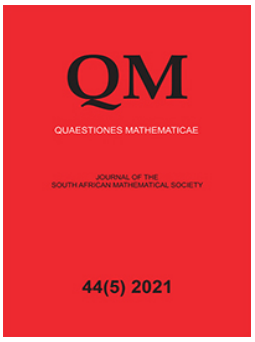

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POSITIONED NUMERICAL SEMIGROUPS

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Abstract. A numerical semigroup S is positioned if for all $s \in \mathbb{N}\backslash S$ we have that $\mathrm{F}(S) + \mathrm{m}(S) - s \in S$. In this paper, we give algorithms to compute the set of positioned semigroups and a criterium to check whether S is or not is positioned. Furthermore, we prove the Wilf's conjecture for this type of numerical semigroups.

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Key words: Numerical semigroups, positioned numerical semigroups, tree, Frobenius number, multiplicity, genus and Wilf's conjecture.

1. Introduction.

A numerical semigroup S is a subset of nonnegative integers $\mathbb{N} = \{0, 1, 2, \ldots\}$ that is closed under addition, $0 \in S$ and $\mathbb{N}\backslash S$ is finite. Numerical semigroups appear in several areas of mathematics and there are several interesting combinatorial invariants of a semigroup (see for example [15]). Notable numerical semigroup invariants include the Frobenius number $\mathrm{F}(S)$ which is the largest integer not belonging to S and the multiplicity $\mathrm{m}(S)$ which is the least positive integer belonging to S.

Let k be a positive integer. A numerical semigroup S is k-positioned if for all $x \in \mathbb{N}\backslash S$ we have that $k - x \in S$. It is clear that if S is a k-positioned numerical semigroup then $k \geq \mathrm{F}(S)$. The $\mathrm{F}(S)$-positioned numerical semigroups are the symmetric numerical semigroups widely studied in [11], [12] and [2].

If S is a nonsymmetric numerical semigroup, then S is not $\mathrm{F}(S)$-positioned, and also it is not $\mathrm{F}(S) + i$-positioned for all $i \in \{1, \ldots, \mathrm{m}(S) - 1\}$ (note that $i \notin S$ and $\mathrm{F}(S) + i - i = \mathrm{F}(S) \notin S$). Therefore, we can conclude that if S is a k-positioned numerical semigroup and not symmmetric then $k \geq \mathrm{F}(S) + \mathrm{m}(S)$.

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We say that S is a positioned numerical semigroup if S is $\mathrm{F}(S)+\mathrm{m}(S)$- positioned. The goal of this survey is to study this type of numerical semigroups.

Let $\mathscr{P}(m,F)$ be the set of all positioned numerical semigroup with multiplicity $m \geq 2$ and Frobenius number F. In Section 2, we see that $\mathscr{P}(m,F) \neq \emptyset$ if and only if $F \geq m-1$ and m does not divide F. In Section 3, (respectively Section 4) we propose an algorithm to compute the set $\mathscr{P}(m,F)$ with $F < 2m$ (respectively $F > 2m$).

Section 5 is dedicated to the study of the Apéry set of a positioned numerical semigroup. The knowledge of Apéry set of a numerical semigroup S gives a criterium to check whether S is or not is positioned. In 1978, H.S. Wilf (see [17]) conjecture an upper bound for the Frobenius number of a numerical semigroup in terms of the other invariants of the semigroup. This question has been solved in some special cases, but remains open in general. We will prove the Wilf's conjecture for positioned numerical semigroups.

2. First results.

A numerical semigroup is irreducible if it cannot be expressed as the intersection of two numerical semigroups properly containing it. In [13] it is proved that the family of irreducible numerical semigroups is equal to the disjoint union of the symmetric and pseudo-symmetric numerical semigroups.

A numerical semigroup is symmetric if $\mathrm{F}(S)-x \in S$ for all $x \in \mathbb{N}\backslash S$. This concept was introduced in [11] by E. Kunz and this class of semigroups has always odd Frobenius number. The translation of this concept for numerical semigroups with even Frobenius number motivates the definition of pseudo-symmetric numerical semigroups (see[2]). A numerical semigroup S is pseudo-symmetric if $\mathrm{F}(S)$ is even and $\mathrm{F}(S)-x \in S$ for all $x \in \mathbb{N}\backslash \left(S \cup \{\frac{\mathrm{F}(S)}{2}\}\right)$.

Proposition 1. *Every irreducible numerical semigroup is positioned.*

Proof. Clearly, if S is a symmetric numerical semigroup then S is $\mathrm{F}(S)$-positioned. Hence $\mathrm{F}(S)-x \in S$ for all $x \in \mathbb{N}\backslash S$ and thus $\mathrm{F}(S)-x+\mathrm{m}(S) \in S$ for all $x \in \mathbb{N}\backslash S$. As result, S is a positioned numerical semigroup.

Suppose now that S is a pseudo-symmetric numerical semigroup then $\mathrm{F}(S)-x \in S$ for all $x \in \mathbb{N}\backslash \left(S \cup \{\frac{\mathrm{F}(S)}{2}\}\right)$. And thus $\mathrm{F}(S)-x+\mathrm{m}(S) \in S$ for all $x \in \mathbb{N}\backslash \left(S \cup \{\frac{\mathrm{F}(S)}{2}\}\right)$. Furthermore, by [13, Lemma 4], $\mathrm{F}(S)+\mathrm{m}(S)-\frac{\mathrm{F}(S)}{2} = \frac{\mathrm{F}(S)}{2}+\mathrm{m}(S) \in S$. □

If S is a numerical semigroup we have that the cardinality of $\mathbb{N}\backslash S$ is known as the genus of S, denoted by $\mathrm{g}(S)$. The next result is [15, Lemma 2.14].

Lemma 2. *If S be a numerical semigroup, then $\frac{\mathrm{F}(S)+1}{2} \leq \mathrm{g}(S)$.*

The irreducible numerical semigroups are those numerical semigroups with the least possible genus in terms of their Frobenius number as we see in the next result [15, Corollary 4.5].

PROPOSITION 3. *Let S be a numerical semigroup.*

(1) *S is symmetric if and only if $\frac{\mathrm{F}(S)+1}{2} = \mathrm{g}(S)$.*

(2) *S is pseudo-symmetric if and only if $\frac{\mathrm{F}(S)+2}{2} = \mathrm{g}(S)$.*

Given a real number r we denote by $\lfloor r \rfloor = \max\{z \in \mathbb{Z} \mid z \leq r\}$ and $\lceil r \rceil = \min\{z \in \mathbb{Z} \mid z \leq r\}$. As a consequence of Proposition 3 we obtain the following result.

COROLLARY 4. *A numerical semigroup S is irreducible if and only if $\lceil \frac{\mathrm{F}(S)+1}{2} \rceil = \mathrm{g}(S)$.*

Given a numerical semigroup S, we define the set

$$Q(S) := \{x \in S \mid 1 \leq x \leq \mathrm{F}(S) + \mathrm{m}(S) - 1\}$$

and its cardinality is denoted by $\mathrm{q}(S)$.

PROPOSITION 5. *Let S be a positioned numerical semigroup. Then*

$$\lceil \frac{\mathrm{F}(S)+1}{2} \rceil \leq \mathrm{g}(S) \leq \lfloor \frac{\mathrm{F}(S)+\mathrm{m}(S)-1}{2} \rfloor.$$

Proof. Since the set $\{1, 2, \ldots, \mathrm{F}(S) + \mathrm{m}(S) - 1\}$ is the disjoint union of the sets $\mathbb{N} \backslash S$ and $Q(S)$ we have that $\mathrm{F}(S) + \mathrm{m}(S) - 1 = \mathrm{g}(S) + \mathrm{q}(S)$. The map

$$\mathbb{N} \backslash S \to Q(S), \; x \to \mathrm{F}(S) + \mathrm{m}(S) - x$$

is injective, which proves that $\mathrm{g}(S) \leq \mathrm{q}(S)$. Hence $2\mathrm{g}(S) \leq \mathrm{g}(S) + \mathrm{q}(S) = \mathrm{F}(S) + \mathrm{m}(S) - 1$, and consequently $\mathrm{g}(S) \leq \lfloor \frac{\mathrm{F}(S)+\mathrm{m}(S)-1}{2} \rfloor$.

Finally, applying Lemma 2, we obtain that $\lceil \frac{\mathrm{F}(S)+1}{2} \rceil \leq \mathrm{g}(S)$. □

We show with an example that the converse of previous proposition is not true.

EXAMPLE 6. Take a numerical semigroup $S = \{0, 10, 12, 13, 14, 15, 16, 17, 20, \rightarrow\}$ (the symbol $\rightarrow$ means that every integer greater than 20 belongs to the set), we have that $\mathrm{F}(S) = 19$, $\mathrm{m}(S) = 10$ and $\mathbb{N} \backslash S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 18, 19\}$ and thus $\mathrm{g}(S) = 12$. Then $12 = \mathrm{g}(S) \leq \lfloor \frac{\mathrm{F}(S)+\mathrm{m}(S)-1}{2} \rfloor = \lfloor \frac{19+10-1}{2} \rfloor = 14$. Since $11 \notin S$ and $\mathrm{F}(S) + \mathrm{m}(S) - 11 = 18 \notin S$ we get that S is not positioned

Let $\mathscr{L}(m, F)$ be the set of all numerical semigroups with multiplicity m and Frobenius number F. The next result appears in [6].

LEMMA 7. *Let m and F be two positive integers such that $m \geq 2$. Then $\mathscr{L}(m, F) \neq \varnothing$ if and only if $F \geq m - 1$ and m does not divide F.*

It is clear that $\{km \mid k \in \mathbb{N}\} \cup \{F+1, \longrightarrow\}$ is the least (with respect to the inclusion ordering) element in $\mathscr{L}(m, F)$. Denote by $\mathscr{M}(m, F)$ the set of maximal elements in $\mathscr{L}(m, F)$. Denote by $\mathscr{I}(m, F)$ the set of all irreducible numerical semigroups with multiplicity m and Frobenius number F. From [4, Proposition 6] we deduce the next result.

Proposition 8. *Let m and F be two positive integers such that $F \geq m-1$ and m does not divide F.*

(1) *If $F = m-1$, then $\mathscr{M}(m, F) = \Big\{\{0, m, \rightarrow\}\Big\}$.*

(2) *If $m < F < 2m$, then $\mathscr{M}(m, F) = \Big\{\{0, m, \rightarrow\} \setminus \{F\}\Big\}$.*

(3) *If $F > 2m$ then $\mathscr{M}(m, F) = \mathscr{I}(m, F)$.*

As a consequence of Propositions 1 and 8 we have the following.

Corollary 9. *Let m and F be two positive integers and $S \in \mathscr{M}(m, F)$. Then S is a positioned numerical semigroup.*

By using Lemma 7 and Corollary 9, we obtain the next result.

Corollary 10. *Let m and F be two positive integers such that $m \geq 2$. The following conditions are equivalent.*

(1) *The set $\mathscr{P}(m, F)$ is not empty.*

(2) *The set $\mathscr{L}(m, F)$ is not empty.*

(3) *$F \geq m-1$ and m does not divide F.*

3. Positioned elementary numerical semigroups. Following the terminology introduced in [5], a numerical semigroup S is elementary if $\mathrm{F}(S) \leq 2\mathrm{m}(S)$. This class of semigroups was also studied in [10], [18] and [14].

Our aim in this section is to give an algorithm method to compute the whole set of positioned elementary numerical semigroups with fixed multiplicity and Frobenius number. The next result is easy to prove (see also [14, Lemma 3.1].

Proposition 11.

(1) *$\mathbb{N}$ is the only positioned numerical semigroup with multiplicity equal to 1.*

(2) *$\{0, 2, \rightarrow\}$ and $\{0, 2, 4 \rightarrow\}$ are the only positioned elementary numerical semigroups with multiplicity 2.*

(3) *If S is a numerical semigroup with $F = \mathrm{m}(S) - 1$ then $S = \{0, \mathrm{m}(S), \rightarrow\}$ and thus S is a positioned numerical semigroup.*

(4) *If S is a numerical semigroup such that $F = \mathrm{m}(S) + 1$ or $F = \mathrm{m}(S) + 2$ then $S = \{0, \mathrm{m}(S), \rightarrow\} \setminus \{F\}$ and thus S is a positioned numerical semigroup.*

Now we will determine the positioned elementary numerical semigroups with $\mathrm{m}(S) \geq 4$ and $\mathrm{F}(S) \geq \mathrm{m}(S) + 3$. By using [18, Proposition 2.1] we deduce the following result.

Proposition 12. *Let m be an integer such that $m \geq 4$, $F \in \{m+3, \ldots, 2m-1\}$ and let A be a subset of $\{m+1, \ldots, F-1\}$. Then $S(A) = \{0, m\} \cup A \cup \{F+1, \rightarrow\}$ is an elementary numerical semigroup with multiplicity m and Frobenius number F. Moreover, every elementary numerical semigroup with multiplicity m and Frobenius number F is of this form.*

We illustrate the above result with an example.

Example 13. Let us compute the set of all elementary numerical semigroups with multiplicity 5 and Frobenius number 8. Since $\Big\{A \mid A \subseteq \{6,7\}\Big\} = \Big\{\emptyset, \{6\}, \{7\}, \{6,7\}\Big\}$. From Proposition 12, we can conclude that the set of numerical semigroups under these conditions is equal to $\Big\{ \{0, 5, 9 \rightarrow\}, \{0, 5, 6, 9 \rightarrow\}, \{0, 5, 7, 9 \rightarrow\}, \{0, 5, 6, 7, 9 \rightarrow\} \Big\}$.

In the following result we see which conditions must the set A (of Proposition 12) fulfill so that $S(A)$ is a positioned numerical semigroup.

Proposition 14. *Let m be an integer such that $m \geq 4$, $F \in \{m+3, \ldots, 2m-1\}$ and let A be a subset of $\{m+1, \ldots, F-1\}$. Then $S(A) = \{0, m\} \cup A \cup \{F+1, \rightarrow\}$ is a positioned elementary numerical semigroup if and only if $F + m - x \in A$ for all $x \in \{m+1, \ldots, F-1\} \setminus A$*

Proof. *Necessity*. By applying Proposition 12, we get that $S(A)$ is a numerical semigroup with $\mathrm{m}\big(S(A)\big) = m$ and $F\big(S(A)\big) = F$. If $x \in \{m+1, \ldots, F-1\} \setminus A$ then $x \notin S(A)$. As $S(A)$ is a positioned numerical semigroup we have that $F + m - x \in S(A)$ and thus $F + m - x \in A$.

Sufficiency. If $x \in \mathbb{N} \setminus S(A)$ then either $x \in \{1, \ldots, m-1\}$ or $x \in \{m+1, \ldots, F-1\} \setminus A$, in both cases $F + m - x \in S(A)$. Hence $S(A)$ is a positioned numerical semigroup. □

For any finite set X, $\#X$ denotes the cardinal of X.

Lemma 15. *Let m be an integer such that $m \geq 4$, $F \in \{m+3, \ldots, 2m-1\}$ and let A be a subset of $\{m+1, \ldots, F-1\}$ such that $S(A)$ is a positioned numerical semigroup. Then $\#A \geq \frac{F-m-1}{2}$.*

Proof. Let $B = \{m+1, \ldots, F-1\} \setminus A$. Then $\{m+1, \ldots, F-1\}$ is the disjoint union of the sets A and B. Hence $\#\{m+1, \ldots, F-1\} = F - m - 1 = \#A + \#B$. By Proposition 14, the correspondence

$$B \longrightarrow A, \; x \longrightarrow F + m - x$$

is an injective map and thus $\#B \leq \#A$. Hence $2.\#A \geq \#A+\#B = F-m-1$. □

Now, we are able to present an algorithm to compute the set $\mathscr{P}(m, F)$ with $F < 2m$, that is, the whole set of positioned elementary numerical semigroups with fixed multiplicity and Frobenius number.

Algorithm 16.
Input: m and F integers such that $m \geq 4$ and $F \in \{m+3, \ldots, 2m-1\}$.
Output: The set $\mathscr{P}(m, F)$

(1) $\mathcal{B} = \big\{A \subseteq \{m+1, \ldots, F-1\} \mid \#A \geq \frac{F-m-1}{2}\big\}$.

(2) $\mathcal{C} = \big\{A \in \mathcal{B} \mid F+m-x \in A \text{ for all } x \in \{m+1, \ldots, F-1\} \setminus A\big\}$.

(3) Return $\{S(A) \mid A \in \mathcal{C}\}$.

Example 17. Let us compute the set $\mathscr{P}(6, 10)$ using Algorithm 16. Then:

(1) $\mathcal{B} = \big\{A \subseteq \{7,8,9\} \mid \#A \geq 2\big\} = \big\{ \{7,8\}, \{7,9\}, \{8,9\}, \{7,8,9\} \big\}$.

(2) $\mathcal{C} = \big\{ \{7,8\}, \{8,9\}, \{7,8,9\} \big\}$.

Then $\mathscr{P}(6,10) = \big\{S(\{7,8\}), S(\{8,9\}), S(\{7,8,9\})\big\} =$
$= \big\{ \{0,6,7,8,11 \rightarrow\}, \{0,6,8,9,11 \rightarrow\}, \{0,6,7,8,9,11 \rightarrow\} \big\}$.

4. Positioned non elementary numerical semigroups. If S is a non-irreducible numerical semigroup then we have that $\big\{h \in \mathbb{N} \backslash S \mid \mathrm{F}(S) - h \notin S \text{ and } h \neq \frac{\mathrm{F}(S)}{2}\big\}$ is not empty. We denote by $\alpha(S) = max\big\{h \in \mathbb{N} \backslash S \mid \mathrm{F}(S) - h \notin S \text{ and } h \neq \frac{\mathrm{F}(S)}{2}\big\}$. The next result is deduced from [15, Lemma 4.3].

Lemma 18. *Let S be a non-irreducible numerical semigroup such that $\mathrm{m}(S) < \frac{\mathrm{F}(S)}{2}$. Then $\overline{S} = S \cup \{\alpha(S)\}$ is a numerical semigroup with $\mathrm{F}(\overline{S}) = \mathrm{F}(S)$ and $\mathrm{m}(\overline{S}) = \mathrm{m}(S)$.*

Assume that S is a numerical semigroup with $\mathrm{m}(S) = m$ and $\mathrm{F}(S) = F > 2m$. As a consequence of the previous lemma, we can define recurrently the following sequence of elements of $\mathscr{L}(m, F)$:

- $S_0 = S$,
- $S_{n+1} = \begin{cases} S_n \cup \{\alpha(S_n)\} & \text{if } S_n \text{ is non-irreducible} \\ S_n & \text{otherwise.} \end{cases}$

It is clear that there exists a sequence of numerical semigroups $S = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k$ and $S_k \in \mathscr{I}(m, F)$. We say that S_k is the irreducible numerical semigroup associated to S and will be denote by $\mathcal{O}(S)$. The following result tells us that if S_0 is a positioned numerical semigroup then every numerical semigroups, in the previous sequence, are positioned.

Proposition 19. *Let S be a non-irreducible positioned numerical semigroup such that* $\mathrm{m}(S) < \frac{F(S)}{2}$. *Then* $\overline{S} = S \cup \{\alpha(S)\}$ *is a positioned numerical semigroup with* $\mathrm{F}(\overline{S}) = \mathrm{F}(S)$ *and* $\mathrm{m}(\overline{S}) = \mathrm{m}(S)$.

Proof. From Lemma 18, we know that $\overline{S} = S \cup \{\alpha(S)\}$ is a numerical semigroup with $\mathrm{F}(\overline{S}) = \mathrm{F}(S)$ and $\mathrm{m}(\overline{S}) = \mathrm{m}(S)$. If $x \in \mathbb{N}\backslash\overline{S}$ then $x \in \mathbb{N}\backslash S$ and thus $\mathrm{F}(S) + \mathrm{m}(S) - x \in S$. Therefore, $\mathrm{F}(\overline{S}) + \mathrm{m}(\overline{S}) - x \in \overline{S}$, and this means that $\overline{S}$ is a positioned numerical semigroup. □

From now on, m and F are two positive integers such that $F \geq 2m$ and m is not a divisor of F.

We define the following equivalence relation over $\mathscr{P}(m, F)$:

$$S \sim T \text{ if and only if } \mathcal{O}(S) = \mathcal{O}(T).$$

We denote the class of $S \in \mathscr{P}(m, F)$ modulo $\sim$ by $[S] = \{T \in \mathscr{P}(m, F) \mid S \sim T\}$ and the quotient set $\mathscr{P}(m, F)/\!\sim = \{[S] \mid S \in \mathscr{P}(m, F)\}$. The following result has immediate proof.

Lemma 20. *If* $S \in \mathscr{P}(m, F)$ *then the equivalence class* $[S]$ *contains a single element of* $\mathscr{I}(m, F)$.

As a consequence of Proposition 1 and Lemma 20 we have the following result.

Proposition 21. *The quotient set* $\mathscr{P}(m, F)/\sim = \{[S] \mid S \in \mathscr{I}(m, F)\}$. *Moreover, if* $S, T \in \mathscr{I}(m, F)$ *and* $S \neq T$ *then* $[S] \cap [T] = \emptyset$.

Remark 22. In view of Proposition 21, in order to determine explicitly the elements in the set $\mathscr{P}(m, F)$ we need

1) an algorithm to compute the set $\mathscr{I}(m, F)$;

2) an algorithm to compute the set $[S]$, for each $S \in \mathscr{I}(m, F)$.

In [6] it was given an efficient algorithm to compute 1). Let us focus in 2) of Remark 22 our goal we will be to give an algorithm to this end.

A graph G is a pair (V, E), where V is a nonempty set and E is a subset of $\{(v, w) \in V \times V \mid v \neq w\}$. The elements of V and E are called vertices and edges of G, respectively. A path of length n connecting the vertices v and w of G is a sequence of distinct edges of the form (v_0, v_1), (v_1, v_2),…, (v_{n-1}, v_n) with $v_0 = v$ and $v_n = w$.

A graph G is a tree if there exists a vertex r (known as the root of G) such that for every other vertex v of G, there exists a unique path connecting v and r. If (v, w) is a edge of the tree then we say that v is a son of w.

Let $\triangle \in \mathscr{I}(m, F)$. Let $G(\triangle)$ be the graph with vertex set $[\triangle]$ such that $(S, T) \in [\triangle] \times [\triangle]$ is an edge if and only if $T = S \cup \{\alpha(S)\}$.

As a consequence of the sequence presented above Proposition 19 we obtain the following result.

Theorem 23. *Let $\triangle \in \mathscr{I}(m,F)$. The graph $G(\triangle)$ is a tree with root equal to $\triangle$.*

Observe that the tree $G(\triangle)$ can be constructed recursively, from the root $\triangle$ in each step we are joining each of the vertices with its sons. In fact, if S is a son of T then (S,T) is an edge of $G(\triangle)$. Hence, $T = S \cup \{\alpha(S)\}$ and thus $S = T \backslash \{\alpha(S)\}$.

Let $\mathcal{X}$ be a nonempty subset of $\mathbb{N}$. We will write $\langle \mathcal{X} \rangle$ for the submonoid of $(\mathbb{N},+)$ generated by $\mathcal{X}$, that is,

$$\langle \mathcal{X} \rangle := \left\{ \sum_{i=1}^{n} \lambda_i x_i \mid n \in \mathbb{N} \backslash \{0\},\ x_1,\ldots,x_n \in \mathcal{X}, \text{and } \lambda_1,\ldots,\lambda_n \in \mathbb{N} \right\}.$$

It is well known (see for example [15]) that $\langle \mathcal{X} \rangle$ is a numerical semigroup if and only if $\gcd(\mathcal{X}) = 1$

If S is a numerical semigroup generated by $\mathcal{X}$ then we say that $\mathcal{X}$ is a system of generators of S. Moreover, if $S \neq \langle \mathcal{X}' \rangle$ for all $\mathcal{X}' \subsetneq \mathcal{X}$, then we say that $\mathcal{X}$ is a minimal system of generators of S. The following result can be deduced of [15, Corollary 2.8].

Corollary 24. *Every numerical semigroup admits a unique minimal system of generators, which in addition is finite.*

This minimal system of generators of S will be denoted by $\mathrm{msg}(S)$ and its cardinality is known as the embedding dimension of S, denoted by $\mathrm{e}(S)$.

As a consequence of [15, Lemma 2.3] we have the following result.

Lemma 25. *Let S be a numerical semigroup and $x \in S$. Then $S \backslash \{x\}$ is a numerical semigroup if and only if $x \in \mathrm{msg}(S)$.*

We know that if S is a son of T in the tree $G(\triangle)$ then $S = T \backslash \{\alpha(S)\}$. From Lemma 25, we can deduce that $\alpha(S) \in \mathrm{msg}(T)$. Moreover, $T \backslash \{\alpha(S)\}$ is a positioned numerical semigroup with multiplicity $\mathrm{m}(T)$ and Frobenius number $\mathrm{F}(T)$.

Proposition 26. *Let $T \in \mathscr{P}(m,F)$ and $x \in \mathrm{msg}(T) \backslash \{m\}$ such that $x < F$. Then $T \backslash \{x\} \in \mathscr{P}(m,F)$ if and only if $F + m - x \in T \backslash \{x\}$.*

Proof. *Necessity.* By applying Lemma 25, we have that $T \backslash \{x\} \in \mathscr{L}(m,F)$. If $x \notin T \backslash \{x\}$ and as $T \backslash \{x\} \in \mathscr{P}(m,F)$, then we obtain that $F + m - x \in T \backslash \{x\}$.

Sufficiency. Let $h \in \mathbb{N}$ such that $h \notin T \backslash \{x\}$. We distinguish two cases.

(1) If $h = x$ then $F + m - x \in T \backslash \{x\}$.

(2) If $h \neq x$ then $h \notin T$. Since T is positioned we get that $F + m - h \in T$. If $F + m - h = x$ then $F + m - x = h \notin T \backslash \{x\}$ a contradiction. Hence $F + m - h \in T \backslash \{x\}$ and thus $T \backslash \{x\} \in \mathscr{P}(m,F)$. □

The next result characterizes the sons of vertex T in the tree $G(\triangle)$. We distinguish between two cases depending whether T is the root of the tree or not.

THEOREM 27. *Let $\triangle \in \mathscr{I}(m,F)$ and let T be a vertex of $G(\triangle)$. Then the sons of T are:*

(1) $\{\,\triangle\backslash\{x\} \mid x \in \mathrm{msg}(\triangle),\ \frac{F}{2} < x < F,\ x \neq \frac{F}{2}+m$ *and* $F+m \neq 2x\}$ *if* $T=\triangle$.

(2) $\{T\backslash\{x\} \mid x \in \mathrm{msg}(T),\ \alpha(T) < x < F$ *and* $F+m-x \in T\backslash\{x\}\}$ *if* $T \neq \triangle$.

Proof. (1) Assume that $x \in \mathrm{msg}(\triangle)$, $\frac{F}{2} < x < F$, $x \neq \frac{F}{2}+m$ and $F+m \neq 2x$. Then $x \in \mathrm{msg}\,(\triangle)\backslash\{m\}$ and $x < F$. By Lemma 25 we obtain that $\triangle\backslash\{x\} \in \mathscr{L}(m,F)$. Since $x \in \mathrm{msg}\,(\triangle)\backslash\{m\}$ then we have that $x-m \notin \triangle$ and as $\triangle$ is irreducible we obtain that either $F-(x-m) \in \triangle$ or $x-m = \frac{F}{2}$. But as $F+m \neq 2x$ we get that $F-(x-m) \in \triangle\backslash\{x\}$ and this implies that $\triangle\backslash\{x\} \in \mathscr{P}(m,F)$. It is clear that $\triangle\backslash\{x\}$ is a son of $\triangle$, because $\alpha\,(\triangle\backslash\{x\}) = x$ and thus $\triangle\backslash\{x\} \cup \{\alpha\,(\triangle\backslash\{x\})\} = \triangle$.

Conversely, if S is a son of $\triangle$ then $S \in \mathscr{P}(m,F)$ and $\triangle = S \cup \{\alpha(S)\}$. We deduce that $S = \triangle\backslash\{\alpha(S)\}$. From Lemma 25 and Proposition 26, we get that $\frac{F}{2} < \alpha(S) < F$ and $F+m-\alpha(S) \in \triangle\backslash\{\alpha(S)\}$. Therefore, $\alpha(S) \neq \frac{F}{2}$ and $F+m \neq 2\alpha(S)$.

(2) Suppose that $x \in \mathrm{msg}(T)$, $\alpha(T) < x < F$ and $F+m-x \in T\backslash\{x\}$. From Proposition 26, we have that $T\backslash\{x\} \in \mathscr{P}(m,F)$. Since $\alpha(T) < x$ then $\alpha(T\backslash\{x\}) = x$. Hence, we deduce that $T\backslash\{x\} \cup \alpha(T\backslash\{x\}) = T$ and thus $T\backslash\{x\}$ is a son of T.

Conversely, if S is a son of T then $S \in \mathscr{P}(m,F)$ and $T = S \cup \{\alpha(S)\}$. Then we get that $S = T\backslash\{\alpha(S)\}$. By applying 25 and Proposition 26, we have that $\alpha(S) \in \mathrm{msg}(T)\backslash\{m\}$ and $F+m-\alpha(S) \in T\backslash\{\alpha(S)\}$. Moreover, $\alpha(S) > \alpha(T)$. □

Now, we are able to give an algorithmic procedure to compute the class $[\triangle]$ from $\triangle \in \mathscr{I}(m,F)$.

ALGORITHM 28.
INPUT: $\triangle \in \mathscr{I}(m,F)$.
OUTPUT: The set $[\triangle]$.

1. $[\triangle] = \{\triangle\}$ and $B = \{\triangle\}$.
2. For each $S \in B$ compute the set $H(S) = \{T \mid T$ is son of $S\}$.
3. $C = \bigcup_{S\in B} H(S)$.
4. If $C = \emptyset$ then return $[\triangle]$.
5. $[\triangle] = [\triangle] \cup C$, $B = C$ go to step 2.

EXAMPLE 29. By Proposition 3 we have that $\triangle = \langle 5,7,9,11\rangle \in \mathscr{I}(5,13)$. Let us compute $[\triangle]$.

- $[\triangle] = \{\,\langle 5,7,9,11\rangle\,\}$ and $B = \{\,\langle 5,7,9,11\rangle\,\}$.

- $\Big\{x \in \mathrm{msg}(\langle 5,7,9,11\rangle),\ \frac{13}{2} < x < 13,\ x \neq \frac{13}{2}+5 \text{ and } 13+5 \neq 2x\Big\}=\{7,11\}$.
  Therefore,
  $H(\langle 5,7,9,11\rangle)=\Big\{\langle 5,7,9,11\rangle \setminus\{7\}, \langle 5,7,9,11\rangle \setminus\{11\}\Big\}=\Big\{\langle 5,9,11,12\rangle,\langle 5,7,9\rangle\Big\}$.
  Furthermore $\alpha(\langle 5,9,11,12\rangle) = 7$ and $\alpha(\langle 5,7,9\rangle) = 11$.

- $C = \Big\{ \langle 5,9,11,12\rangle , \langle 5,7,9\rangle \Big\}$.

- $[\triangle] = \Big\{ \langle 5,7,9,11\rangle , \langle 5,9,11,12\rangle , \langle 5,7,9\rangle \Big\}$.

- $B = \Big\{ \langle 5,9,11,12\rangle , \langle 5,7,9\rangle \Big\}$.

- $C = \emptyset$.

And thus $[\triangle] = \Big\{ \langle 5,7,9,11\rangle , \langle 5,9,11,12\rangle , \langle 5,7,9\rangle \Big\}$.

5. Apéry set and Wilf's conjecture. Let S be a numerical semigroup and let n be one of its nonzero elements. The Apéry set (named so in honour of [1]) of n in S is

$$\mathrm{Ap}(S,n) := \{s \in S \mid s-n \notin S\}.$$

The following result appears in [15, Lemma 2.4].

Lemma 30. *If S is a numerical semigroup and $n \in S\backslash\{0\}$, then $\mathrm{Ap}(S,n) = \{0 = w(0), w(1), \ldots, w(n-1)\}$, where $w(i)$ is the least element in S congruent with i modulo n, for all $i \in \{0,\ldots,n-1\}$.*

Note that the above lemma in particular implies that the cardinality of $\mathrm{Ap}(S,n)$ is n. Moreover, we easily deduce that for all $x \in S$ there exist a unique $(k,w) \in \mathbb{N}\times\mathrm{Ap}(S,n)$ such that $x = kn+w$. Therefore, we get that $\mathrm{Ap}(S,n)\cup\{n\}$ is a finite system of generators of S.

Proposition 31. *Let S be a numerical semigroup. Then S is positioned if and only if $\mathrm{F}(S)+2\mathrm{m}(S)-w \in S$ for all $w \in \mathrm{Ap}(S,\mathrm{m}(S))$.*

Proof. *Necessity.* If $w \in \mathrm{Ap}(S,\mathrm{m}(S))\backslash\{0\}$ then $w-\mathrm{m}(S) \in \mathbb{N}\backslash S$. As S is positioned, we obtain that $\mathrm{F}(S)+2\mathrm{m}(S)-w \in S$

Sufficiency. If $x \in \mathbb{N}\backslash S$ then there exist $k \in \mathbb{N}\backslash\{0\}$ such that $x+k\mathrm{m}(S) \in \mathrm{Ap}(S,\mathrm{m}(S))$. Then, by hypothesis, we have that $\mathrm{F}(S)+2\mathrm{m}(S)-(x+k\mathrm{m}(S)) \in S$ and thus $\mathrm{F}(S)+\mathrm{m}(S)-x-(k-1)\mathrm{m}(S) \in S$. Hence, $\mathrm{F}(S)+\mathrm{m}(S)-x \in S$ and consequently S is positioned. □

The following result improves the previous proposition.

Corollary 32. *Let S be a numerical semigroup. The following conditions are equivalent.*

(1) *S is positioned.*

(2) *if* $w \in \mathrm{Ap}(S, \mathrm{m}(S))$ *then either* $\mathrm{F}(S) + 2\mathrm{m}(S) - w \in \mathrm{Ap}(S, \mathrm{m}(S))$ *or* $\mathrm{F}(S) + \mathrm{m}(S) - w \in \mathrm{Ap}(S, \mathrm{m}(S))$.

Proof. 1) *implies* 2). From Proposition 31 we obtain that $\mathrm{F}(S)+2\mathrm{m}(S)-w \in S$. If $\mathrm{F}(S)+2\mathrm{m}(S)-w \notin \mathrm{Ap}(S, \mathrm{m}(S))$ then $\mathrm{F}(S)+\mathrm{m}(S)-w \in S$. Since $\mathrm{F}(S)-w \notin S$ we have that $\mathrm{F}(S) + \mathrm{m}(S) - w \in \mathrm{Ap}(S, \mathrm{m}(S))$.

2) *implies* 1). Follows trivially applying Proposition 31. □

Given a numerical semigroup S, in [7] is given an algorithm to obtain the set $\mathrm{Ap}(S, \mathrm{m}(S))$. This fact combined with Corollary 32 gives us a method to conclude whether S is or is not positioned. Note that $F(S) + \mathrm{m}(S) = \max\big(\mathrm{Ap}(S, \mathrm{m}(S))\big)$

EXAMPLE 33. We will find out if $S = \langle 5, 7, 9\rangle$ is a positioned numerical semigroup. Hence $\mathrm{Ap}(S, 5) = \{0, 7, 9, 16, 18\}$ and $18 = \mathrm{F}(S) + \mathrm{m}(S) = \max\big(\mathrm{Ap}(S, 5)\big)$. Since $\{18 - 0, 18 - 9, 18 - 18\} \subseteq \mathrm{Ap}(S, 5)$ and $\{23 - 7, 23 - 16\} \subseteq \mathrm{Ap}(S, 5)$, by Corollary 32, we have that S is positioned.

For S a numerical semigroup, if we denote by $N(S) = \{s \in S \mid s < \mathrm{F}(S)\}$ then $\{\mathrm{F}(S) - s \mid s \in N(S)\} \subseteq \mathbb{N}\backslash S$. Following the notation of [9] (see also for example [3]) the elements in the previous set will be called the gaps of the first type. To the remaining gaps of S, that is, $L(S) = \{x \in \mathbb{N}\backslash S \mid \mathrm{F}(S) - x \in \mathbb{N}\backslash S\}$ we will call the gaps of the second type. Let us denote by $\mathrm{n}(S)$ and $\mathrm{l}(S)$ the cardinal of $N(S)$ and $L(S)$, respectively.

The relation between $\mathrm{n}(S)$ and $\mathrm{l}(S)$ in a numerical semigroup is quite interesting. In fact, in [17] Wilf conjectured in 1978 that if S is a numerical semigroup then $\mathrm{g}(S) \leq (\mathrm{e}(S) - 1)\mathrm{n}(S)$. This question is still widely open and it is one of the most important problems in numerical semigroups theory. It is clear that $\mathrm{g}(S) = \mathrm{n}(S) + \mathrm{l}(S)$ and thus $\mathrm{l}(S) \leq (\mathrm{e}(S) - 2)\mathrm{n}(S)$ is another way to present the Wilfs Conjecture.

It is easy to prove the next result.

PROPOSITION 34. *If S is a numerical semigroup, then* $\mathrm{g}(S) = \frac{\mathrm{F}(S)+1+\mathrm{l}(S)}{2}$.

As a consequence of Propositions 5 and 34 we have the following.

COROLLARY 35. *If S is a positioned numerical semigroup, then* $\mathrm{l}(S) \leq \mathrm{m}(S) - 2$.

The following result improves the corollary above.

PROPOSITION 36. *Let S be a positioned numerical semigroup and $x \in \mathbb{N}$. Then $x \in L(S)$ if and only if* $\{x + \mathrm{m}(S), \mathrm{F}(S) - x + \mathrm{m}(S)\} \subseteq \mathrm{Ap}(S, \mathrm{m}(S))$.

Proof. *Necessity*. Suppose that $x \in L(S)$, then $x \in \mathbb{N}\backslash S$ and $\mathrm{F}(S) - x \in \mathbb{N}\backslash S$. Hence $\mathrm{F}(S) + \mathrm{m}(S) - x \in S$ and $\mathrm{F}(S) + \mathrm{m}(S) - (\mathrm{F}(S) - x) \in S$. Consequently, both $\mathrm{F}(S) - x + \mathrm{m}(S)$ and $x + \mathrm{m}(S)$ belong to $\mathrm{Ap}(S, \mathrm{m}(S))$.

Sufficiency. If $\mathrm{F}(S) - x + \mathrm{m}(S)$ and $x + \mathrm{m}(S)$ belong to $\mathrm{Ap}(S, \mathrm{m}(S))$ then $x \in \mathbb{N}\backslash S$ and $\mathrm{F}(S) - x \in \mathbb{N}\backslash S$. Therefore $x \in L(S)$. □

As a consequence of Proposition 36, we obtain the following result.

Corollary 37. *If S is a positioned numerical semigroup, then* $L(S) = \{w - \mathrm{m}(S) \mid w \in \mathrm{Ap}(S, \mathrm{m}(S))$ *and* $\mathrm{F}(S) + 2\mathrm{m}(S) - w \in \mathrm{Ap}(S, \mathrm{m}(S))\}$.

The next result gives us an effective way to compute $L(S)$ whenever S is a positioned numerical semigroup.

Proposition 38. *If S is a positioned numerical semigroup, then* $L(S) = \{w - \mathrm{m}(S) \mid w \in \mathrm{Ap}(S, \mathrm{m}(S))$ *and* $\mathrm{F}(S) + \mathrm{m}(S) - w \notin \mathrm{Ap}(S, \mathrm{m}(S))\}$.

Proof. If $x \in L(S)$, by using Corollary 37, then $x = w - \mathrm{m}(S)$ with $w \in \mathrm{Ap}(S, \mathrm{m}(S))$ and $\mathrm{F}(S) + 2\mathrm{m}(S) - w \in \mathrm{Ap}(S, \mathrm{m}(S))$. Hence, $\mathrm{F}(S) + \mathrm{m}(S) - w \notin S$ and so $\mathrm{F}(S) + \mathrm{m}(S) - w \notin \mathrm{Ap}(S, \mathrm{m}(S))$.

Conversely, if $w \in \mathrm{Ap}(S, \mathrm{m}(S))$ and $\mathrm{F}(S) + \mathrm{m}(S) - w \notin \mathrm{Ap}(S, \mathrm{m}(S))$, by Corollary 32, $\mathrm{F}(S) + 2\mathrm{m}(S) - w \in \mathrm{Ap}(S, \mathrm{m}(S))$. Applying Corollary 37, we obtain that $w - \mathrm{m}(S) \in L(S)$. □

Example 39. By Example 33 we have that $S = \langle 5, 7, 9 \rangle$ is a positioned numerical semigroup. We have that $\mathrm{Ap}(S, 5) = \{0, 7, 9, 16, 18\}$ and $\mathrm{F}(S) + \mathrm{m}(S) = 18$. Since $\{w \in \mathrm{Ap}(S, 5) \mid 18 - w \notin \mathrm{Ap}(S, 5)\} = \{7, 16\}$, by applying Proposition 38, we get that $L(S) = \{2, 11\}$.

Now our aim in this section is to prove the Wilf's Conjecture for positioned numerical semigroups. The conditions 1) 2) and 3) in the next result are deduced from [10], [16] and [8], respectively.

Lemma 40.

(1) *If S is an elementary numerical semigroup then S verifies the Wilf's Conjecture.*

(2) *If S is a numerical semigroup with* $\mathrm{e}(S) = 2$ *then S verifies the Wilf's Conjecture.*

(3) *If S is a numerical semigroup with* $\mathrm{e}(S) = 3$ *then S verifies the Wilf's Conjecture.*

Theorem 41. *If S is a positioned numerical semigroup then S verifies the Wilf's Conjecture.*

Proof. From Lemma 40, we can suppose that $\mathrm{F}(S) > 2\mathrm{m}(S)$ and $\mathrm{e}(S) > 3$. If $x \in L(S)$, by using Proposition 36, then $\{x + \mathrm{m}(S), \mathrm{F}(S) - x + \mathrm{m}(S)\} \subseteq S$. Since $x + \mathrm{m}(S) + \mathrm{F}(S) - x + \mathrm{m}(S) = \mathrm{F}(S) + 2\mathrm{m}(S)$ we obtain that $\min\{x + \mathrm{m}(S), \mathrm{F}(S) - x + \mathrm{m}(S)\} \leq \frac{\mathrm{F}(S)+2\mathrm{m}(S)}{2} < \mathrm{F}(S)$. Therefore, the correspondence $f : L(S) \to N(S)$, defined by $f(x) = \min\{x + \mathrm{m}(S), \mathrm{F}(S) - x + \mathrm{m}(S)\}$ is a map, such that, if $x, y \in L(S)$ and $f(x) = f(y)$ then $x = y$ or $y = \mathrm{F}(S) - x$. Hence if $s \in N(S)$ then $\#\{x \in L(S) \mid f(x) = s\} \leq 2$. Consequently, $\mathrm{l}(S) \leq 2\mathrm{n}(S)$ and thus $\mathrm{l}(S) \leq \big(\mathrm{e}(S) - 2\big)\mathrm{n}(S)$. This proves that a positioned numerical semigroup verifies the Wilf's Conjecture. □

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