The role of representations in promoting the quantitative reasoning

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In this communication, we discuss the role of representations in the development of conceptual knowledge of 2nd grade students involved in additive quantitative reasoning through the analysis of the resolutions of two tasks that present transformation problems. Starting to discuss what is meant by additive quantitative reasoning and mathematical representation, we present after some empirical results in the context of a teaching experiment developed in a public school. The results show the difficulties with the inverse reasoning present in both situations proposed to students. Most students preferably use the symbolic representation, using also the written language as a way to express the meaning attributed to its resolution. The iconic representation was used only by a pair of students. Representations have assumed a dual role, that of being the means of understanding the students' thinking, and also supporting the development of their mathematical thinking.

Keywords: Mathematical representations, transformation problems, additive quantitative reasoning, inversion.

Introduction

This paper is part of the Project "Adaptive thinking and flexible computation: Critical issues" being developed by the Schools of Education of Lisboa, Setúbal and Portalegre. Its main goal is to discuss the role of representations in the development of 2nd grade students' conceptual knowledge that is present in different levels of understanding of numerical operations/relations when they solved two tasks. These were conceived with the aim to develop quantitative additive reasoning, as a means of understanding this same reasoning. The tasks present transformation problems included in the classes of search of initial state (Vergnaud, 2009). They are the last ones of a sequence of six tasks that was applied in the context of a teaching experiment developed in a public school in Lisbon. The empirical data analysis focused on representations aims to discuss the inferences we make in the reasoning of the students but also their role in the development of students' reasoning.

Mathematical representations and quantitative reasoning

Quantitative reasoning, within the additive structure, focuses mainly on relations between quantities (Thompson, 1993), being that the problems of transformation aimed at finding the initial state have increased cognitive complexity for 2nd grade students by requiring an inverse reasoning (Vergnaud, 2009). The representations are interconnected with the reasoning given the relevance of their role in the understanding of students' reasoning (NCTM, 2000). But, the representations also assume an important role in students' learning, constituting cognitive means with which they develop their mathematical thinking (NCTM, 2000; Ponte & Serrazina, 2000). In a broad sense, a representation is a setting that can represent something somehow (Goldin, 2008). The term "representation" refers both to the process of representing and the result of this process. In mathematics education, representations are privileged tools for students express their mathematical ideas, still working as helpers in the construction of new knowledge (NCTM, 2000). However, a mathematical representation cannot be understood or interpreted in isolation, since only makes sense when part of a more comprehensive and structured system in which different representations are related (Goldin & Shteingold, 2001).

According to Stylianou (2010), the way as representations are used in classroom has impact in students learning and this largely depends on the role of the teacher, using “student-generated work as a launching point for discussions” (p. 339). This idea is reinforced by Ponte and Serrazina (2000) when they say that how mathematical ideas are represented influences profoundly the way they are understood and used. For example, according to Vergnaud (2009), the inverse transformation can be represented by two symbolic representations -- the algebraic one and the arrow diagram -- considering, however, that while the algebraic representation is not suitable for children in elementary school, using the diagram representation the teacher can help students connect, immediately, the different components of the relationship, namely the direct and inverse transformations, giving meaning to the temporal motion go forward and backward.

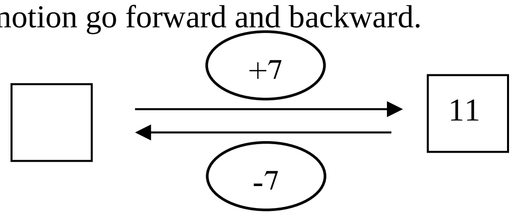


Figure 1: Arrow diagram (Vergnaud, 2009, p. 87)

Figure 1 presents a representative diagram of subtracting 7 to the final state in the situation "John has just won 7 marbles in playing with Meredith; now he has 11 marbles; how many marbles did he have before playing?" (Vergnaud, 2009, pp. 86-87). However, while recognizing the importance of this representation, the author states that children need several examples of the inverse transformation in order to be able to effectively understand it. "Several kinds of awareness are needed: you lose what you have just won; or you win what you have just lost; you go backwards as many steps as you have gone forwards and reciprocally" (p. 87).

For Ponte and Serrazina (2000), the main forms of representation used in the primary education are: (i) *the oral and written language*; (ii) *symbolic representations*, like numbers or the signs of the four operations and the equal sign; (iii) *iconic representations*, like figures or graphics; and (iv) *active representations*, like manipulative materials or other objects. It is through the analysis of the representations used by students that the teacher can become aware of their thinking and help them in the construction of own representations in mathematical language.

NCTM (2000) also emphasizes the role of idiosyncratic representations constructed by students when they are solving problems and investigating mathematical ideas, in that it can help them in understanding and solving problems and provide "meaningful ways to record a solution method and to describe the method to others" (p. 68). Observing these representations, teachers and researchers can understand the ways of interpreting and reasoning of students.

Methodology

This study follows a qualitative approach within an interpretive paradigm. Its methodology of *design research* is part of a perspective of learning design, in order to produce local theories of teaching and learning sequences that are resources and references available to inform the practices of teachers and researchers (Gravemeijer, 2015).

The data were collected in a second grade classroom (7-8 years old), with 26 students, of a public primary school in Lisboa. The Project team defined a sequence of tasks with the aim to develop the calculation flexibility in addition and subtraction problems. The process of tasks elaboration included previous testing of some (namely the ones focused in this paper), through clinical interviews (Hunting, 1997) with students of the same grade. It is a technique that is directed by the researcher and seeks a description of the ways of thinking of respondents. The task sequence was previously discussed and analyzed with the classroom teacher having been made minor adjustments. Classroom teacher explored the task sequence with her students (a task every week). During their schooling the empty number line had been used regularly both by the teacher and students.

The data collection was made through participant observation of the authors of this paper, which drew up field notes and supported by video recording, subsequently transcribed. The written records of the students were also collected. All these data were analyzed and triangulated. By ethical reasons, the students' names were changed to ensure confidentiality.

In this communication we analyze two tasks (Figure 2), proposed to the students in the same class (given the similarity between them) and presented on the same sheet of paper, with space for the respective resolution.

Game of marbles I

Ana and Luís played a game of marbles together. At the beginning both had the same number of marbles.

Ana won 3 marbles from Luís and had 7 at the end of the game.

How many did Luís have at the end of the game, knowing that he did not win marbles?

Game of marbles II

Ana and Luís made a game of marbles.

Ana won 6 marbles from Luís and had 10 marbles at the end of the game.

Luís won nothing and had 3 marbles at the end of the game.

Compare the number of marbles of Ana and Luis before the game and at the end of the game.

Figure 2: Performed tasks

These were the last ones of the sequence solved by students. With the previous exploration of the other tasks, students had already worked the relationship between wins and losses over a marbles game, realizing that what a player wins, the other loses. All the tasks were first solved in pairs. In this class, after all pairs had solved the two tasks, the teacher promoted a collective discussion with whole class, from six pairs resolutions (three on each task) who presented their work on the blackboard.

Exploring the tasks

In this section we present and discuss some examples of tasks' resolutions. Their choice was made taking into account the diversity of representations presented by the different pairs and being representative of what happened in class.

Game of marbles I

The resolution of Alexandre and Rosa shows the inversion of reasoning as a critical aspect. Thus, they took first 10 marbles for both players from the sum of 3 ("Ana won 3 marbles") with 7, not mobilizing an inverse reasoning to determine the initial number of marbles.

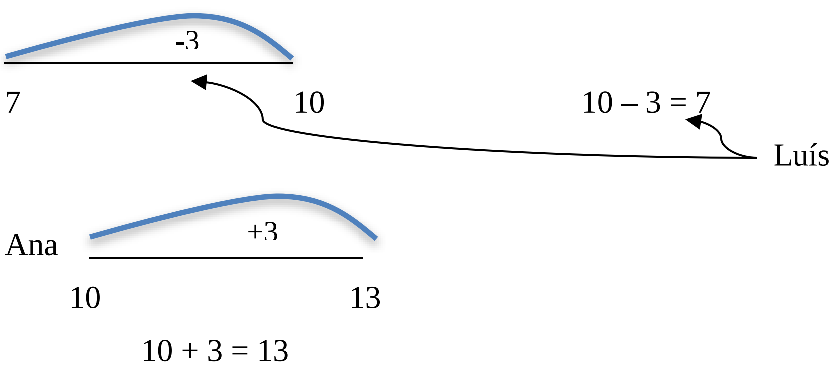


Figure 3: Alexandre's diagram for Game of marbles I

Alexandre's diagram (Figure 3) represents the wins and losses of marbles as they are thought by the pair of students. But, when asked by the teacher they erased all they did and made it again on their worksheets (this diagram was seen in the video records). The new representation (Figure 4) reveals the necessary inversion to find the initial number of marbles. They got it for Ana, but they seem to forget that both players had at the beginning the same number, as in Luis' allusive representation, they retired 3 marbles from 7.

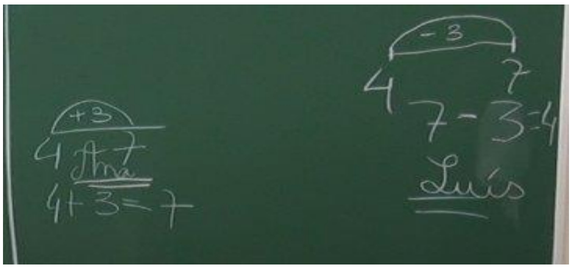
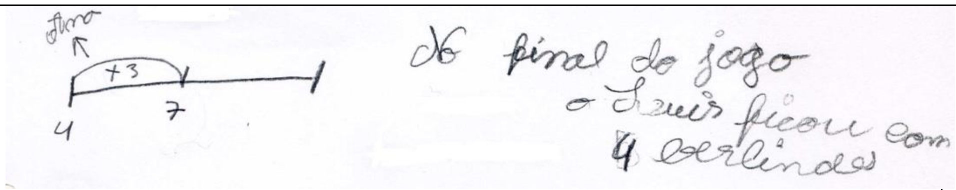


Figure 4: New Alexandre's diagram for Game of marbles I

Vítor and Joana can reverse the reasoning to the case of Ana but they show a lack of understanding of the situation alluding to Luís (Figure 5).



At the end Luis had 4 marbles

Figure 5: Vítor's diagram for Game of marbles I

This diagram helped the children's thinking in the sense that it respects the temporal order of the game: the initial marbles on the left and the final ones on the right side. However, they did not look again for the initial situation, that is, if Ana won, Luís has to have less at the end of the game.

Rui and António used symbolic representations presenting a number line with two curve lines that represent the inversion of addition and subtraction (Figure 6). Like the anterior pair, they reversed their reasoning to Ana but they assumed the same situation to the player Luís, stating that he won 7 marbles at the end of the game. Here the term "won" means the absolute number of marbles at the end of the game and not a quantitative difference. The distinction between the quantitative difference and the result of an arithmetical operation is a critical aspect that emerges from this resolution.

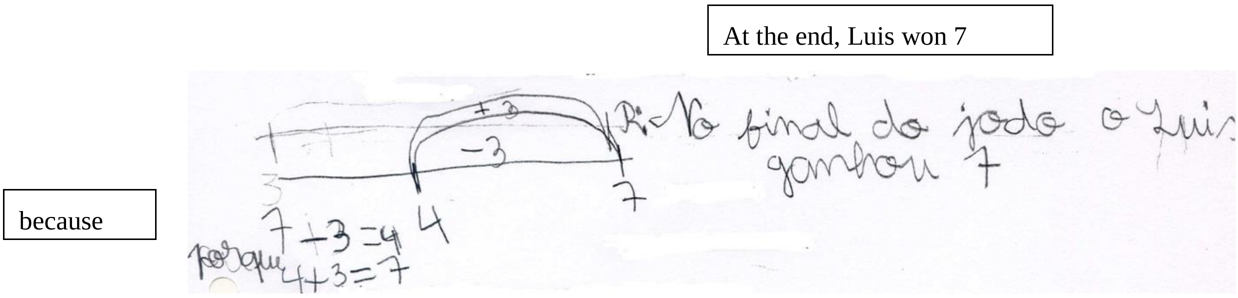


Figure 6: Rui's diagram for Game of marbles I

The word "because" explains the inversion used to determine the number of initial marbles, justifying it with the inverse relationship between addition and subtraction. The order in which they placed the operations shows the inverse reasoning process: first, they determined the initial number of marbles (the initial state), and after they confirmed the resultant final state with the inverse relationship.

Tiago and João used a table disposition (Figure 7), with columns for each of the two players and the lines for the different moments of the game, the top line to the beginning of the game and the bottom to the end.

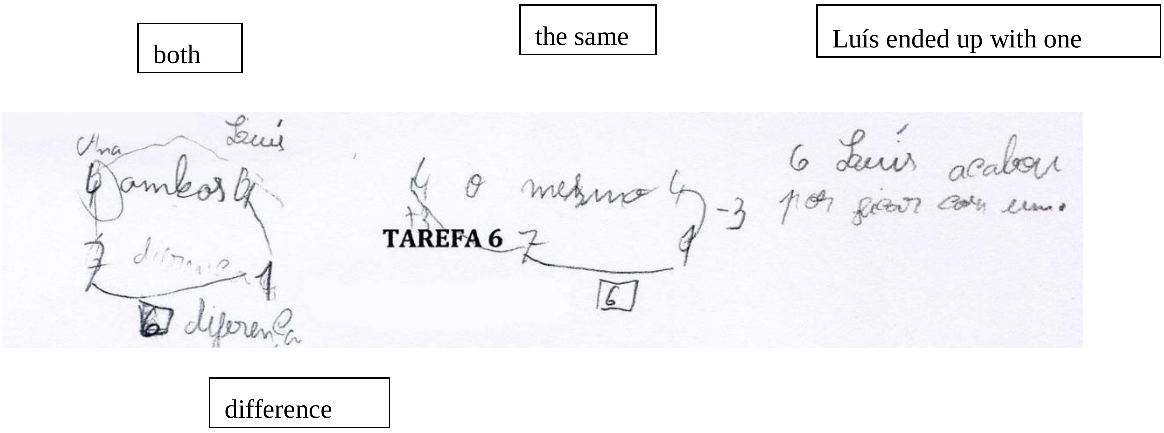


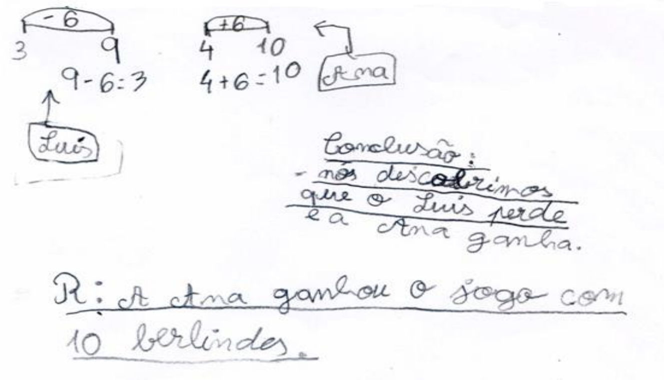
Figure 7: Tiago's diagram for Game of marbles I

It was the only pair in the class, which established the difference between the final numbers of marbles, although not required, focusing on the difference as an additive comparison of quantities. It seems that the representation used allowed them to manage two data at the same time, the quantity of marbles and the transformation after game.

Game of marbles II

Alexandre and Rosa read the problem. Immediately, Alexandre said: "Luís started with 9 and Ana started with 4." Once Alexandre had understood the previous problem when the teacher questioned their resolution, as mentioned before, here he already coped well with the unknown initial state, solving mentally the problem, through an inverse reasoning.

On the diagram's lines (Figure 8), the numbers are placed in ascending order, getting the initial number of marbles on the right for Luís and on the left for Ana. Their answer is focused on the absolute amount of marbles of Ana at the end of the game and not on the comparison.



Conclusion: we discover that Luís loses and Ana wins.

R: Ana won the game with 10 marbles.

Figure 8: Alexandre's diagram for Game of marbles II

Vítor and Joana used the line representation (Figure 9), adopting the temporal criterion as they did in the previous task, putting the initial marbles on the left and the final ones on the right side and at this time they got the right solution.

Figure 9: Vítor's diagram for Game of marbles II

Rui used an iconic representation of Ana's marbles: first, the six marbles won from Luís; then the initial 4 marbles (probably counting them until the total 10); and finally the three final marbles of Luís. He was not able to reverse his thinking in order to determine the initial marbles of Luís.

Ana won

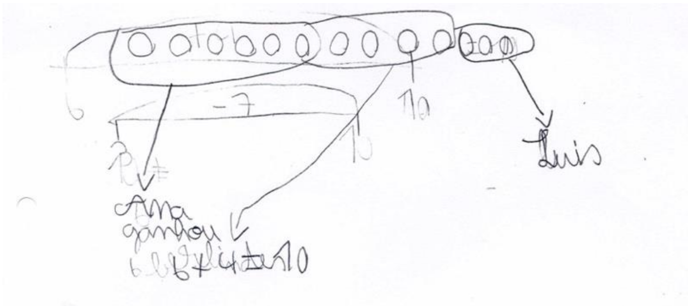
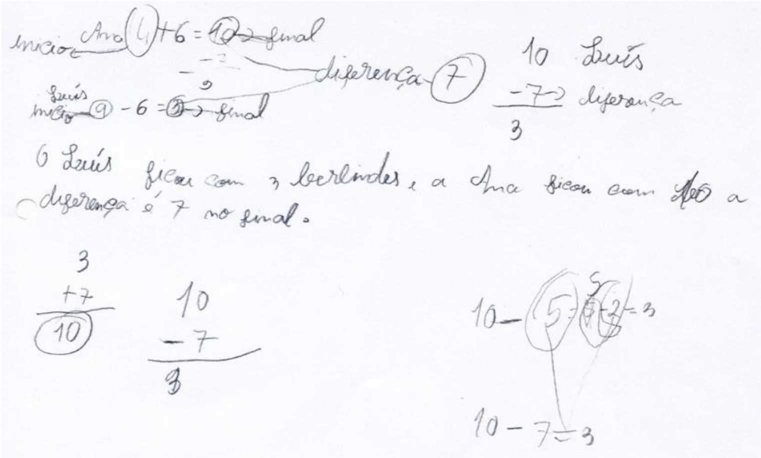


Figure 10: Rui's diagram for Game of marbles II

Tiago surrounded the numbers to assign their meaning, recording the player and the time of the game to which they relate (Figure 11) and respecting also the temporal order of the game.



Luis got 3 marbles and Ana got 10, the difference is 7 at the end.

Figure 11: Tiago's diagram for Game of marbles II

Tiago focused the additive comparison of the resultant final states, recording that "the difference is 7 at the end".

Final remarks

The inverse transformation is a critical aspect in solving the tasks (Vergnaud, 2009). Thus, the first representations used by Alexandre in *Game of marbles I* show a reasoning associated with prototypical situations of addition asking for final states and not initial ones. Students seem to cope more easily with this kind of transformation in the second task after having understood the inversion involved in *Game of marbles I*. For this understanding the teacher’s questions seem to have been fundamental.

An example is the case of Alexandre and, although he did not fully solve the first problem, managed to overcome the obstacle of inversion in the second task solving it mentally very fast. Despite the inherent difficulty of the inverse transformation, the productions of the students in the class reveal a widespread understanding of an aspect of inverse thinking: what a player wins, the other loses.

Students seem to have essentially privileged two forms of representation (Ponte & Serrazina, 2000): the written language and symbolic representations. Just a pair of students used the iconic representation in support of symbolic representation (Rui and António). Among the symbolic representations used, there was a predominance of horizontal dispositions of the calculations, although the use of the empty number line also had had a significant expression, helping to think about the transformations involved in problems. We should stress that empty number line had been used by these students and her teacher since the first grade. Thus, the curved lines, which represented the transformation, supported the thought around the wins and losses, as well the temporal motion. The table disposition also seems to have helped the students (Tiago and João) to structure and relate the various elements of the problem: the two players and the two time points of the game.

Analyzing student productions, we can infer different levels of quantitative additive reasoning. While most learners focused on the absolute amounts of marbles, Tiago focused on quantitative difference as a quantitative result of comparing two quantities additively to find the relative change (Thompson, 1993). So, for the majority of students, the notion of difference as an additive comparison of quantities is a problematic aspect.

The representations used by the students had a dual role. On the one hand, they were windows to interpret their reasoning. On the other hand, they were scaffolds that helped to think mathematically demanding situations, taking into account their ages. It should be stressed that the teacher's role, making questions, not giving answers, while students were doing their work was also essential. As stated by Vergnaud (2009), the development of a conceptual field involves not only situations and schemes but also symbolic tools of representation.

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