Developing prospective primary teachers' knowledge of mathematical reasoning processes in the context of a geometry task

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This paper aims to discuss the prospective primary teachers' knowledge of reasoning processes, namely the way they relate several reasoning processes, when solving a didactical task involving geometry. Data were collected by audio and video records of lessons, participant observation and the collection of written records of the prospective teachers. The results show how a group of prospective primary teachers may reach a high level of knowledge when involved in didactical tasks that are supported by relevant mathematical tasks and real classroom episodes, while working collaboratively. In particular, geometry tasks that involve spatial structuring favor the emergence of different reasoning processes and its relationships.

Keywords: Mathematical reasoning processes, Prospective primary teachers, Geometry, Spatial structuring

Introduction.

Teacher education should give special attention to mathematical reasoning, considering both the ability to reason, and knowledge about the reasoning processes (Stylianides & Stylianides, 2006). In particular, developing mathematical reasoning processes in the domain of geometry, in early years classrooms, implies specifically developing visualisation and spatial reasoning (Moss et al., 2015) since the central processes of generalising and justifying (Rodrigues et al., 2021), in this case, are founded on the geometric properties and on the objects´ structure.

This paper is part of the *Mathematical Reasoning and Teacher Education* (REASON) project, which aims to study the mathematical and didactical knowledge teachers need to carry out a practice that promotes pupils' mathematical reasoning and to study the ways to foster its development in prospective and practicing teachers of primary, middle and secondary school. In our communication we intend to discuss the knowledge of reasoning processes of a group of prospective primary teachers, when solving a task involving geometry, namely the way they relate several reasoning processes. This task was the fifth one implemented in a teacher education experiment carried out in 2019/20.

Conceptual framework

Reasoning in geometry

Reasoning geometrically about a spatial entity (object, diagram or concept) implies constituting an adequate mental model, that is, one that captures its relevant spatial structure and its geometric properties. Battista et al. (2018) state that "spatial and geometric structuring are types of spatial and geometric reasoning, respectively, that play vital roles in the construction of appropriate mental models for geometric reasoning" (p. 202). For spatial reasoning to adequately support geometric reasoning, these mental models must incorporate operational knowledge of relevant geometric properties and concepts, using mental models that integrate geometric properties into their structure and operation (Battista, 2007).

Lannin et al. (2011) distinguish two aspects in the generalisation: (i) identify common elements in different cases; (ii) extend reasoning beyond the domain for which common elements were initially identified, that is, thinking about a relationship, idea, representation, rule, pattern or other mathematical property considering it in a broader domain. For example, when, at the beginning of school, a student identifies squares as the figures that have four equal sides, he is making a generalisation that is false, but it is a generalisation. For these authors, the process of justifying consists of building a logical sequence of statements, each one relying on established knowledge in order to reach a conclusion. Thus, constructing a valid justification for a generalisation is not easy as it has to be verified that the generalisation is true for all cases in the domain, resorting to valid implicit relations. A valid justification must explain why by offering a view of the underlying relationships that exist in all cases.

Thus, we consider that the process of generalising is fundamental in Mathematics when we intend to "make general statements about properties, concepts or procedures" and that "justification is central to making it possible to mathematically validate" those statements (Mata-Pereira & Ponte, 2018, p. 783). These two processes interact with each other, as in many situations the language used in justification has to be general so that its applicability to the entire domain is clear; on the other hand, when some generalisations are established, it is because, at least implicitly, there are already justifications for them. For Jeannotte and Kieran (2017), exemplifying is an auxiliary process of generalising and justifying, which allows inferring data about a problem by generating elements that support those processes. In the process of generalising, it is essential to look for similarities and differences through the production of examples, in which case it is necessary to mobilize the process of comparing. In turn, in justifying the examples can be critical, for example when using counterexamples. For these authors, classifying consists, through the search for similarities and differences, identifying common and distinct points in different objects, joining them or separating them into a class of objects based on mathematical properties or definitions. This process involves comparing and, by stating that all elements of the class obey certain characteristics, it establishes a generalisation (Brunheira, 2019). For Mason (2001), “classification is not just about making distinctions and describing properties, but about justifying conjectures that all possible objects with those properties have been described or otherwise captured” (p.7). Mariotti and Fischbein (1997) state in the following way what means to classify in geometry:

> A classification task consists of stating an equivalence among similar but figurally different objects, towards a generalisation. That means overcoming the particular case and consider this particular case as an instance of a general class. In other terms, the process of classification consists of identifying pertinent common properties, which determine a category. (p. 244)

Thus, in addition to identifying the different reasoning processes, it is essential to have a deep understanding of the meaning of each one in order to establish relationships between them, thus reaching a high level of knowledge (Rodrigues et al., 2021).

Reasoning in preservice teacher education

Several studies (Lannin et al., 2011; Stylianides & Ball, 2008; Stylianides & Stylianides, 2009) indicate that prospective elementary school teachers must have opportunities to develop their mathematical reasoning if they are to work on it with their students, particularly in geometry.

In the field of geometry, Battista (2007) states that reasoning is strongly based on the spatial structuring of objects or situations, that is, on mental models that are activated to interpret and reason about these objects or situations. In the context of preservice teacher education in geometry, Brunheira (2019) suggests that processes such as classifying and justifying generalisations about geometric figures are influenced by the quality of spatial reasoning, but they are also promoters of its development. For Lehrer et al. (2013), geometric concepts such as shapes and relationships between them, for example congruence, constitute opportunities to build those relationships.

In addition to developing their own reasoning, Francisco and Maher (2011) refer to the need to create opportunities for teachers to learn about how to develop mathematical reasoning in students. In the same sense, Stylianides and Ball (2008) defend the need to develop in teachers the ability to plan and implement tasks that promote the development of reasoning in their students.

Methodology

The study reported here followed a qualitative-interpretative approach (Erickson, 1986) since it aims to understand the way prospective teachers relate several reasoning processes. Its context is a teacher education experiment developed in 2019/20 with 31 prospective primary teachers, attending a Master Degree certifying for teaching in primary schools (grade 1 to 4) and teaching Mathematics and Natural Sciences in grades 5 and 6. It took place during six lessons, one per week, each lasting two hours and 30 min and focused on mathematical reasoning, addressing specialised mathematics knowledge for teaching. All the tasks were initially explored autonomously by the prospective teachers, organised into eight groups, and were subsequently discussed by the class as a whole.

The data were collected through participant observation of the lessons, by the team Project, using audio and video recordings of the autonomous work carried out by two groups of prospective teachers (Groups 1 and 2) and the whole class discussions, and documents collection (all tasks resolutions). The data reported here are from Group 2. According to the ethical criterion of confidentiality, all the prospective teachers signed a free and informed consent form, in relation to the data collection methods, and are given fictitious names.

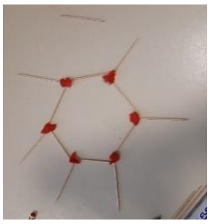
This paper refers to a didatical task about reasoning in geometry. Figure 1 presents an excerpt of the task.

Consider the task *Let's learn about pyramids*, proposed to 3rd grade students. In the previous year, the class had already come into contact with pyramids and prisms, in a first approach to their characteristics. So the teacher introduced the task by projecting the image below and asking *What is the intruder?*

After the initial discussion, the students started solving the task in pairs, using some models of pyramids in cardboard and wood, match sticks, toothpicks and plasticine balls.

3. Identify the reasoning processes involved.

4. Read the following dialog. The students had already analysed the possibility of building a pyramid with 13 edges using the material, as shown in the image. They were currently analysing the same issue for 15 edges.



Teacher — So, with 15 toothpicks, 15 edges, what happened?
Student — It would be missing 1.
Teacher — So and how many toothpicks did you put in the base?
Student — Eight.
Teacher — Eight. And now how many do you have to put on the side edges?
Student — Oh my God...
Teacher — OK, you can look at what you've done!
Student — Seven.
Teacher — So, can we build with 15?
Student — No… there was a toothpick missing… with odd numbers I couldn't do it.
Teacher — Ah! So tell me why it's not possible with odd numbers.
Student — Because one is missing or one is left.
Teacher — And why does this happen? What happens to the edges in the pyramids?
Student — Hmm... Here (points to the base) and here (points to the place where the side edges would be) must have the same number.

4.1. Discuss how the student's reasoning evolved, relating it to the interaction she established with the teacher.

4.2. Explain the role of the manipulative material in this situation and throughout the task.

Figure 1: An excerpt of the task

We used content analysis (Bardin, 2010) of the data using the framework we elaborated before (Rodrigues et al., 2021) concerned with knowledge of reasoning processes. The categories correspond to the reasoning processes worked in the teacher education experiment: generalising, justifying, exemplifying, comparing and classifying. Each of these categories were divided into subcategories corresponding to six levels of specialised mathematical knowledge of the content, presented in hierarchical form (Table 1).

Table 1: Framework for knowledge of mathematical reasoning processes

Category	Subcategories
Knowledge of the reasoning process	5. Knowledge of the process fits the definition presented, and includes its relationship with the other reasoning processes
	4. Knowledge of the process fits the definition presented, and is explicitly outlined by enunciating the properties of the process
	3. Knowledge of the process fits the definition presented, and is explicitly outlined through illustrative example(s)
	2. Recognising a reasoning process though considering only ‘correct’ processes
	1. Knowledge of the process takes on the meaning of the term in everyday language
	0. The process is confused with other processes

Results

Episode 1

In this episode, we analyse the reasoning processes that the group of prospective teachers identified to be used by the 3rd grade students during the solution of the intruder task (Figure 1). This lesson intended to support the introduction of the classification process which is, in our perspective, the main process in question. In fact, to find the intruder, students must identify pertinent common properties that determines the pyramids as a class (for example, they all have triangular faces that converge at a vertex) where the prism does not belong. For this, they should also ignore their particularities (for example, distinct bases) and consider them as representatives of a more general class (Mariotti & Fischbein, 1997).

Nuno:	OK. "Identify the reasoning processes involved". Here, I think that . . . the most obvious common is generalisation. They identify a property that fits all the pyramids.
Lara:	Yes.
Daniela:	Yes, Yes.
Nuno:	So one of them is to generalise. To compare...
Daniela:	Also to compare. You don't think so?
Nuno:	Between several...
Daniela:	Between figures, yes.
Lara:	To generalise they compare, don't they?
Nuno:	Yes, yes. They exemplify, here it does not...
Lara:	No.
Daniela:	No.
Daniela:	Compare between what?
Lara:	Among the different figures so that you can generalise.
Helena:	For example, here they made a comparison. When they had to select what it was [the intruder].
Nuno:	Yes, that's a fact. Between the ones that are and the ones that are not [pyramids].
Helena:	Exactly.

As expected, the prospective teachers did not report the classification process, but their analysis clearly identifies the generalising process that is strongly related to that process (Jeannotte & Kieran, 2017; Mariotti & Fischbein, 1997). Furthermore, Nuno explains the substantiation of this generalisation by saying that it corresponds to the property that “fits all pyramids”, an idea that gathers consensus. Associated with the process of generalising, the group also refers to the process of comparing as a support process because, as Lara says, it is necessary to compare the different figures “so that you can generalise”, which is also consistent with the literature.

The group agrees and is sure about the two identified processes, when one of the elements raises the hypothesis that the process of justifying may be also involved:

Lara:	To justify I don't know if it makes sense. Okay, you find properties that you can justify, but properties are generalisations.
Nuno:	Yes, yes. So, to generalise.
Helena:	I think it's enough to generalise and to compare.
Nuno:	Although, in order to generalise, they will also have to justify first, generalising is the most comprehensive of all. They will say first that the pyramid is a pyramid...
Helena:	Because so, so and so. Exactly.

Nuno:	It is a quadrangular pyramid, because the base is a square and because the faces are triangles, it has x vertices...
Daniela:	I think that, in order to reach the generalisation, they start with justification.

What started out as a tentative hypothesis from one of the prospective teachers turned out to be a meaningful possibility for everyone. The idea that the process of justifying may be involved derives from the perspective that, in this case, when we generalise we already have justification in mind or, to put it another way, we generalise because we know why. This idea is consistent with the suggestion by Mason (2001) when he states that the process of classifying also involves justifying conjectures that all possible objects with those properties have been described. Furthermore, in the particular case of geometry, as Brunheira (2019) states, the justification of generalisations concerning a class of geometric figures is based on a mental model of the class of objects, that is, its spatial structure that often presides over the formulation of generalisations.

In this way, we consider that the group's dialogues are quite relevant as they identify interactions between the processes of generalising and justifying, also showing understanding about all processes already dealt with, which corresponds to level 5.

Episode 2

Lara:	She started, she realized first that with 13 [toothpick] it would be left with one, right? That was the first thing she noticed. With 13 there would be one left, then with 15…
Daniela:	Yes, but here…
Lara:	One would be missing.
Lara:	She only realized the 15… she only gave the answer to 15 so quickly, because she had already done the one for 13.
Helena:	Because she had already done for 13.
Lara:	Because she even said there was one missing. Do you understand?
Daniela:	Yes…
Nuno:	Then we have to make a comparison with what the teacher was saying. Right here at the beginning, the teacher refers to another example, so she can...
Lara:	So, she can make a generalisation.
Nuno:	A generalisation, exactly.
Daniela:	So, the student began by understanding that with 13 toothpicks, one would be missing to complete the pyramid.
Nuno:	Then the teacher … encourages the student to go further…
Lara:	By giving another example.
Nuno:	…and it presents a new example, in this case with 15 toothpicks. Then the teacher encourages the student to go further by presenting a new example. She presents a new example, enabling the student to use the reasoning process, to generalise. She gave this example so that she could later conclude that it couldn't be an odd number.

In the written record with the answers to the task, the group summarizes the previous ideas and adds:

> In order to lead the student to a generalisation, the teacher guided her, leading her to understand that the number of toothpicks could not be odd. Also, the teacher asks why this happens, prompting justification. In a first moment, the student does not justify it, she only describes what happened. After the teacher's insistence, the student points to the material and justifies why her generalisation is valid. (Group’s record on question 4)

In this episode, the prospective teachers elect three processes that are mobilized: to generalise (that there are no pyramids with an odd number of edges), to exemplify (for 13 and 15 edges) and to justify (why an odd number of edges is impossible). Regarding the first process, the group correctly identifies that it is a generalisation when the student extends her conclusion (about 13 and 15) to the domain of odd numbers, corresponding to the definition of the generalisation process that was established. With regard to the process of justifying, it is noteworthy that the group is able to distinguish a simple description of an event (when the student says "Because one is missing or one is left") from a justification, relating this process with the investigation of the underlying reasons why it is true (Lannin et al., 2011). Finally, about the process of exemplifying, actually the examples used (with 13 and 15 edges) are suggested in the task. However, the group recognizes the support these examples provide for both the processes of generalising and justifying (Jeannotte & Kieran, 2017). In the first case, they consider that it is based on the attempt to build a pyramid with 13 edges that the student quickly concludes that it is impossible to construct a pyramid with 15 edges. In addition, they also realize that these two examples are fundamental to generalise and to justify, as they recognise the understanding of the situation they generate, allowing the student to understand why the number of edges cannot be odd.

Conclusion

The didactical task led the group of future teachers in a discussion about the reasoning processes involved in task on the properties of pyramids. In this discussion, the group was able to easily recognize, in context, the characteristics of the process of generalising, as well as its relationship with the process of comparing and exemplifying. However, the richness of the context involved–the establishment of the class of pyramids–enhances the emergence of various reasoning processes that occur in a non-linear way, generating a discussion about the distinction between generalising and justifying. Despite some hesitation, participants used the association between the process of justifying and the understanding of why a relationship works (Lannin et al., 2011) as a selection criterion for that process, which is found to be appropriate. Furthermore, they are also able to understand the supporting role that the process of exemplifying assumes in establishing a justification without confusing the role of empirical examples in establishing a statement, which is very common (Stylianides & Stylianides, 2009). On the contrary, the group seems to recognize that, in geometry, a justification must be associated with the spatial structure of objects (Brunheira, 2019).

This paper focused on a group of prospective teachers who shows a maximum level of knowledge about reasoning processes and its relationships, highlighting the importance and potential of didactical tasks that promote this knowledge, including the idea that different kinds of tasks can offer different kinds of opportunities for reasoning (Stylianides & Stylianides, 2006), which may be promoted using real classroom episodes. This research should be developed further highlighting the difficulties felt by other future teachers and ways to address them.

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References

Bardin, L. (2010). *Análise de conteúdo* (4.ª ed). Edições70.

Battista, M. T. (2007). The development of geometric and spatial thinking. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 843-908). Information Age.

Battista, M. T., Frazze, L. M., & Winer, M. L. (2018). Analyzing the relation between spatial and geometric reasoning for elementary and middle school students. In K. S. S. Mix and M. K. Battista (Eds.). *Visualizing Mathematics The Role of Spatial Reasoning in Mathematical Thought* (pp. 195-229). Springer.

Brunheira, L. (2019). *O desenvolvimento do raciocínio geométrico na formação inicial dos professores dos primeiros anos*. Instituto de Educação da Universidade de Lisboa. http://hdl.handle.net/10451/38922.

Erickson, F. (1986). Qualitative methods in research on teaching. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed., pp. 119-161). Macmillan.

Francisco, J. M., & Maher, C. A. (2011). Teachers attending to students' mathematical reasoning: lessons from na after-school research program. *Journal of Mathematics Teacher Education, 14*(1), 49-66.

Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. *Educational Studies in Mathematics, 96*(1), 1-16. doi:10.1007/s10649-017-9761-8.

Lannin, J. K., Elliott, R., & Ellis, A. B. (2011). *Developing essential understanding of mathematical reasoning for teaching mathematics in prekindergarten-grade 8*. NCTM.

Lehrer, R, Kobiela, M., & Weinberg, P.J. (2013). Cultivating inquiry about space in a middle school mathematics classroom. *ZDM Mathematics Education, 45*, 385-376. Doi:10.1007/s11858-012-0479-x

Mariotti, M. A., & Fischbein, E. (1997). Defining in classroom activities. *Educational Studies in Mathematics, 34*(3), 219–248.

Mata-Pereira, J., & Ponte, J. P. (2018). Promover o raciocínio matemático dos alunos: Uma investigação baseada em design. *Bolema, 32*(62), 781–801. doi:10.1590/1980-4415v32n62a02.

Moss, J., Hawes, Z., Naqvi, S., & Caswell, B. (2015). Adapting Japanese Lesson Study to enhance the teaching and learning of geometry and spatial reasoning in early years classrooms: A case study. *ZDM Mathematics Education, 47*, 377–390. https://doi.org/10.1007/s11858-015-0679-2

Rodrigues, M., Brunheira, L., & Serrazina, L. (2021). A framework for prospective primary teachers' knowledge of mathematical reasoning processes. International Journal of Educational Research, 107, 101750-101761. https://doi.org/10.1016/j.ijer.2021.101750

Stylianides, A. J., & Ball, D. L. (2008). Understanding and describing mathematical knowledge for teaching knowledge about proof for engaging students in the activity of proving. *Journal of Mathematics Teacher Education, 11*(4), 307-332.

Stylianides, A. J., & Stylianides, G. J. (2006). Content knowledge for mathematics teaching: The case of reasoning and proving. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th PME International Conference* (Vol. 5, pp. 201–208). PME.

Stylianides, G. J., & Stylianides, A. J. (2009). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education, 40*, 314–352.