (i) pupils that usually express what they think, and (ii) pupils with reasonable performance in Mathematics. For the ethical principle of confidentiality, we use fictitious names for the children interviewed. The interviews were audiotaped and occurred in a room outside pupils' classrooms and had lasting less than 30 minutes. We also used the observation technique in the course of interviews, recording after its end the children's performance observed in field notes.

Each pupil solved three/four tasks but not all solved the same tasks. The task *boxes with balls*, inspired by Cobb, Boufi, McClain and Whitenack (1997) (Figure 1), was proposed to two first graders (Ana and Rui) and two second graders (João and Diogo).

The task *game of marbles,* adapted from Thompson (1993) (Figure 2), was only solved by João of 2nd grade.

Because the limited size of the paper we do not present all five tasks proposed in the interview. We chose these two tasks for the paper because they have relevant features to reveal adaptive thinking. The task *boxes with balls* was chosen because it reveals the children's thinking about flexible partitioning (e.g., a collection of 9 items conceptualized in imagination as: five and four, three and six, etc.). It is embedded in a fairytale context with the purpose of captivating young children and appealing to their imaginary world in which inanimate objects (such as balls) come to life and jump from one box to another, without human interference. The idea of movement (change of state) was central for the task design. So, consideration of the dynamic part of that movement will induce the pupils to explore different possibilities of decomposition of 9, since the balls are not distributed statically into two boxes but continue to jump from one box to the other, varying in number at each moment. The existence of two and no more boxes relates to the fact that it is desired to induce the representation of 9 into two groups, facilitating the development of the additive structure of **N** and the obtaining of certainty of the totality of solutions by the use of some organization in the disposition of them. For instance, in this scheme, we can see a structure of increasing and decreasing sequences and a central symmetry that support the exhaustion of splittings (Freudhental, 1983):

$$9 = + \frac{9}{0}\quad\frac{8}{1}\quad\frac{7}{2}\quad\frac{6}{3}\quad\frac{5}{4}\quad\frac{4}{5}\quad\frac{3}{6}\quad\frac{2}{7}\quad\frac{1}{8}\quad\frac{0}{9}$$

The other task was chosen because it aims to emphasize the notion of quantitative difference as signifi-

Once upon a time two boxes, one blue and one red, and 9 magical bouncy balls that spent their lives jumping from one box to another.

The balls will continue to bounce.
Imagine the different ways that we can see all nine balls in the two boxes.

Figure 1: Task *boxes with balls*

Ana, Luís and André played two games of marbles together.
Ana won 3 marbles from Luís and 7 marbles from André.
Luís won 4 marbles from André and 9 marbles from Ana.
André did not win marbles.
a) What is the least number of marbles that André had at the beginning of the games?
b) Compare the number of Luís's marbles before and after these two games.

Figure 2: Task *game of marbles*

cantly independent of the knowledge of the values of the additive and subtractive. It reveals the children's ability to reason inversely and to distinguish between quantitative difference and the absolute value. This is a fundamental aspect in flexible calculation.

Based on our adopted approach to flexible calculation and adaptive thinking in which we integrate the theoretical framework of Threlfall (2009) and Thompson (1993) that is presented in the previous section, the data were analyzed trying to understand how the children were able to establish a network of connections through their reasoning about different representations of the numbers, and about relationships between numbers and quantities. The analytical categories built from the theoretical framework and inductively emerged from the data focused on pupils' process of solving the tasks: applied relationships (relating the numbers, relating the operations, using the inverse relationship, comparing quantities); applied properties of addition (exhausting all possibilities).

SOME EMPIRICAL RESULTS

Addition and subtraction: From concrete to abstract thinking

In the task *boxes with balls* the collection of balls represented in the sheet of paper remained visible throughout the interview so that each child might be able to propose various ways in which the balls could be in the boxes. A sheet of paper with two drawn boxes was available for the pupils.

Once the first grade pupils had trouble with reading, the researcher read the task to be sure that there were not problems of understanding. Ana answered immediately:

Ana: 5 + 4

Researcher: And other ways?

Ana wrote immediately on the paper: 2+7, 7+2, 8+1, 1+8, 4+5, 3+6, 6+3.

After these records, Ana added there could still be 9 + 0. When questioned if it would be needed to write 2 + 7 and 7 + 2, Ana replied: "It gives the same, but it is not the same. Here [*pointing to the firs box*] there are 2 and there 7 [*pointing to the second box*] and here [7 + 2] is the opposite."

It seems that Ana thought about the concrete balls, and looked at the sums as ordered pairs of numbers.

In the case of Rui, after registering 5 + 4, the sequence of registration was: 4+5, 6+3, 3+6, 8+1, 1+8, 9+0.

After a moment, Rui wrote on the paper: 7+2 and 2+7.

Rui also responded to the question whether or not 6 + 3 is the same as 3 + 6: "It is. Only that is unlike."

Both children, Ana and Rui, represented first the situation of 5+4. Both pupils were able to visualize all decompositions of 9 thinking about the real situation — boxes and balls, although they did not need to draw them in the boxes. So, it seems that both thought about ordered pairs of numbers, trying to write all the pairs, using their different images of 9. Both seem to identify the commutative property because they wrote commutative pairs of numbers in a consistent way. Rui, after having written 5+4 and 4+5, he wrote 6+3, 3+6, using the increase/decrease 1 property. The same seems to happen with Ana, when she wrote 8+1, 1+8 following 2+7, 7+2.

In case of the two second grade pupils, João wrote "4+5, 3+6, 2+7, 1+8" and then stopped. Asked if there were more chances, João replied: "No, because I could change the order of numbers, but it would be the same thing, the sum is the same." Diogo wrote on the paper: 5 + 4, 6 + 3, 8 + 1, 7 + 2.

When the researcher asked if he already had written all the possibilities, he said: "Yes. If I change, its sum is the same, 9".

Both second graders made all possible non-empty decompositions of 9, not having considered the possibility of 9 + 0 (or 0 + 9). It should be noted that João used the increase/decrease 1 property to write all the decompositions, while Diogo appeared to start in that way, but changed his strategy when wrote 8+1 after 6+3 (increase/decrease 2?) and then seems to come back to the first one. But they did not express their ways of thinking.

These pupils seem to be able to think about the numbers abstracting from real situations. More, it appears that they have already understood the commutative property of addition.

Thus, we see that the consideration of the contextual situation was taken over by first graders but not by second graders who ignored the fact that the parcels play different roles in the proposed situation. The first grade pupils understand the concrete situation, and their thoughts are close to real situations since they considered ordered pairs of numbers. Instead, the second graders overcame the concrete situation.

Quantitative difference

In the task *game of marbles,* João used a tabulated registration. He began to register the total wins of marbles for each player: "+10"; "+13"; "+0". After, he put the total losses for each player by reading the sentences allusive to the wins: "-9"; "-3"; "-11". For that, he mobilized an inverse reasoning, understanding that the number of marbles won from someone is the number of marbles that someone had lost. His resolution can be seen in Figure 3.

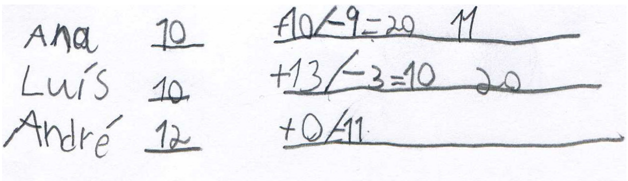

Figure 3: João's resolution of the task *game of marbles*

Then he focused on the item a) of the task, considering that André would have the minimum number of 12 marbles at the beginning of the games to have lost 11. He raised other hypotheses for this initial number like 20 or 30.

Researcher: And less than 12, no?

João: No, he had to have marbles.

This question focuses the absolute quantity of marbles. João held various hypotheses for the number of marbles of André before the games, all above 11, but assumed that at least André would have had 12 marbles. Probably, he did not equate the hypothesis of 11 marbles for having discarded the possibility of André having no marbles at the end of the games.

After, João did the balance of wins and losses of Luís's marbles, concluding that Luís would have 10 marbles more at the end of the games and recorded "=10" ("+13/-3=10"). In the trace corresponding to the beginning of the games, João wrote "10".

João: [At the end] Luís got ten more (...).

Researcher: At the beginning, did Luís have 10 or 10 more?

João: He had ten marbles.

João wrote "20" in the final of Luís's line corresponding to the total number of Luís's marbles at the end of the games. Next, the researcher guided João to the case of Ana:

Researcher: And Ana? She won ten and lost nine. After all, did she have more or less marbles at the end of the game?

João: Less. Before, she had ten marbles.

Researcher: If she started the games with ten marbles, won 10, with how many would she have at the end?

João: Twenty.

Researcher: And then she lost nine...

João: She had eleven.

João failed to make the balance between wins and losses of Ana's marbles, and concluded that Ana would have less marbles without mentioning how many. He wrote "10" in the trace corresponding to the beginning of the games. After, he registered "=20" ("+10/-9=20") as the number of Ana's marbles after 10 marbles won in the two games, and finally recorded "11" after losses, corresponding to the total number of Ana's marbles at the end of the games. João did not confront this final record with his previous statement ("Less") or verbalized that Ana had one more marble at the end of the games.

The critical issue inherent to this task is the distinction between quantitative difference and the absolute value. João did not make confusion between one thing and another, distinguishing the relative change (plus 10) from the absolute amount (20) in the second trace of the Luís's line, allusive to the end of the games. However he was not able to express the quantitative difference for the start of the two games (minus 10), needing to put absolute numbers for each player. João used the same equal symbol in the lines

of Ana and Luís ("=20"; "=10"), but he attributed different meanings to the numbers: in the case of Ana, 20 is the absolute amount; in the case of Luís, 10 is the quantitative difference, that is to say, it is the result of relative change — the amount by which one quantity fell short or exceeded of another. The need to refer to the absolute amount — the concrete number of marbles — is also evident in the way João determines the absolute value of the number of marbles of Ana and Luís at the end of the games ("11"; "20").

FINAL REMARKS

In the task *boxes with balls* pupils established different decompositions of the number 9 using their network of connections to have 9 based on properties of addition or making a direct subtraction and realizing that, for instance, by taking 4 from 9 they get 5. In both situations, pupils dealt with the operations addition and subtraction as being intrinsically inverse to one another (Greer, 2012). The quantity of nine balls can be symbolized by the number nine (Thompson, 1993) expressed by different sums representing the partition of a collection of objects or the decomposition of the number nine. The results show that all pupils dealt with the number, and not properly with the quantity. They did not divide the set of balls. Although the collection of balls represented in the sheet of paper remained visible, the pupils ignored them, thinking in a higher level about the number 9 conceptualized in imagination as sums representing its decompositions. For that, they seem to understand the relationships between those sums.

Although all the pupils started with the numbers 4 and 5 (or 5 and 4) adopting the approach of double through the decomposition of 9 into almost equal groups, the first and second graders solved the first task in different ways. Even though, first grade pupils did not need to materialize the situation with manipulatives or drawings, they solved the task very close to its context. We suspect that they were sure that they had generated all the possibilities because they applied some organization in the generation of the pairs of numbers writing consistently commutative pairs. So they looked for ordered pairs of numbers that together make 9. Instead, the second graders disengaged from the concrete situation, and only considered the fact that they had to obtain 9 from a sum. João did it systematically (increase/decrease 1) while Diogo did not do it for all decompositions. As they already know the commutative property (though not in a formal way) they applied it to justify that they found all the cases. So, they solved the problem in mathematics terms but not the actual proposed problem, where it should be considered the two different boxes and, in this perspective, it is not the same to have the balls in the red box or in the blue box. All the pupils already seem to understand the commutative property, but use it in a different way: the first graders use it to exhaust all possibilities through the symmetry of ordered commutative pairs of numbers (Ana: "It gives the same but it is not the same"), and the second graders use it not to present the commutative pairs, whereas the symmetrical parts would be the same (Diogo: "If I change, its sum is the same 9"). De Corte and Verschaffel (1987) stress that younger children are more influenced by the semantic aspects of the tasks. So it seems that the differences in the answers of first and second graders may be related with their age and their level of mathematical thinking.

In the task *game of marbles*, João shows an additive adaptive thinking as he seems to apply the inverse relationship between the wins and losses of marbles. Unlike the pupils of 5th grade reported by Thompson (1993), João did not confuse the notions of quantitative difference and absolute value of marbles nor needed to know the initial number of marbles to be able to think about wins and losses. However, he needed to be anchored in concrete numbers of marbles as happened with the pupils studied by Thompson (1993). Because João felt the need to attribute absolute values to the initial numbers of marbles, he showed to reason in terms of difference as quantitative operation without separating it from the involved particular arithmetical calculations. It is to say, João could not speak of relative changes associated to additive comparisons without referring to absolute amounts, showing conceiving the quantitative difference as an invariant numerical relationship. Thompson (1993) argues that it is important to conceive a quantitative difference independently of numerical information about quantities and relationships. However, we suspect that young pupils of 2nd grade are not able to conceive the independence of the values of the additive and subtractive when they reason quantitatively. For that reason, we consider that this task is not suitable for pupils of this grade, and we will not implement it in the teaching experiment.

The results reported here support the idea that flexible calculation is related to the knowledge and use of numerical relationships, being richer as pupils are developing their number sense, and are able to use the network of relationships that are building (Baroody & Rosu, 2006; Rathgeb-Schnierer & Green, 2013).

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