Information Measures and Synchronization in Complete Networks

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Abstract

- The main purpose of this talk is to present **information measures and synchronization of complete networks** with local identical chaotic dynamical systems.
- The networks topologies are characterized by **circulant matrices** and the conditional Lyapunov exponents are explicitly determined.
- For different types of **local dynamics**, necessary and sufficient conditions for the occurrence of synchronization with or without the negativity of the conditional Lyapunov exponents are presented.
- Some properties of the **mutual information rate** and the **Kolmogorov-Sinai entropy** are established, depending on the topological entropy of the individual chaotic nodes and on the synchronization interval.
- Numerical studies are included.

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- Numerical studies are included.

Preliminars: complete networks

- An active channel is an active network constructed using N elements that have some intrinsic dynamics and can be described by classical dynamical systems, such as chaotic oscillators, neurons, phase oscillators, and so on.
- Consider a network of N identical chaotic dynamical oscillators or units, described by a connected and unoriented graph $G = (V, E)$, where V represents the vertices (nodes) and E the edges of G, with no loops and no multiple edges.
- Throughout this work we will study complete networks of order N, with $\frac{N(N-1)}{2}$ edges and every vertex of the associated graph G has degree $N - 1$.
- The space of complete networks with N nodes will be denoted by K_N.
- In each node the dynamic of the oscillators is defined by $\dot{x}_i = f(x_i)$, with $f : \mathbb{R}^n \to \mathbb{R}^n$ and $x_i \in \mathbb{R}^n$ is the state variables of the node i.
- The local dynamics considered at each node establish the topological, metrical and chaotic complexity of the network that is being studied.

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Preliminars: laplacian and jacobian matrices

- Consider A the adjacency matrix of K_N and $D = diag(N-1, \ldots, N-1)$, then $L = [l_{ij}] = A - D$ represents the laplacian matrix of the complete graph and is written in the following form,

$$L = \begin{bmatrix} -(N-1) & 1 & 1 & \ldots & 1 \\ 1 & -(N-1) & 1 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & \ldots & 1 & -(N-1) \end{bmatrix}.$$

- The dynamics of these N coupled oscillators can be expressed by the following system of differential equations,

$$\dot{x}_i = f(x_i) + \sigma \sum_{j=1}^{N} l_{ij} x_j, \qquad (1)$$

where $i = 1, 2, ..., N$ and $\sigma > 0$ is the coupling parameter.

- Let f' be the derivative of f, then the jacobian matrix of the networks in K_N is written as follows,

$$J = \begin{bmatrix} f' - (N-1)\sigma & \sigma & \ldots & \sigma \\ \sigma & f' - (N-1)\sigma & \ldots & \sigma \\ \ldots & \ldots & \ldots & \ldots \\ \sigma & \sigma & \ldots & f' - (N-1)\sigma \end{bmatrix}.$$

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Preliminars: circulant matrices

- Every matrix associated with a complete network K_N has a certain regularity, so we are able to determine its spectra and the associated eigenspaces.
- The laplacian matrix L and the jacobian matrix J are circulant matrices.
- The laplacian matrix L has exactly two eigenvalues $\mu_1 = 0$, a simple root, and $\mu_2 = -N$, with multiplicity $N - 1$.
- The jacobian matrix J has also two eigenvalues $\lambda_1 = f'$, a simple root, and $\lambda_2 = f' - N\sigma$, with multiplicity $N - 1$.
- In the context of the study of information measures, the eigenvalue λ_1 measures the exponential divergence of nearby trajectories in the direction of the synchronization manifold.
- The eigenvalue λ_2 measures the exponential divergence of nearby trajectories in the direction transversal to the synchronization manifold, [Baptista et al., 2016].

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Preliminars: information measures

- In an active network, every pair of elements form a communication channel and the rate with which information is exchanged between a transmitter S_i and a receiver S_j, is given by the mutual information rate, defined by,

$$I_C(S_i, S_j) = \lambda_{\|} - \lambda_{\perp}$$

1. where $\lambda_{\|}$ denotes the positive Lyapunov exponents (parallel), associated to the synchronization manifold;
2. $\lambda_{\perp}$ denotes the positive Lyapunov exponents (transversal), associated to the transversal manifold.

- For complete networks K_N, the mutual information rate is given by,

$$I_C = \begin{cases} \lambda_{\|} - \lambda_{\perp}, & \text{if } \lambda_{\perp} > 0 \\ \lambda_{\|}, & \text{if } \lambda_{\perp} \leq 0 \end{cases}. \quad (2)$$

- The Kolmogorov-Sinai entropy, denoted by H_{KS}, provides a global measure of the amount of information that can be simultaneously transmitted among the network. For complete networks K_N, the Kolmogorov-Sinai entropy is given by,

$$H_{KS} = \begin{cases} \lambda_{\|} + \lambda_{\perp}, & \text{if } \lambda_{\perp} > 0 \\ \lambda_{\|}, & \text{if } \lambda_{\perp} \leq 0 \end{cases}. \quad (3)$$

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Preliminars: synchronization interval

- From several publications in the last decades we can say that the information theory and synchronization are directly related in a network.
- Other central point of our investigation is the phenomenom of synchronization in the space of complete networks K_N and its relations with the information measures I_C and H_{KS}, just mentioned in Eqs.(2) and (3), respectively.
- The synchronization interval is given by,

$$\sigma_1 = \frac{1 - e^{-\chi(f)}}{|\mu_2|} < \sigma < \frac{1 + e^{-\chi(f)}}{|\mu_N|} = \sigma_2, \tag{4}$$

where $0 = \mu_1 < |\mu_2| \leq \ldots \leq |\mu_N|$ are the eigenvalues of the laplacian matrix L and $\chi(f)$ is the Lyapunov exponent of the local dynamics f.

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Local dynamics: piecewise linear maps with $s > 1$

- In this section we consider the space of all the complete networks K_N, given by Eq.(1), where the local dynamics in each node is defined by $f : I \subset \mathbb{R} \to I$, a continuous piecewise linear map with constant slope $s > 1$ everywhere.
- Thus, throughout this section we consider the following parameters space,

$$\Sigma^+ = \left\{ (N, s, \sigma) \in \mathbb{R}^3 : N \in \mathbb{N} \setminus \{1\}, s > 1, \sigma > 0 \right\}. \tag{5}$$

- Since each complete network K_N has identical chaotic nodes and $|\mu_2| = |\mu_N| = N$, then the synchronization interval is nonempty, for all $s > 1$. Moreover, from Eq.(4), the synchronization interval may be expressed in terms of the topological entropy of f, i.e., $h_{top}(f) = \chi(f) = \log(s)$, [Milnor et al., 1988].

Property

Consider the (K_N, Σ^+) space. Let $f : I \to I$ be a continuous piecewise linear map with slope $s > 1$ everywhere. The synchronization interval of K_N is given by,

$$\sigma_1 = \frac{s-1}{Ns} < \sigma < \frac{s+1}{Ns} = \sigma_2. \tag{6}$$

Local dynamics: piecewise linear maps with $s > 1$

- Considering that the local dynamics f is a continuous piecewise linear map with slope $s > 1$ everywhere, the jacobian matrix J has only two distinct eigenvalues, $\lambda_1 = s$ and $\lambda_2 = s - N\sigma$, with multiplicity $N - 1$.
- So, the parallel Lyapunov exponent is given by,

$$\lambda_{||} = \int_I \ln|\lambda_1|\, dx = |I| \ln(s), \tag{7}$$

where $|I|$ represents the amplitude of the interval I.
- The transversal Lyapunov exponent is given by,

$$\lambda_{\perp} = \int_I \ln|\lambda_2|\, dx = |I| \ln|s - N\sigma|\,. \tag{8}$$

- We remark that for each complete network K_N, there is a single transversal Lyapunov exponent.
- The following proposition provides necessary and sufficient conditions for the existence of synchronization with or without negative transversal Lyapunov exponents.

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Proposition

Consider the (K_N, Σ^+) space. Let $f : I \to I$ be a continuous piecewise linear map with slope $s > 1$ everywhere, I_σ be the synchronization interval, given by Eq.(6), and $I_{\lambda_\perp^-}$ be the interval where $\lambda_\perp \leq 0$, with $\lambda_\perp$ given by Eq.(8). It is verified that:

(i) $I_\sigma \cap I_{\lambda_\perp^-} \neq \emptyset$ *if and only if* $1 < s < 1 + \sqrt{2}$;

(ii) $I_\sigma \cap I_{\lambda_\perp^-} = \emptyset$ *if and only if* $s \geq 1 + \sqrt{2}$.

- Proposition 1 bring up to the discussion the complete synchronization versus negativity of the conditional or transversal Lyapunov exponents, [Caneco et al., 2014], [Cao et al., 2006], [Pecora et al., 1991] and [Shuai et al., 1997].
- The negativity of the conditional Lyapunov exponents is a necessary condition for the stability of the synchronized state, [Boccaletti et al., 2002].
- In our study we present necessary and sufficient conditions which illustrate two classic cases of this discussion:
  (i) in item (i) of Proposition 1, there is lack of synchronization in the region where all transversal Lyapunov exponents are negative, see Fig.1;
  (ii) in item (ii) of Proposition 1, it is possible to achieve synchronization without the negativity of all conditional Lyapunov exponents, see Fig.2.

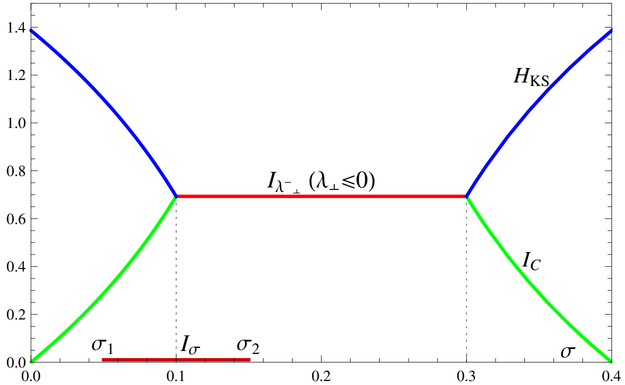

Figure: Scheme for Proposition 1 (i) and Proposition 2 (i) ($1 < s < 1 + \sqrt{2}$), with $N = 10$, $s = 2$, the synchronization interval is $I_\sigma =]1/20, 3/20[$, $I_{\lambda_\perp^-} = [1/10, 3/10]$, where $I_\sigma \cap I_{\lambda_\perp^-} \neq \emptyset$, but there is lack of synchronization in the interval $I_{\lambda_\perp^-}$, where all transversal Lyapunov exponents are negative, [Cao et al., 2006].

Local dynamics: piecewise linear maps with $s > 1$

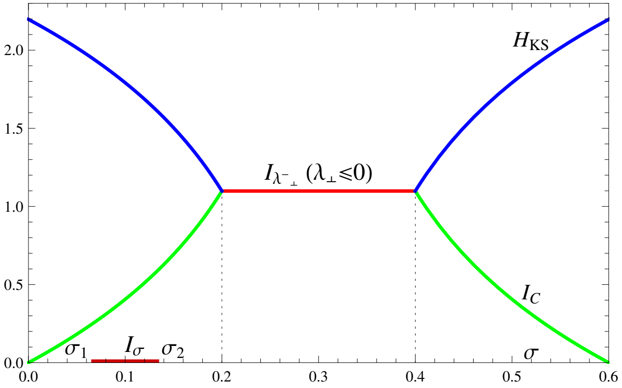

Figure: Scheme for Proposition 1 (ii) and Proposition 2 (ii) ($s > 1 + \sqrt{2}$), with $N = 10$, $s = 3$, the synchronization interval is $I_\sigma =]1/15, 2/15[$, $I_{\lambda_\perp^-} = [1/5, 2/5]$, where $I_\sigma \cap I_{\lambda_\perp^-} = \emptyset$, there is synchronization without the negativity of all conditional Lyapunov exponents, [Shuai et al., 1997].

Local dynamics: piecewise linear maps with $s > 1$

- Taking into account the expressions of the parallel and transversal Lyapunov exponents, the mutual information rate and the Kolmogorov-Sinai entropy, defined by Eqs.(2) and (3), respectively, are explicitly written by the following expressions:

$$I_C = \begin{cases} |I| \ln\left(\frac{s}{|s-N\sigma|}\right), & \text{if } \lambda_\perp > 0 \\ |I| \ln(s), & \text{if } \lambda_\perp \leq 0 \end{cases} \quad (9)$$

and

$$H_{KS} = \begin{cases} |I| \ln\left(s|s-N\sigma|\right), & \text{if } \lambda_\perp > 0 \\ |I| \ln(s), & \text{if } \lambda_\perp \leq 0 \end{cases}. \quad (10)$$

Proposition

Consider the (K_N, Σ^+) space. Let $f : I \to I$ be a continuous piecewise linear map with slope $s > 1$ everywhere, I_σ be the synchronization interval, given by Eq.(6), and $I_{\lambda_\perp^-}$ be the interval where $\lambda_\perp < 0$, with $\lambda_\perp$ given by Eq.(8). It is verified that:

(i) *for $1 < s < 1 + \sqrt{2}$,*
   1. *if $\sigma \in I_\sigma^- = I_\sigma \cap I_{\lambda_\perp^-}$, then $I_C = H_{KS}$;*
   2. *if $\sigma \in I_\sigma^+ = I_\sigma \setminus I_\sigma^-$, then I_C increases and H_{KS} decreases;*

(ii) *if $s \geq 1 + \sqrt{2}$, then I_C increases and H_{KS} decreases, with $I_C \neq H_{KS}, \forall \sigma \in I_\sigma$.*

Local dynamics: piecewise linear maps with $|s| > 1$

- Now we consider the study of complete networks K_N, where the local chaotic dynamics in each node of K_N is defined by $f : I = [b_1, b_2] \subset \mathbb{R} \to I$, a continuous piecewise linear map, such that there exist points $b_1 = d_0 < d_1 < \ldots < d_p < d_{p+1} = b_2$, where f is linear in each subinterval $I_i = [d_i, d_{i+1}]$, $i = 0, \ldots, p$, with constant slope $|s| > 1$ everywhere.
- Generally, we consider that f has k subintervals with slope $s > 1$, denoted by $\bar{I}_j$, with $j = 1, \ldots, k$, and $p + 1 - k$ subintervals with slope $s < -1$, denoted by $\tilde{I}_j$, with $j = 1, \ldots, p + 1 - k$.
- The parameters space considered in this section is,

$$\Sigma^{\pm} = \left\{ (N, s, \sigma) \in \mathbb{R}^3 : N \in \mathbb{N} \setminus \{1\}, |s| > 1, \sigma > 0 \right\}.$$

- We remark that, with this local dynamics, the synchronization interval in the $(K_N, \Sigma^{\pm})$ space is the same as established in Property 1, given by Eq.(6), i.e.,

$$\sigma_1 = \frac{s-1}{Ns} < \sigma < \frac{s+1}{Ns} = \sigma_2.$$

Local dynamics: piecewise linear maps with $|s| > 1$

- The jacobian matrix J has the eigenvalues $\lambda_1 = |s|$ and $\lambda_2 = |s| - N\sigma$, with multiplicity $N-1$. Consequently, the parallel Lyapunov exponent is given by,

$$\lambda_{||} = \int_I \ln|\lambda_1|\, dx = |I| \ln(s) \qquad (11)$$

where $|I| = b_2 - b_1$.
- On the other hand, the transversal Lyapunov exponent is given by,

$$\begin{aligned} \lambda_\perp &= \int_I \ln|\lambda_2|\, dx \\ &= \sum_{j=1}^{k} \left|\bar{I}_j\right| \ln|s - N\sigma| + \sum_{j=1}^{p+1-k} \left|\tilde{I}_j\right| \ln(s + N\sigma) \end{aligned}$$

  1. $a^+ = \sum_{j=1}^{k} \left|\bar{I}_j\right|$ is the amplitude of the subintervals $\bar{I}_j$ with slope $s > 1$;
  2. $a^- = \sum_{j=1}^{p+1-k} \left|\tilde{I}_j\right|$ is the amplitude of the subintervals $\tilde{I}_j$ with slope $s < -1$;
- Thus, if the local chaotic dynamics is a continuous piecewise linear map f with slope $|s| > 1$ everywhere, then the transversal Lyapunov exponent is given by,

$$\lambda_\perp = \ln|s - N\sigma|^{a^+} + \ln(s + N\sigma)^{a^-} . \qquad (12)$$

The transversal Lyapunov exponent $\lambda_\perp$ depends on the amplitudes a^+ and a^-.

Equal amplitudes of the subintervals ($a^+ = a^-$)

- Let r_1 be the only positive real root of the polynomial $s^4 - 2s - 1 = 0$ and r_2 be the only positive real root of the polynomial $s^4 - 2s^2 - 2s - 1 = 0$. Notice that $1 < r_1 < r_2$.

Proposition

Consider the $(K_N, \Sigma^{\pm})$ space. Let $f : I \to I$ be a continuous piecewise linear map with slope $|s| > 1$ everywhere, I_σ be the synchronization interval, given by Eq.(6), and $I_{\lambda_\perp^-}$ be the interval where $\lambda_\perp < 0$, with $\lambda_\perp$ given by Eq.(12). For $a^+ = a^-$, it is verified that:

(i) $I_{\lambda_\perp^-} \subset I_\sigma$ *if and only if* $1 < |s| < r_1$;

(ii) $I_\sigma \cap I_{\lambda_\perp^-} \neq \emptyset$ *if and only if* $r_1 \leq |s| \leq r_2$;

(iii) $I_\sigma \cap I_{\lambda_\perp^-} = \emptyset$ *if and only if* $|s| > r_2$.

- The result of (i) is the case of total synchronization with all transversal Lyapunov exponents negative, see Fig.3.
- The results of (ii) and (iii) are analogous to the results of Proposition 1.

Equal amplitudes of the subintervals ($a^+ = a^-$)

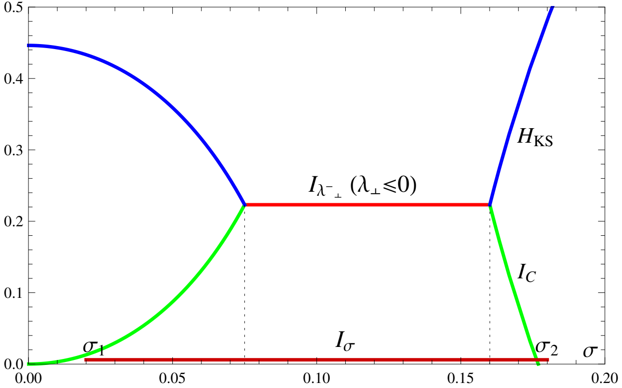

Figure: Scheme for Proposition 3 (i) ($1 < |s| < r_1$), for $a^+ = a^-$ with $N = 10$, $s = 1.25$, the synchronization interval is $I_\sigma =]0.02, 0.18[$, $I_{\lambda_\perp^-} = [0.075, 0.16]$, where $I_{\lambda_\perp^-} \subset I_\sigma$, there is total synchronization with all transversal Lyapunov exponents negative, [Pecora et al., 1991].

Equal amplitudes of the subintervals ($a^+ = a^-$)

- When $a^+ = a^-$, the I_C and H_{KS} are explicitly written by the expressions:

$$I_C = \begin{cases} |I| \left[\ln(s) - \frac{1}{2} \ln\left(|s - N\sigma|(s + N\sigma) \right) \right], & \lambda_\perp > 0 \\ |I| \ln(s), & \lambda_\perp \leq 0 \end{cases} \quad (13)$$

and

$$H_{KS} = \begin{cases} |I| \left[\ln(s) + \frac{1}{2} \ln\left(|s - N\sigma|(s + N\sigma) \right) \right], & \lambda_\perp > 0 \\ |I| \ln(s), & \lambda_\perp \leq 0 \end{cases} . \quad (14)$$

Proposition

Consider the $(K_N, \Sigma^\pm)$ space. Let $f : I \to I$ be a continuous piecewise linear map with slope $|s| > 1$ everywhere, I_σ be the synchronization interval, given by Eq.(6), and $I_{\lambda_\perp^-}$ be the interval where $\lambda_\perp < 0$, with $\lambda_\perp$ given by Eq.(12). For $a^+ = a^-$ and $1 < |s| < r_1$, it is verified that:

(i) *if $\sigma \in I_{\lambda_\perp^-}$, then $I_C = H_{KS}$;*

(ii) *if $\sigma \in I_\sigma \setminus I_{\lambda_\perp^-}$ and $\sigma_1 < \frac{\sqrt{s^2-1}}{N}$, then I_C increases and H_{KS} decreases, with $I_C \neq H_{KS}, \forall \sigma$;*

(iii) *if $\sigma \in I_\sigma \setminus I_{\lambda_\perp^-}$ and $\sigma_2 > \frac{\sqrt{s^2+1}}{N}$, then I_C decreases and H_{KS} increases, with $I_C \neq H_{KS}, \forall \sigma$.*

Different amplitudes of the subintervals ($a^+ \neq a^-$)

- When $a^+ \neq a^-$, the I_C and H_{KS} are written as follows:

$$I_C = \begin{cases} |I| \ln(s) - \ln\left(|s - N\sigma|^{a^+}(s + N\sigma)^{a^-}\right), & \lambda_\perp > 0 \\ |I| \ln(s), & \lambda_\perp \leq 0 \end{cases} \tag{15}$$

and

$$H_{KS} = \begin{cases} |I| \ln(s) + \ln\left(|s - N\sigma|^{a^+}(s + N\sigma)^{a^-}\right), & \lambda_\perp > 0 \\ |I| \ln(s), & \lambda_\perp \leq 0 \end{cases}. \tag{16}$$

- Clearly, the expressions given by Eqs.(15) and (16) are more complex than those previously studied. These are dependent on the amplitudes of the subintervals with slope $s > 1$ and slope $s < -1$, with $a^+ \neq a^-$.
- The following proposition provides necessary conditions for the negativity of transversal Lyapunov exponent $\lambda_\perp$.

Proposition

Consider the $(K_N, \Sigma^\pm)$ space. Let $f : I \to I$ be a continuous piecewise linear map with slope $|s| > 1$ everywhere. Consider the measures I_C and H_{KS} defined by Eqs.(15) and (16), respectively, with $a^+ \neq a^-$. If $I_C = H_{KS}$, then $|s - N\sigma| < 1$ and $a^+ > a^-$.

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Open questions?

- There are sufficient conditions to guarantee the negativity of the conditional Lyapunov exponents, for different slopes of f $(a^+ \neq a^-)$?

- Under what conditions the chaotic signals transmitted through filters produce an output with higher dimension, due to the appearance of a fractal set?