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ORIGINAL PAPER

Three-player NIM with podium rule

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Abstract

If a combinatorial game involves more than two players, the problem of coalitions arises. To avoid the problem, Shuo-Yen Robert Li analyzed three-player NIM with the podium rule, that is, if a player cannot be last, he should try to be last but one. With that simplification, he proved that a disjunctive sum of NIM piles is a $\mathcal{P}$-position if and only if the sum modulo 3 of the binary representations of the piles is equal to zero. In this paper, we extend the result in order to understand the complete characterization of the outcome classes, the possible reductions of the game forms, the equivalence classes under the equality of games and related canonical forms.

Keywords Combinatorial game theory · Impartial games · NIM · Three-player games · Podium rule

1 Introduction

Traditional combinatorial game theory (Albert et al. 2007; Berlekamp et al. 1982; Conway 1976; Siegel 2013) studies perfect information games in which there are no chance devices (e.g. dice) and two players take turns moving alternately. Under *normal play*, the last player to move wins; under *misère play*, the last player to move

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loses. The *options* of a game are all those positions which can be reached in one move. Using the notation of combinatorial game theory, where Left and Right are the players, game forms can be expressed recursively as $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$ where $G^{\mathcal{L}}$ are the Left options and $G^{\mathcal{R}}$ are the Right options of G. Often, games decompose into components during the play. In those situations, a player has to choose a component in which to play and, so, the concept of *disjunctive* sum is formalized: $G + H = \{G^{\mathcal{L}} + H, G + H^{\mathcal{L}} \mid G^{\mathcal{R}} + H, G + H^{\mathcal{R}}\}$. A game belongs to one of four outcome classes: $\mathcal{L}$—$\mathcal{L}$eft wins, regardless of moving first or second; $\mathcal{R}$—$\mathcal{R}$ight wins, regardless of moving first or second; $\mathcal{N}$—$\mathcal{N}$ext player wins regardless of whether this is Left or Right; $\mathcal{P}$—$\mathcal{P}$revious player wins regardless of whether this is Left or Right. When G is impartial, Left options and Right options are the same for the game and all its subpositions. In that case, instead of Left and Right, we simply speak about Previous and Next. Also, in that case, we only have two outcome classes, $\mathcal{P}$ and $\mathcal{N}$ and, instead of $G = \{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$, we use $G = \{G'\}$ where G' is the set of options of G (the options are the same for both players).

An example of a combinatorial *impartial* ruleset is the classic game of NIM, first studied by Bouton (1902). NIM is played with piles of matches. On his turn, each player can remove any number of matches from any pile. The *nim-sum* of two nonnegative integers is the exclusive or (XOR), written $\oplus$, of their binary representations. It can also be described as adding the numbers in binary without carrying. With normal play convention, if we have the disjunctive sum of NIM piles of sizes $g_1, \ldots, g_n$, we only need to compute the nim-sum $g_1 \oplus g_2 \oplus \cdots \oplus g_n$ in order to check the outcome class of the position. If the result is 0, we have a $\mathcal{P}$-position; if the result is different than zero, we have a $\mathcal{N}$-position. Here we are concerned with a version of NIM for *three players*, played under *normal play*. As we will see, there are some similarities with the case of two players.

It is a challenging problem to extend combinatorial game theory to three or more players. Of course, games represent a conflict of interests between the players: if the conflict involves more than two players then a new problem, coalitions, arises. A crazy example, where the least skilled player has the advantage, is discussed in Kilgour and Steven (1997). In that "Truel", the advantage of the worst player could be dissipated through ingenious coalitions. Basically, if we want to analyze three-player games, we have to clarify first the following fundamental question:

What is the strategy of a player who can no longer win?

There are some different ways to answer. Those answers give us possible *conventions* to have a starting point for a mathematical research. Two possibilities follow.

(1) Free strategy;
(2) Attempt to be second-to-last.

In order to analyze three-player impartial games, James Propp adopted (1), a kind of "agnostic approach" (Propp 2000). He defined the possible outcome classes and, with that, he understood in what circumstances one player has a winning strategy against the combined forces of the other two. With no assumptions about how the players behave, there are four outcome classes:

– G is an $\mathcal{N}$-position if it has some $\mathcal{P}$-position as an option.

- G is an $\mathcal{O}$-position ($\mathcal{O}$ther) if all of its options are $\mathcal{N}$-positions, *and* it has at least one option (this proviso prevent us from mistakenly classifying 0 as an $\mathcal{O}$-position).
- G is an $\mathcal{P}$-position if all of its options are $\mathcal{O}$-positions.
- G is an $\mathcal{Q}$-position ($\mathcal{Q}$ueer) if none of the above conditions is satisfied.

Regarding that, Propp studied the "behaviors" of the disjunctive sums in terms of the outcome classes. For instance, $\mathcal{P} + \mathcal{P} = \mathcal{Q}$ is possible (e.g. $\{\{2\}\} + \{\{2\}\}$).

In order to analyze three-player NIM, Shuo-Yen Robert Li adopted (2), a kind of "lesser evil approach" (Li 1978). First, there are fewer outcome classes:

- G is an $\mathcal{N}$-position if it has some $\mathcal{P}$-position as an option.
- G is an $\mathcal{O}$-position ($\mathcal{O}$ther) if it has no $\mathcal{P}$-positions as options, *and* it has at least one $\mathcal{N}$-position as an option.
- G is a $\mathcal{P}$-position if all of its options are $\mathcal{O}$-positions.

Second, he proved the following elegant result:

Theorem 1.1 (Li's characterization of $\mathcal{P}$-positions) *Let $G = g^1 + \cdots + g^n$ be a disjunctive sum of* NIM *piles. Let S be the sum mod 3 of the binary representations of the g^i. The game G is a $\mathcal{P}$-position if and only if $S = 0$.*

In fact, using the sum mod N of the binary representations of the g^i, the result works for N players, as Li proved. We can already observe the similarities with the case of two players (see also Benesh and Gaetz 2016).

In this paper, we extend the result in order to understand the complete characterization of the outcome classes. After, we present our main contributions: reductions of the game forms and the characterization of the equivalence classes modulo three-player NIM with podium rule. That is, for each G we identify a smallest form G' such that for any three-player NIM position X, both $G + X$ and $G' + X$ have the same outcomes.

This paper is self-contained.

2 Three-player NIM with podium rule: $\mathcal{G}$-values and outcomes

First, we define NIM positions (or NIM forms). The definition works for both two-player NIM or three-player NIM.

Definition 2.1 A star(0), or simply 0, is the form $\{\,\}$ (no options). A star(1), or simply $*$, is the form $\{0\}$ (0 is the only option). In general, a star(n), or simply $*n$, is the form defined recursively by $\{0, *, *2, \ldots, *(n-1)\}$.

Definition 2.2 A NIM pile is a $*n$ form—n is the size of the pile. A NIM position G is a disjunctive sum of NIM piles, $p(G)$ is the number of piles with positive sizes of G, and $m(G)$ is the standard sum of the sizes of the piles of G.

As we have mentioned in the introduction, in this paper, we assume the following convention:

PODIUM RULE: *If a player cannot be last to move, he should try to be last but one.*

With that, there are three outcome classes, $\mathcal{N}$, $\mathcal{O}$, and $\mathcal{P}$. We extend Li's characterization for all three outcome classes.

NOTATION:

(1) $G = g^1 + \cdots + g^n$ is the disjunctive sum of n piles.
(2) $g^i = g^i_k \cdots g^i_0$ is the binary representation of g^i ($g^i_j \in \{0, 1\}$). We assume $g^i_k \neq 0$ for $g^i > 0$. If $g^i = 0$, then $k = 0$ and $g^i_0 = 0$. We use an analogous approach for ternary representations.
(3) We write $o(G) = \mathcal{N}$ or $G \in \mathcal{N}$ if the outcome of G is $\mathcal{N}$. The same for $\mathcal{P}$ and $\mathcal{O}$.
(4) If a is a positive ternary or a binary representation, then $\mathrm{lmd}(a)$ means *left-most non-zero digit* of a. If $a = 0$, we define $\mathrm{lmd}(a) = 0$.
(5) Let a and b be non-negative integers with $2^k \leq a < 2^{k+1}$. If $c = b \times 2^{k+1} + a$, then we call a a *suffix* of c. In addition, if S is a set of numbers, then a *is a suffix of* S means that a is the suffix of a' for some $a' \in S$.

For example, 3 is a suffix of 23 since $23 = 5 \times 2^2 + 3$, or, in binary, 11 is a suffix of 10111. Let $S = \{15, 8, 23, 5\}$ then 3 is a suffix of S but 2 is not. (In binary, $S = \{1111, 1000, 10111, 101\}$.)

We want to study the particular case of three-player NIM with podium rule. Therefore, as usual in combinatorial game theory, we define the equality of forms *modulo* three-player NIM.

Definition 2.3 Consider two forms of three-player NIM, $G = g^1 + \cdots + g^n$ and $H = h^1 + \cdots + h^m$. We say that $G \equiv H$ (mod 3- NIM) if

$$o(G + X) = o(H + X), \text{ for all } X = x^1 + \cdots + x^k \text{ (an arbitrary 3- NIM game).}$$

Observation 2.4 If X were any game form, we would have the equality of games in general. In that case, G would act like H in any context (in the presence of any distinguish form X). If we consider the restriction modulo 3- NIM, "any context" means "any 3- NIM context", that is, "in the presence of any distinguish 3- NIM form X". When $G \equiv H$ (mod 3- NIM), it is possible to have G different than H in general terms; however, it is impossible to distinguish G from H with a 3- NIM form. From now on, instead of "$\equiv$ (mod 3- NIM)", we just write "$\equiv$".

It is a well-known fact that, for two players, all impartial games are equal, under the general equality of games, to some NIM pile [Sprague–Grundy Theory, (Grundy 1939; Sprague 1935)]. The values involved in two-player NIM are the already defined stars (or nimbers). If $G = g^1 + \cdots + g^n$ with k equaling $g^1 \oplus \cdots \oplus g^n$, then we define $\mathcal{G}(G)$ to be k, and we conclude that G is equivalent to a NIM pile of size k. For three-player NIM with podium rule, the information given by the generalization of the $\mathcal{G}$-value is not so strong. First, we show how it works.

Definition 2.5 Let $G = g^1 + \cdots + g^n$ be a form of three-player NIM. Then,

$$\mathcal{G}_3(G) = g^1 \oplus_3 \cdots \oplus_3 g^n$$

is the *ternary sum* without carrying of the *binary representations* of g^i. We say that $\mathcal{G}_3(G)$ is the "three-player $\mathcal{G}$-value of G" or simply the "$\mathcal{G}_3$-value of G".

Example 2.6 Consider $G = 15 + 14 + 23 + 5$. Then $\mathcal{G}_3(G) = 12100$.

$$\begin{array}{rccccc} & & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 & 0 \\ & 1 & 0 & 1 & 1 & 1 \\ \oplus_3 & & & 1 & 0 & 1 \\ \hline & 1 & 2 & 1 & 0 & 0 \end{array}$$

◇

Definition 2.7 Let $a = a_k \cdots a_0$ ($a_i \in \{0, 1, 2\}$) be a ternary representation and $b = b_j \cdots b_0$ ($b_i \in \{0, 1\}$) be a binary representation. We say that b is *linked to* a if, for all $0 \le i \le k$, $a_i = 1 \Rightarrow b_i = 1$ and $a_i = 2 \Rightarrow b_i = 0$.

Example 2.8 Both $b = 10111$ and $c = 10101$ are linked to $a = 12100$. ◇

Observation 2.9 Of course, if b is linked to a and b is a suffix of b', then b' is also linked to a.

Definition 2.10 Let $G = g^1 + \cdots + g^n$ be a form of three-player NIM. The set of *linked* options of G is

$$\langle G \rangle = \begin{cases} \emptyset, & \text{if } \operatorname{lmd}(\mathcal{G}_3(G)) \neq 1, \\ \left\{ g^i : g^i \text{ is linked to } \mathcal{G}_3(G) \right\}, & \text{if } \operatorname{lmd}(\mathcal{G}_3(G)) = 1. \end{cases}$$

Example 2.11 Consider again $G = 15 + 14 + 23 + 5$. Then, $\langle G \rangle = \{23\}$. ◇

Example 2.12 Consider $H = 31 + 29 + 23 + 18$. Then, $\langle H \rangle = \{18, 23\}$. ◇

The importance of the previous definition is expressed in the following theorem.

Theorem 2.13 *Let* $G = g^1 + \cdots + g^i + \cdots + g^n$ *be a position of three-player* NIM. *The following statements are equivalent:*

(1) $g^i \in \langle G \rangle$;
(2) *there exists* $g'^i < g^i$ *such that* $\mathcal{G}_3(g^1 + \cdots + g'^i + \cdots + g^n) = 0$.

Proof In both implications, let k be the place value of the $\operatorname{lmd}(\mathcal{G}_3(G))$.
($\Rightarrow$) Let $g^i \in \langle G \rangle$. By definition, because $\langle G \rangle$ is not empty, $\operatorname{lmd}(\mathcal{G}_3(G)) = 1$. Also, by definition, $g^i > 0$ (we cannot have at the same time $g^i = 0$, $g^i \in \langle G \rangle$, and $\operatorname{lmd}(\mathcal{G}_3(G)) = 1$). Construct g'^i in the following way:

(a) Replace the kth digit "1" of g^i by "0". That makes $\operatorname{lmd}(\mathcal{G}_3(G')) = 0$ and justifies why we can change a "0" to a "1" in other smaller place values of g^i.
(b) For the digits of g^i at places $k' < k$:

- if the k'th digit of $\mathcal{G}_3(G)$ is "0", do nothing;
- if the k'th digit of $\mathcal{G}_3(G)$ is "1", replace the k'th digit "1" of g^i by "0";
- if the k'th digit of $\mathcal{G}_3(G)$ is "2", replace the k'th digit "0" of g^i by "1".

By construction, $\mathcal{G}_3(g^1 + \cdots + g'^i + \cdots + g^n) = 0$.
($\Leftarrow$) Suppose that there exists $g'^i < g^i$ such that $\mathcal{G}_3(g^1 + \cdots + g'^i + \cdots + g^n) = 0$. If $\mathrm{lmd}(\mathcal{G}_3(G)) = 0$, then $\mathrm{lmd}(g^1 + \cdots + g'^i + \cdots + g^n) \neq 0$ since $g^i \neq g'^i$, which is a contradiction. If $\mathrm{lmd}(\mathcal{G}_3(G)) = 2$, then we have $g^i_k = 0$ and $g'^i_k = 1$. Since $g'^i < g^i$, we must have $0 = g_{k+l} \neq g'_{k+l} = 1$ for some $l > 0$. But this implies $\mathrm{lmd}(\mathcal{G}_3(G)) \neq 0$ and $\mathcal{G}_3(G) \neq 0$. This is also a contradiction, so we conclude that $\mathrm{lmd}(\mathcal{G}_3(G)) = 1$.

Consider $k' \leq k$. If the k'th digit of $\mathcal{G}_3(G)$ is “2”, then we must have $g^i_{k'} = 0$ and $g'^i_{k'} = 1$; if the k'th digit of the $\mathcal{G}_3(G)$ is “0”, then we must have $g^i_{k'} = 1$ and $g'^i_{k'} = 0$. So, g is linked to $\mathcal{G}_3(G)$ and $g^i \in \langle G \rangle$. □

Now, we are ready to expand Li’s characterization for all outcomes.

Theorem 2.14 *Let $G = g^1 + \cdots + g^n$ be a position of three-player* NIM.

$$o(G) = \begin{cases} \mathcal{P}, & \text{if } \mathcal{G}_3(G) = 0, \\ \mathcal{N} & \text{if } \mathcal{G}_3(G) \neq 0 \text{ and } \langle G \rangle \neq \emptyset, \\ \mathcal{O} & \text{if } \mathcal{G}_3(G) \neq 0 \text{ and } \langle G \rangle = \emptyset. \end{cases}$$

Proof If G is 0, 1, 1 + 1, or 2, the theorem is trivially true because $0 \in \mathcal{P}$, $1 \in \mathcal{N}$, $1 + 1 \in \mathcal{O}$ and $2 \in \mathcal{N}$.

Case 1: Consider $\mathcal{G}_3(G) = 0$ and a move to $G' = g^1 + g^2 + \cdots + g'^i + \cdots + g^n$. Let k be the place value of the left-most “1” that is replaced by “0” in g^i. At the kth place, $\mathrm{lmd}(\mathcal{G}_3(G')) = 2$ and, by definition, $\langle G' \rangle = \emptyset$. By induction, $G' \in \mathcal{O}$ and, because G' is arbitrary, $G \in \mathcal{P}$.

Case 2: Consider $\mathcal{G}_3(G) \neq 0$ and $\langle G \rangle \neq \emptyset$. By Theorem 2.13, there is a G' such that $\mathcal{G}_3(G') = 0$. By induction, $G' \in \mathcal{P}$ and, so, $G \in \mathcal{N}$.

Case 3: Consider $\mathcal{G}_3(G) \neq 0$ and $\langle G \rangle = \emptyset$. There are two possibilities:

(1) $\mathrm{lmd}(\mathcal{G}_3(G)) = 2$;
(2) $\mathrm{lmd}(\mathcal{G}_3(G)) = 1$ and there is no g^i linked to $\mathcal{G}_3(G)$.

Case 3.1: We have that $\mathrm{lmd}(\mathcal{G}_3(G)) = 2$. Let k be the place value of the $\mathrm{lmd}(\mathcal{G}_3(G))$. It is impossible to adjust the digit at the kth place with a single move in order to get an option $G' \in \mathcal{P}$. On the other hand, without loss of generality, consider g^1 and g^2 with a “1” at the kth place. We will show that we can always move in g^1, changing G into G', such that $g^2 \in \langle G' \rangle$. In order to choose such a move, we don’t change the digits at places larger than k (those digits will be the same in $g^{1'}$ and g^1), and we execute the following procedure.

(a) We replace the kth digit of g^1 by “0”. That makes $\mathrm{lmd}(\mathcal{G}_3(G')) = 1$ and justifies why we can change a “0” to a “1” in other smaller place values.
(b) For the digits of g^1 at places $j \in \{0, \ldots, k - 1\}$, we summarize the procedure in the following table, where $\mathcal{G}_3(G)_j$ and $\mathcal{G}_3(G')_j$ are, respectively, the jth digits of the ternary representations of $\mathcal{G}_3(G)$ and $\mathcal{G}_3(G')$. For instance, the last row of the table is the case where $\mathcal{G}_3(G)_j = 2$, $g^2_j = 1$, and $g^1_j = 1$. In that case, we make g'^1_j to be 0, resulting in $\mathcal{G}_3(G')_j = 1$. Then, $\mathcal{G}_3(G')_j = 1$ and $g^2_j = 1$, so the jth digits of $\mathcal{G}_3(G)_j$ and g^2_j are properly linked.

$\mathcal{G}_3(G)_j$	g_j^2	g_j^1	$g_j'^1$	$\mathcal{G}_3(G')_j$
0	0	0	0	0
0	0	1	1	0
0	1	0	0	0
0	1	1	1	0
1	0	0	1	2
1	0	1	0	0
1	1	0	0	1
1	1	1	1	1
2	0	0	0	2
2	0	1	1	2
2	1	0	1	0
2	1	1	0	1

There is no $G' \in \mathcal{P}$ (Theorem 2.13) but, by induction, there is at least one $G' \in \mathcal{N}$. Hence, $G \in \mathcal{O}$.

Case 3.2: We have that $\text{lmd}(\mathcal{G}_3(G)) = 1$. Let k be the place of the $\text{lmd}(\mathcal{G}_3(G))$. There is no $g^i \in \langle G \rangle$ and, because of that, there is no $G' \in \mathcal{P}$ (Theorem 2.13). Consider g^1, a pile of G such that $g_k^1 = 1$ (its existence is mandatory because $\mathcal{G}_3(G)_k = 1$). For some place $k' < k$, we must have ($g_{k'}^1 = 0$ and $\mathcal{G}_3(G)_{k'} = 1$) or ($g_{k'}^1 = 1$ and $\mathcal{G}_3(G)_{k'} = 2$); if not, g^1 would be linked to $\mathcal{G}_3(G)$. Let k' be the largest under one of these conditions, and let g^2 be a pile such that $g_{k'}^2 = 1$ (again, because of the conditions, its existence is mandatory). We can move in g^2, changing G into G' such that $g^1 \in \langle G' \rangle$. In order to choose such a move, after a removal of the "1" at the k'th place, replace in the previous table g_j^2 by g_j^1, g_j^1 by g_j^2, and $g_j'^1$ by $g_j'^2$, and, then, execute the procedure for the orders $j < k'$.
Again, there is no $G' \in \mathcal{P}$ and, by induction, there is at least one $G' \in \mathcal{N}$. Hence, $G \in \mathcal{O}$. □

Example 2.15 Consider again $G = 15 + 14 + 23 + 5$ ($\mathcal{G}_3(G) = 12100$). The binary representation of 23 is 10111. Because 10111 is linked to 12100 we have that $\langle G \rangle \neq \emptyset$ and we are in the second case of the Theorem 2.14. Hence, G is a $\mathcal{N}$-position. Naturally, the winning move corresponds to replacing the binary 10111 by the binary 1011, that is, to replace the pile of size 23 by a pile of size 11. ◇

Observation 2.16 In three-player NIM it is possible to have two different games with the same $\mathcal{G}$-value. For instance, $\mathcal{G}_3(2+2+2) = \mathcal{G}_3(0) = 0$. However, $2+2+2 \neq 0$ because, with $X = 3 + 1$, we have that $2+2+2+X \in \mathcal{N}$ ($1+1+2+2+3$ is a winning move) and $X \in \mathcal{O}$ ($\mathcal{G}_3(X) \neq 0$, and there is no winning move).

With Theorem 2.14, we can argue that we don't have a Sprague-Grundy Theory for impartial games with podium rule. The next theorem shows why.

Theorem 2.17 *The game form* $\{2\}$*, the form that only has* 2 *as an option, is not equal to any form of three-player* NIM $g^1 + \cdots + g^n$.

Proof It is easy to check that

(i) $\{2\}$ is a $\mathcal{O}$-position;
(ii) $\{2\}+1$ is a $\mathcal{P}$-position (both $2+1$ and $\{2\}$ are $\mathcal{O}$-positions);
(iii) $\{2\}+2+2$ is a $\mathcal{N}$-position (because $2+2+2$ is a $\mathcal{P}$-position).

Suppose $\{2\}=g^1+\cdots+g^n$.
If $\mathcal{G}_3(g^1+\cdots+g^n)\neq 2$, then letting $X=1$ gives $o(\{2\}+X)=\mathcal{P}$ and $o(g^1+\cdots+g^n+X)\neq\mathcal{P}$, contradicting $\{2\}=g^1+\cdots+g^n$. Therefore, $\mathcal{G}_3(g^1+\cdots+g^n)=2$, and it follows that $\mathcal{G}_3(g^1+\cdots+g^n+2+2)=22$. Because $\langle\mathcal{G}_3(G+2+2)\rangle=\emptyset$, by Theorem 2.14, $g^1+\cdots+g^n+2+2$ is an $\mathcal{O}$-position. Letting $X=2+2$, $\{2\}+X$ is a $\mathcal{N}$-position, and $g^1+\cdots+g^n+X$ is an $\mathcal{O}$-position, contradicting $\{2\}=g^1+\cdots+g^n$. □

3 Equivalence classes under the equality of games and related canonical forms

In this section, we show when two forms are equivalent *modulo* 3-NIM. With this knowledge, it will be possible to define canonical forms for the equivalence classes of $[G]_{\equiv}$. First, an important particular case.

Theorem 3.1 $1+1+1\equiv 0$.

Proof Let $X=x_1+\cdots x_n$ be an arbitrary game.

First, we argue that $1+1+1+X\in\mathcal{P}$ if and only if $X\in\mathcal{P}$. This follows because $\mathcal{G}_3(1+1+1+x_1+\cdots x_n)=0$ if and only if $\mathcal{G}_3(x_1+\cdots x_n)=0$.

Second, we argue that $1+1+1+X\in\mathcal{N}$ if and only if $X\in\mathcal{N}$.

If X has an option X' such that $\mathcal{G}_3(X')=0$, then $X\in\mathcal{N}$. If so, $1+1+1+X\in\mathcal{N}$ as well, because $\mathcal{G}_3(1+1+1+X')=0$ since $1\oplus_3 1\oplus_3 1=0$. On the other hand, if $1+1+1+X'$ is an option of $1+1+1+X$ such that $\mathcal{G}_3(1+1+1+X')=0$, then $\mathcal{G}_3(X')=0$, and $X\in\mathcal{N}$. Finally, if $\mathcal{G}_3(1+1+X)=0$, then in $1+1+1+X$ there is also a winning removal of a "1" in the units place of X. That move is a winning move in X when played alone. Hence, $X\in\mathcal{N}$. Therefore, $1+1+1+X\in\mathcal{N}$ if and only if $X\in\mathcal{N}$.

Third, $1+1+1+X\in\mathcal{O}$ if and only if $X\in\mathcal{O}$ is just a consequence of the previous cases and the fact that there are only three outcome classes. □

To prove the general case, we need some definitions and a lemma.

Definition 3.2 Let $G=g^1+\cdots+g^n$ and $H=h^1+\cdots+h^m$ be two forms, indexed so that $g^1=h^1, g^2=h^2,\ldots,g^k=h^k$, and $g^i\neq h^j$ for all $i,j>k$. The set $G\cap H=\{g^1,\ldots g^k\}=\{h^1,\ldots h^k\}$ are the *common piles of G and H*. The sets $G\backslash H=\{g^{k+1},g^{k+2},\ldots,g^n\}$ and $H\backslash G=\{h^{k+1},h^{k+2},\ldots,h^m\}$ are the *extra G-piles with respect to H* and the *extra H-piles with respect to G*.

Lemma 3.3 *Let $G=g^1+\cdots+g^n$ and $H=h^1+\cdots+h^m$ be two forms, neither with three piles of size 1. Then $G\equiv H$ if and only if $\mathcal{G}_3(G)=\mathcal{G}_3(H)$, all elements of $G\backslash H$ are suffixes of $\{h^1,\ldots,h^m\}$ and all the elements of $H\backslash G$ are suffixes of $\{g^1,\ldots,g^n\}$.*

To show that this relation is not trivial, let $G = 7 + 6 + 3$ and $H = 7 + 6 + 2 + 1$. Now $\mathcal{G}_3(G) = \mathcal{G}_3(H)$, the set of common piles is $\{7, 6\}$, and for the extra piles, both 1 and 3 are suffixes of 7, and 2 is a suffix of 6.

Proof ($\Leftarrow$) Consider an arbitrary game X. By Theorem 2.14, it is clear that $G+X \in \mathcal{P}$ if and only if $H + X \in \mathcal{P}$ due the fact $\mathcal{G}_3(G) = \mathcal{G}_3(H)$. Also, $G + X \in \mathcal{N}$ if and only if $H + X \in \mathcal{N}$ because every move in $G\backslash H$ has a corresponding move in a pile of $\{h^1, \ldots, h^m\}$ with the same suffix, and every move in $H\backslash G$ has a corresponding move in a pile of $\{g^1, \ldots, g^n\}$ with the same suffix (recall from Observation 2.9). Again, the fact that $G + X \in \mathcal{O}$ if and only if $H + X \in \mathcal{O}$ is a consequence of the previous cases and the fact that there are only three outcome classes.

($\Rightarrow$) Suppose that $G \equiv H$ (so $\mathcal{G}_3(G) = \mathcal{G}_3(H)$), and let $g \in G\backslash H$ that is not a suffix of any $h^1, \ldots, h^m$.

Case 1: g is not a power of 2. Let b be the binary representation of g. Let $X = x^1 + \cdots + x^n$ be a sum of powers of two such that $\mathcal{G}_3(G + X) = b'$, where b' is a ternary representation gotten by replacing all the "0"s of b by "2"s. The game X exists because the powers of two form a basis of $\mathbb{Z}_2^\infty$. A pile is linked to b' if and only if it has b as suffix. Because of that, $G + X \in \mathcal{N}$ and $H + X \notin \mathcal{N}$. This is a contradiction.

Case 2: g is a power of 2 greater than 1. Let b be the binary representation of g. Let $X = x^1 + \cdots + x^n$ be a sum of elements of the form $2^j - 1$ such that $\mathcal{G}_3(G + X) = b'$, where b' is a ternary representation gotten by replacing all the "0"s of b by "2"s. The game X exists because the elements $2^j - 1$ form a basis of $\mathbb{Z}_2^\infty$. A pile is linked to b' if and only if it has b as suffix. Because of that, $G + X \in \mathcal{N}$ and $H + X \notin \mathcal{N}$. This is a contradiction.

Case 3: $g = 1$. In this case, all h^i are even (or else g would be a suffix). Therefore, because $\mathcal{G}_3(G) = \mathcal{G}_3(H)$ it is mandatory that the number of 1's in the units place of the piles of G is a multiple of 3. If we have three singletons in G, we have a contradiction, regarding the statement of the theorem. If not, we have an odd pile larger than 1 in $G\backslash H$ and a situation already studied in the previous cases. □

The characterization of the equality is the following theorem.

Theorem 3.4 *Let G and H be two forms with no three singular piles. Then, $G \equiv H$ if and only if*

(a) $\mathcal{G}_3(G) = \mathcal{G}_3(H)$, *and*
(b) *all elements of $G\backslash H$ and all the elements of $H\backslash G$ are suffixes of $G \cap H$.*

Proof Due Lemma 3.3, we only have to prove that all elements of $G\backslash H$ are suffixes of $G \cap H$ (without loss, we prove the same for $H\backslash G$). Suppose that there is $g^i \in G\backslash H$ that is a suffix of $H\backslash G$, but not of $G \cap H$. Assume that g^i is the largest possible in those conditions. That means that g^i is a suffix of some $h^j \in H\backslash G$. Again, by Lemma 3.3, h^j must be a suffix of $\{g^1, \ldots, g^n\}$. If h^j is a suffix of $G \cap H$, then g^i is a suffix of $G \cap H$ as well, which is a contradiction. Therefore, h^j is just a suffix of $g' \in G\backslash H$. If $g' = g^i$ then $g' = g^i = h^j$ and $g = h^j \in G \cap H$ which is a contradiction. We must have $g' > g^i$, contradicting the maximality of g^i. □

Finally, we define canonical forms for three-player NIM. In the following definition, we use the notation established in Definition 2.2.

Definition 3.5 (*order for equivalence classes*) Let G, H be two NIM forms such that $G \equiv H$. Then $G \leq_R H$ if one the following items holds:

(1) $p(G) < p(H)$;
(2) $p(G) = p(H)$ and $m(G) < m(H)$;
(3) $p(G) = p(H)$ and $m(G) = m(H)$ and $G \leq_L H$, where $\leq_L$ is the lexicographic order.

Example 3.6 Let $G = 16 + 9 + 4$ and $H = 16 + 10 + 3$. Then, $p(G) = p(H)$, $m(G) = m(H)$, and $G \leq_L H$ because (16, 9, 4) is smaller than (16, 10, 3) regarding the lexicographic order. Hence, $G \leq_R H$. ◇

Both the usual order of $\mathbb{N}_0$ and the lexicographic order are total orders satisfying the well-ordering principle. Therefore, by construction, $\leq_R$ is a total order satisfying the well-ordering principle. We can define canonical forms.

Definition 3.7 Let G be a NIM form. The *canonical form* of G is the smallest form, considering the order $\leq_R$, of the equivalence class $[G]_{\equiv}$.

Definition 3.8 Let G be a NIM form. We say that a NIM form G' is obtained from G by *reduction* if $G' \equiv G$ and $G' <_R G$.

There are two natural reduction procedures.

Definition 3.9 Let G be a NIM form. We define the following two reduction procedures.

(R1) *Removal* of matches from some piles of G, obtaining $G' \equiv G$ such that $G' <_R G$.
(R2) *Different organization* of G, obtaining $G' \equiv G$ with the same number of piles and matches, and such that $G' <_R G$.

Example 3.10 Simplifying from $2+2+2+3+3+3+3$ to $2+2+2+2+2+2+3$ is a *Removal*. ◇

Example 3.11 Simplifying from $7+7+6+2$ to $7+6+6+3$ is a *Different organization*. ◇

A natural question appears:

Is it true that if we methodically simplify a form G with R1 and R2, we reach its canonical form?

Unfortunately, it is not. Consider $H = 11 + 11 + 8 + 6 + 5 + 5 + 3$ and $G = 11+8+8+6+6+5+2$. By Theorem 3.4, $G \equiv H$ and $G <_R H$ since $p(G) = p(H)$ and $m(G) < m(H)$. In fact, if we subtract 3 from 11, and move 1 from 3 to 5, we reduce H, obtaining G, the canonical form of H. However, we observe that this procedure *is not* a composition. The two actions can be done *at once*. If we try to split the actions, performing, for instance, R1∘R2, after changing H to $11 + 11 + 8 + 6 + 5 + 6 + 2$, we lose the equality because, due to Theorem 3.4, $11 + 11 + 8 + 6 + 5 + 5 + 3 \not\equiv 11 + 11 + 8 + 6 + 5 + 6 + 2$. We can check that it is impossible to use a single R1 or a single R2 to simplify H.

4 Future work

A complete characterization of the equivalence classes under equality of games was presented. As shown, impartial forms like {2} are not equal to any form of three-player NIM $g^1 + \cdots + g^n$. Therefore, two natural questions arise.

(1) Is it possible to present a more general characterization for all impartial forms? Or, at least, can we present a more general characterization for some equality more general than $\equiv$ *modulo* three-player NIM?
(2) With a suitable answer for the first question, it is possible to extend the work for three-player subtraction games with podium rule?

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