# Constructing Good Deals in Discrete Time Arbitrage-Free Dynamic Pricing Models

Beatriz Balbás

University of Castilla la Mancha. Avda. Real Fábrica de Seda s/n. 45600 Talavera, Toledo, Spain. beatriz.balbas@uclm.es

Raquel Balbás

University Complutense of Madrid. Department of Actuarial and Financial Economics. Somosaguas Campus.

28223 Pozuelo de Alarcón, Madrid, Spain. raquel.balbas@ccee.ucm.es

April 27, 2011

**Abstract**

Recent literature has proved that many classical pricing models (Black and Scholes, Heston, etc.) and risk measures ($VaR$, $CVaR$, etc.) may lead to "pathological meaningless situations", since traders can build sequences of portfolios whose risk level tends to $-\infty$ and whose expected return tends to $+\infty$, *i.e.*, $(risk = -\infty, return = +\infty)$. Such a sequence of strategies may be called "good deal". This paper focuses on the risk measures $VaR$ and $CVaR$ and analyzes this caveat in a discrete time complete pricing model. Under quite general conditions the explicit expression of a good deal is given, and its sensitivity with respect to some possible measurement errors is provided too. We point out that a critical property is the absence of short sales. In such a case we first construct a "shadow riskless asset" ($SRA$) without short sales and then the good deal is given by borrowing more and more money so as to invest in the $SRA$. It is also shown that the $SRA$ is interested by itself, even if there are short selling restrictions.

**Key words**. Risk Measure, Discrete Time Pricing Model, Good Deal.

**J.E.L. Classification**. G11, G12, G14.

# 1 Introduction

Since Artzner *et al.* (1999) introduced the axioms and properties of the "Coherent Measures of Risk" many authors have extended the discussion. So, among many other interesting contributions, Goovaerts *et al.* (2004) introduced the Consistent Risk Measures, Rockafellar *et al.* (2006) defined the Expectation Bounded Risk Measures, Brown and Sim (2009) introduced the Satisfying Measures, Balbás *et al.* (2009) studied the Adapted Risk Measures, and Aumann and Serrano (2008) and Foster and Hart (2009) defined Indexes of Riskiness. All of these measures are more and more used by researchers, practitioners, regulators and supervisors.

Many risk measures provide regulators and supervisors with the capital reserve that a manager must add in order to protect the wealth of her/his clients. It is usually assumed that this capital requirement will be invested in a risk-free asset. Nevertheless, several theoretical and empirical papers have shown that alternative investments may outperform the risk-free asset effectiveness (Balbás *et al.*, 2010*b*).

The notion of "Good Deal" was introduced in the seminal paper by Cochrane and Saa-Requejo (2000). Mainly, a good deal is an investment strategy providing traders with a "very high return/risk ratio", in comparison with the value of this ratio for the Market Portfolio. In that paper risk is measured with the standard deviation, and the absence of good deals is imposed in an arbitrage-free model so as to price in incomplete markets. This line of research has been extended for more general risk functions (see, for instance, Staum, 2004). Moreover, some recent papers impose conditions that are strictly stronger than the absence of arbitrage (Dana and Le Van, 2010, Stoica and Lib, 2010, etc.). They fix a risk measure and its subgradient must contain "Equivalent Risk Neutral Probabilities". Thus, the existence of "Equivalent Risk Neutral Probabilities" (or the absence of arbitrage) is not sufficient. Some of them must belong the risk measure subgradient.

However, the fulfillment of these assumptions stronger than the arbitrage absence is not so obvious in very important Pricing Models of Financial Economics. Balbás *et al.* (2010*a*) have shown the existence of "pathological results" when combining some risk measures (Conditional Value at Risk or $CVaR$, Dual Power Transform or $DPT$, etc.) and very popular pricing models (Black and Scholes, Heston, etc.). Indeed, for the examples above the Stochastic Discount Factor ($SDF$) of the pricing model does not belong to the risk measure subgradient, which implies the existence of sequences of portfolios whose expected returns tend to plus infinite and whose risk levels tend to minus infinite ($risk = -\infty$, $return = +\infty$). The analysis of Balbás *et al.* (2010$a$) has been extended in Balbás *et al.* (2010$b$), where the authors present explicit constructions of the sequences above for the $CVaR$ and the Black and Sholes model. Balbás *et al.* (2010$b$) use the expression "good deal" to indicate such a sequence. We will also give this meaning to the expression "good deal".

The present paper seems to present two major contributions with respect to the literature above. The first one is the effective construction (or explicit expression) of good deals (*i.e.*, sequences whose $(risk, return)$ tends to $(-\infty, +\infty)$) for the Value at Risk ($VaR$) and the $CVaR$ in a discrete time complete pricing model. The second novelty is a sensitivity analysis, *i.e.*, we measure the effect of errors when testing key variables such as the $SDF$ of the pricing model.

The article's outline is as follows. Section 2 will present the notations and the general framework we are going to deal with, as well as some important background that will be applied. Section 3 will be devoted to studying the special properties of discrete time complete pricing models. In particular, we will give the conditions of the model leading to the existence of good deals. Concrete examples such as the well-known Binomial Model or some Risk Adverse Pricing Models will be included in the analysis. Discrete time models are very important for several reasons, since they are easy to use in practice and give good approximations of every continuous time pricing model. In this sense, the analysis of this paper may be very useful to traders, since it will allow then to build practical good deals in a easy way.

Sections 4 and 5 are the most important of the paper. The first one yields the strategy optimizing the investment of the capital requirements in a framework with short selling restrictions (this strategy will be called "shadow riskless asset", $SRA$). In particular, Theorem 7 gives the closed formula of this strategy, and its remarks highlight very important consequences. On the one hand, the $SRA$ is similar to some classical portfolio insurance strategies, which is consistent with the empirical findings of Annaert *et al.* (2009). These authors reveal that some put option-linked portfolio insurance strategies are not outperformed by other hedging methods if one draws on stochastic dominance criteria or $VaR$ and $CVaR$. On the other hand, if the manager can borrow as much money as desired and invest this money in the $SRA$ then we have the explicit construction of a "good deal". By borrowing more and more money so as to invest in the $SRA$ we get a sequence of portfolios such that there is no lower/upper bound for the ($risk$, $return$) couple ($risk = -\infty$, $return = +\infty$), so every hedging strategy may be outperformed by a new one, and that leads to sequences of hedging strategies with unlimited potential gains.

Section 5 is devoted to measuring the sensitivity of our solutions with respect to estimation errors. In particular, Theorem 8 gives a general formula when the initial capital requirements, the random final wealth of the manager, and/or the pricing model (the $SDF$) are modified. Its remarks analyze important consequences and particularize the findings for special examples. For instance, in the Binomial Model we can measure the sensitivity with respect to interest rates, volatilities, etc.

Section 7 presents the most important conclusions of the paper.

## 2 Preliminaries, notations and theoretical background

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ composed of the set of "states of the world" $\Omega$, the $\sigma-$algebra $\mathcal{F}$ and the probability measure $\mathbb{P}$. Consider also a couple of conjugate numbers $p \in [1, \infty)$ and $q \in (1, \infty]$ (*i.e.*, $1/p + 1/q = 1$). As usual $L^p$ ($L^q$) denotes the space of $\mathbb{R}-$valued random variables $y$ on $\Omega$ such that $\mathbb{E}\left(|y|^p\right) < \infty$, $\mathbb{E}()$ representing the mathematical expectation ($\mathbb{E}\left(|y|^q\right) < \infty$, or $y$ essentially bounded if $q = \infty$). According to the Riesz Representation Theorem (Horvàth, 1966), we have that $L^q$ is the dual space of $L^p$.

As usual, we will assume that prices are in $L^2$. Thus, consider a time interval $[0, T]$, a subset $\mathcal{T} \subset [0, T]$ of trading dates containing 0 and $T$, and a filtration $(\mathcal{F}_t)_{t \in \mathcal{T}}$ providing the arrival of information and such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. Assume that the market is complete, *i.e.*, every final pay-off $y \in L^2$ may be reached by the price process $(S_t)_{t \in \mathcal{T}}$ of a self-financing portfolio. This process is adapted to the filtration $(\mathcal{F}_t)_{t \in \mathcal{T}}$ and satisfies the equality $S_T = y$, $a.s.$ Consequently, suppose also that there is a linear and continuous pricing rule $\Pi : L^2 \longrightarrow \mathbb{R}$ providing us with the initial (at $t = 0$) price $\Pi(y)$ of every $y \in L^2$.

The completeness of the model implies the existence of a risk-free asset. Thus, if $r_f \geq 0$ is the risk-free rate, Equality

$$\Pi(k) = ke^{-r_f T} \tag{1}$$

must hold for every $k \in \mathbb{R}$. Besides, according to the Riesz Representation Theorem there exists a unique $z_\pi \in L^2$ such that

$$\Pi(y) = e^{-r_f T} \mathbb{E}(y z_\pi) \tag{2}$$

for every $y \in L^2$. Moreover, to prevent the existence of arbitrage, the strict inequality

$$z_\pi > 0 \tag{3}$$

$a.s.$ must hold (Duffie, 1988). $z_\pi$ is usually called "Stochastic Discount Factor" ($SDF$), and it is closely related to the Market Portfolio of the $CAPM$ (Duffie, 1988).

Expressions (1) and (2) imply that $ke^{-r_f T} = \Pi(k) = e^{-r_f T} k \mathbb{E}(z_\pi)$, which leads to

$$\mathbb{E}(z_\pi) = 1. \tag{4}$$

We will deal with risk measures that may be extended beyond $L^2$. Let $p \in [1, 2]$ and consider its conjugate number $q \in [2, \infty]$. Let $\rho : L^p \longrightarrow \mathbb{R}$ be the general risk function that a trader uses in order to control the risk level of his final wealth at $T$. Denote by

$$\Delta_\rho = \{z \in L^q; -\mathbb{E}(yz) \leq \rho(y), \ \forall y \in L^p\}. \tag{5}$$

$\Delta_\rho$ is usually called the "subgradient of $\rho$". We will assume that $\Delta_\rho$ is convex and $\sigma(L^q, L^p)$ −compact, and

$$\rho(y) = Max \ \{-\mathbb{E}(yz) : z \in \Delta_\rho\} \tag{6}$$

holds for every $y \in L^p$. Furthermore, we will also impose

$$\Delta_\rho \subset \{z \in L^q; \mathbb{E}(z) = 1\}. \tag{7}$$

Then, we have:

**Assumption 1**. The set $\Delta_\rho$ given by (5) is convex and $\sigma(L^q, L^p)$ −compact, $z = 1$ *a.s.* is in $\Delta_\rho$, (6) holds for every $y \in L^p$, and (7) holds. $\square$

The assumption above is closely related to the Representation Theorem of Risk Measures stated in Rockafellar *et al.* (2006). Following their ideas, it is easy to prove that Assumption 1 holds if and only if $\rho$ is continuous and satisfies

$$\rho(y+k) = \rho(y) - k \tag{8}$$

for every $y \in L^p$ and $k \in \mathbb{R}$.

$$\rho(\alpha y) = \alpha\rho(y) \tag{9}$$

for every $y \in L^p$ and $\alpha > 0$.

$$\rho(y_1 + y_2) \leq \rho(y_1) + \rho(y_2) \tag{10}$$

for every $y_1, y_2 \in L^p$.

$$\rho(y) \geq -\mathbb{E}(y) \tag{11}$$

for every $y \in L^p$.

It is easy to see that if $\rho$ is continuous and satisfies Properties (8), (9), (10), and (11) then it is also coherent in the sense of Artzner *et al.* (1999) if and only if

$$\Delta_\rho \subset L_+^q = \{z \in L^q; \mathbb{P}(z \geq 0) = 1\}. \tag{12}$$

Particular interesting examples are the Conditional Value at Risk ($CVaR$, Rockafellar *et al.,* 2006), the Weighted Conditional Value at Risk ($WCVaR$, Cherny, 2006), the Compatible Conditional Value at Risk ($CCVaR$, Balbás *et al.,* 2010$a$), the Dual Power Transform ($DPT$) of Wang (2000) and the Wang Measure (Wang, 2000), among many others. Furthermore, following the original idea of Rockafellar *et al.* (2006) to identify their Expectation Bounded Risk Measures and their Deviation Measures, it is easy to see that

$$\rho(y) = \sigma(y) - \mathbb{E}(y) \tag{13}$$

is continuous and satisfies (8), (9), (10), and (11) if $\sigma : L^p \longrightarrow \mathbb{R}$ is a continuous deviation, that is, if $\sigma$ is continuous and satisfies (9), (10),

$$\sigma(y+k) = \sigma(y)$$

for every $y \in L^p$ and $k \in \mathbb{R}$, and

$$\sigma(y) \geq 0$$

for every $y \in L^p$. Particular examples are the classical $p-$deviation given by

$$\sigma_p(y) = \left[\mathbb{E}\left(\left|\mathbb{E}(y) - y\right|^p\right)\right]^{1/p},$$

or the downside $p-$semi-deviation given by

$$\sigma_p^-(y) = \left[\mathbb{E}\left(\left|Max\ \left\{\mathbb{E}(y) - y, 0\right\}\right|^p\right)\right]^{1/p}.$$

Let us now introduce the hedging problem of this paper. Suppose that the random variable $y_0 \in L^2$ represents a trader's final (at $T$) wealth. Its final risk will be given by $\rho(y_0)$, which justifies that this quantity may be an adequate final value (at $T$) of the capital requirement. Indeed, (8) leads to

$$\rho(y_0 + \rho(y_0)) = 0 \tag{14}$$

and the risk will vanish if the additional amount $\rho(y_0)e^{-r_fT}$ is invested in the risk-free security. Nevertheless, Balbás *et al.* (2010*b*) have proved that this investment in the risk-free security may be outperformed by alternative hedging strategies $y \in L^2$, in the sense that the current price of $y$ is still $\rho(y_0)e^{-r_fT}$ but the global risk $\rho(y_0 + y)$ is negative. More accurately, these authors consider the pay-off $y \in L^2$ added by the trader to his initial portfolio $y_0 \in L^2$, they suppose that

$$C > 0 \tag{15}$$

gives (the value at $T$ of) the highest amount of money devoted to reducing the risk level,[1] and they finally propose the following optimization problems so as to select Portfolio $y$:

$$\begin{cases} Min\ \rho(y + y_0 - \mathbb{E}(yz_\pi)) \\ \mathbb{E}(yz_\pi) \leq C \\ y \geq 0 \end{cases}. \tag{16}$$

and

$$\begin{cases} Min\ \rho(y + y_0 - \mathbb{E}(yz_\pi)) \\ \mathbb{E}(yz_\pi) \leq C \end{cases} \tag{17}$$

[1]If $\rho(y_0) > 0$ then (14) shows that $C = \rho(y_0)$ could be a suitable choice for $C$.

Problem (16) involves the global risk $\rho\left(y+y_0-\mathbb{E}\left(yz_\pi\right)\right)$ that the trader is facing, so it has to incorporate the value $\mathbb{E}\left(yz_\pi\right)$ of the added portfolio, that will have to be paid and will reduce the trader's wealth. Constraint $y \geq 0$ may be indicating the presence of short-selling restrictions. Since we are minimizing risk, one can consider that short sales must be allowed if they do not make the riskiness increase, so Problem (17) also makes sense.

Since $y=0$ satisfies the constraints of (16) and (17) both problems are feasible. However, the paper above presents examples illustrating that (16) and (17) may be unbounded, *i.e.* there may be sequences $(y_n)_{n=1}^{\infty}$ of feasible portfolios such that $\rho\left(y_n+y_0-\mathbb{E}\left(yz_\pi\right)\right) \rightarrow -\infty$. Furthermore, as we will prove in Proposition 1 below, if the existence of this sequence holds then it provides us with returns converging to $+\infty$. Henceforth, if there are sequences of (17) or (17)-feasible portfolios whose riskiness converges to $-\infty$ (and therefore their expected return converges to $+\infty$) then we will say that Problem (16) or (17) admits good deals.

**Proposition 1** *If the sequence* $(y_n)_{n=1}^{\infty}$ *satisfies* $Lim_{n\rightarrow-\infty}\rho\left(y_n+y_0-\mathbb{E}\left(y_nz_\pi\right)\right) = -\infty$, *then* $Lim_{n\rightarrow-\infty}\mathbb{E}\left(y_n+y_0-\mathbb{E}\left(y_nz_\pi\right)\right) = +\infty$.

**Proof.** (11) shows that $\mathbb{E}\left(y_n+y_0-\mathbb{E}\left(y_nz_\pi\right)\right) \geq -\rho\left(y_n+y_0-\mathbb{E}\left(y_nz_\pi\right)\right) \rightarrow +\infty$. $\square$

Following Balbás *et al.* (2010*b*), the solution $y^*$ of (16), if it exists, will be called "shadow riskless asset" ($SRA$).

The rest of this section is devoted to summarizing the findings of Balbás *et al.* (2010*b*) that will apply henceforth. The proofs may be found in this reference.

Problem

$$\begin{cases} Max \ \ -C\lambda-\mathbb{E}\left(y_0z\right) \\ z \leq (1+\lambda)\,z_\pi \\ \lambda \in \mathbb{R},\ \lambda \geq 0,\ \ z \in \Delta_\rho \end{cases} \tag{18}$$

is the dual of (16), $\lambda \in \mathbb{R}$ and $z \in \Delta_\rho$ being the decision variables. Similarly,

$$\begin{cases} Max \ \ -\mathbb{E}\left(y_0z\right) \\ z = z_\pi \\ z \in \Delta_\rho \end{cases} \tag{19}$$

is the dual of (17), $z \in \Delta_\rho$ being the decision variable.

The following primal-dual relationships hold

**Theorem 2** *Suppose that* $y^* \in L^2$ *and* $(\lambda^*, z^*) \in \mathbb{R} \times L^2$*. Then, they solve (16) and (18) if and only if the following Karush-Kuhn-Tucker conditions*

$$\begin{cases} \lambda^* \left(C - \mathbb{E}\left(y^* z_\pi\right)\right) = 0 \\ C - \mathbb{E}\left(y^* z_\pi\right) \geq 0 \\ \mathbb{E}\left(\left(y^* + y_0\right) z\right) \geq \mathbb{E}\left(\left(y^* + y_0\right) z^*\right), & \forall z \in \Delta_\rho \\ \left(\left(1 + \lambda^*\right) z_\pi - z^*\right) y^* = 0 \\ \left(1 + \lambda^*\right) z_\pi - z^* \geq 0 \\ y^* \in L^2,\ y^* \geq 0,\ \lambda^* \in \mathbb{R},\ \lambda^* \geq 0,\ z^* \in \Delta_\rho \end{cases} \tag{20}$$

*are fulfilled. Moreover, if (16) is bounded then (18) is feasible and bounded, both optimal values coincide, and the dual solution is attainable.* □

**Theorem 3** *If* $z_\pi \notin \Delta_\rho$ *then the following conditions hold:*

*a) Problem (17) is unbounded, i.e., there are good deals.*

*b) If the solution of (16) exists, i.e., if a SRA exists, then it is not a risk-free asset.*

*c) If (16) is bounded then the solution* $(\lambda^*, z^*) \in \mathbb{R} \times L^2$ *of (18) satisfies* $\lambda^* > 0$*. Consequently, the second constraint in (20) becomes a equality.* □

Consider the $CVaR_{\mu_0}$ of Rockafellar *et al.* (2006), $\mu_0 \in (0,1)$ being the level of confidence. These authors proved the equality

$$\Delta_{CVaR_{\mu_0}} = \left\{ z \in L^\infty; \mathbb{E}\left(z\right) = 1,\ 0 \leq z \leq \frac{1}{1 - \mu_0} \right\}. \tag{21}$$

**Theorem 4** *Suppose that* $\rho = CVaR_{\mu_0}$*,* $\mu_0 \in (0,1)$ *being the level of confidence.*

*a) Problem (16) is bounded.*

*b) If* $y_0$ *is bounded from below, i.e., if* $y_0$ *has a finite essential infimum, then (16) is solvable (it attains its minimum value).*

*c) Suppose that* $y^* \in L^2$ *and* $(\lambda^*, z^*) \in \mathbb{R} \times L^2$*. Then, they solve (16) and (18) if and only if there exist* $\alpha \in \mathbb{R}$*,* $\alpha_1, \alpha_2 \in L^2$ *and a measurable partition* $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ *such that* *the following Karush-Kuhn-Tucker conditions*

$$\begin{cases} \lambda^* \left( C - \mathbb{E}\left( y^* z_\pi \right) \right) = 0 & \\ C - \mathbb{E}\left( y^* z_\pi \right) \geq 0 & \\ y^* + y_0 = \alpha - \alpha_1 + \alpha_2 & \\ \alpha_i \geq 0, & i = 1, 2 \\ \alpha_1 = \alpha_2 = 0 & on\ \Omega_0 \\ z^* = \dfrac{1}{1 - \mu_0}\ and\ \alpha_2 = 0 & on\ \Omega_1 \\ z^* = 0\ and\ \alpha_1 = 0 & on\ \Omega_2 \\ \left( \left( 1 + \lambda^* \right) z_\pi - z^* \right) y^* = 0 & \\ \left( 1 + \lambda^* \right) z_\pi - z^* \geq 0 & \\ y^* \in L^2,\ y^* \geq 0,\ \lambda^* \in \mathbb{R},\ \lambda^* \geq 0,\ z^* \in \Delta_\rho & \end{cases} \tag{22}$$

*are fulfilled. Moreover, if $z_\pi \notin \Delta_{CVaR_{\mu_0}}$ then the dual solution $(\lambda^*, z^*)$ satisfies $\lambda^* > 0$ and the first and second conditions simplify to $C - \mathbb{E}\left( y^* z_\pi \right) = 0$.* □

**Remark 1** *As pointed out by Balbás et al. (2010b), Theorem 3 implies that the $SRA$ $y^*$ (if it exists) is frequently a risky asset, as well as the existence of good deals in absence of short-selling restrictions. Indeed, suppose that $\rho$ may be extended to the whole space $L^1$. Then (5) implies that $\Delta_\rho \subset L^\infty$. Very important expectation bounded risk measures may be extended to $L^1$. Among others, the $CVaR$, the measure (13) if $\sigma$ is the $1-$deviation (or absolute deviation) or the $1-$down-side semi-deviation (or down-side absolute semi-deviation) and the $DPT$ of Wang (2000). Also the $WCVaR$ may be often extended to $L^1$. Combine the previous risk measures and a pricing model with unbounded $SDF$. Many important examples satisfy this requirement. For instance, the Black and Scholes model (Wang, 2000). Also the Heston model and other stochastic volatility models often have an unbounded $SDF$. In these cases the $SRA$ (if it exists) is not risk-free, and there are good deals available (Theorem 3).*

# 3 Discrete pricing models and the CVaR

According to Remark 1 above the $CVaR$ and the Black and Scholes model lead to the existence of a non risk-free $SRA$ (this asset exists due to Theorem $4a$) and the presence of good deals. Explicit constructions of both the $SRA$ and good deals may be found in Balbás *et al.* (2010*b*). In this paper we will deal with discrete pricing models for several reasons. Firstly, this framework significantly simplifies the mathematical exposition of the paper. Secondly, discrete pricing models are very realistic in practice since traders must face "ticks" when checking the real evolution of markets. Thirdly, most of the continuous time pricing models have an appropriate discrete time approximation.

Similarly, we are going to deal with $\rho = CVaR_{\mu_0}$, $\mu_0 \in (0,1)$ being the level of confidence. Bearing in mind (12) and (21), $CVaR_{\mu_0}$ is a coherent and expectation bounded measure of risk. Moreover, Ogryczak and Ruszczynski (2002) have shown that $CVaR_{\mu_0}$ is consistent with the second order stochastic dominance. These properties provoke that the $CVaR_{\mu_0}$ is becoming a very popular risk measure for researchers, practitioners, regulators and supervisors. Furthermore, expression

$$CVaR_{\mu_0}(y) \geq VaR_{\mu_0}(y), \tag{23}$$

trivially implies that every good deal generated by $CVaR_{\mu_0}$ also becomes a good deal if risks are given by $VaR_{\mu_0}$, despite the fact that this risk measure is not expectation bounded. This is important because $VaR_{\mu_0}$ is one of the most applied risk measures in practice.

Hence let us consider that

$$\Omega = \{0, 1, 2, ..., n\}. \tag{24}$$

Without loss of generality we can consider that $y_0(\omega)$ increases as so does $\omega \in \Omega$. Moreover, this constraint naturally holds if $n$ represent the number of trading dates and $\omega \in \Omega$ represents the number of growths of the manager portfolio price within the time interval $[0, T]$. For instance, in the binomial model

$$y_0 = W u^{\omega} d^{n-\omega}, \tag{25}$$

$W > 0$ denoting the initial (at $t = 0$) value of the portfolio and $u > 1$ and $d < 1$ denoting the usual factors affecting the portfolio price between two consecutive trading dates.

Analogously, let us assume that $z_{\pi}(\omega)$ decreases as $\omega \in \Omega$ increases. This is the usual situation if we assume that the market is risk adverse. For instance, in the binomial model, if $\nabla t$ represents the time length between consecutive trading dates and $R = e^{r_f \nabla t} \in (d, u)$

represents the capitalization factor of the risk-free asset, then

$$z_\pi = \left(\frac{R-d}{R_{y_0}-d}\right)^{\omega}\left(\frac{u-R}{u-R_{y_0}}\right)^{n-\omega}, \tag{26}$$

$R_{y_0}$ denoting the expected return of Portfolio $y_0$ between two consecutive trading dates. Obviously, since $y_0$ is in general a risky asset, in a risk adverse world we have that $u > R_{y_0} > R > d > 0$, and $z_\pi$ is decreasing.

Beyond the binomial model, in a general risk adverse framework, if we assume that Portfolio $y_0$ is efficient in a return/variance setting then there exists a couple of strictly positive real numbers $\eta_1$ and $\eta_2$ such that

$$z_\pi = \eta_1 - \eta_2 y_0, \tag{27}$$

and therefore $z_\pi$ is strictly decreasing if $y_0$ is strictly increasing. In practice, $\eta_1$ and $\eta_2$ may be easily computed from (4) and taking into account the current price $W$ of Portfolio $y_0$. We get System

$$\begin{cases} \eta_1 - \eta_2 \mathbb{E}\left(y_0\right) = 1 \\ \eta_1 \mathbb{E}\left(y_0\right) - \eta_2 \mathbb{E}\left(y_0^2\right) = W \end{cases}. \tag{28}$$

Finally, let us assume that

$$z_\pi\left(0\right) > \frac{1}{1-\mu_0}, \tag{29}$$

or equivalently, $z_\pi \notin \Delta_{CVaR_{\mu_0}}$ due to (21). Summarizing we have

**Assumption 2**. Henceforth $\rho = CVaR_{\mu_0}$, $\mu_0 \in (0,1)$ being the level of confidence, $\Omega$ is given by (24), $y_0$ is a strictly increasing function of $\omega \in \Omega$, $z_\pi$ is a strictly decreasing function of $\omega \in \Omega$, and (29) holds. □

**Remark 2** *Under the conditions above Theorem* 3 *implies the existence of good deals, while Theorem* 4 *implies the existence of a SRA $y^*$ which is not risk-free. Furthermore, Theorem* 4 *also leads to the equality*

$$C - \mathbb{E}\left(y^* z_\pi\right) = 0 \tag{30}$$

*and the inequality*

$$\lambda^* > 0. \tag{31}$$

*$(\lambda^*, z^*)$ denoting the solution of (18). Henceforth $y^*$ and $(\lambda^*, z^*)$ will denote a primal and a dual solution, and their existence is guaranteed by the arguments above.* □

# 4 Constructing good deals

Let us give several properties that will allow us to solve (16) and (17).[2] First of all, though (16) and (17) are not linear, notice that in our discrete framework (see (24)) Problem (18) is linear and can be solved by standard well-known methods. Thus we can assume that $(\lambda^*, z^*)$ is known.

**Theorem 5** *$z^*$ is decreasing, i.e., if $\omega, \tilde{\omega} \in \Omega$, $\omega < \tilde{\omega}$, then $z^*(\omega) \geq z^*(\tilde{\omega})$.*

**Proof.** Suppose that $z^*(\omega) < z^*(\tilde{\omega})$. Then, since $z_\pi$ is strictly decreasing, the ninth condition in (22) leads to

$$(1+\lambda^*)\, z_\pi(\omega) > (1+\lambda^*)\, z_\pi(\tilde{\omega}) \geq z^*(\tilde{\omega}) > z^*(\omega),$$

and the eighth condition in (22) implies that $y^*(\omega) = 0$. Besides, $\omega \notin \Omega_1$ in (22) and $\tilde{\omega} \notin \Omega_2$. Hence,

$$y_0(\tilde{\omega}) \leq y_0(\tilde{\omega}) + y^*(\tilde{\omega}) = \alpha - \alpha_1(\tilde{\omega}) \leq$$
$$\alpha \leq \alpha + \alpha_2(\omega) = y_0(\omega) + y^*(\omega) = y_0(\omega),$$

which contradicts that $y_0$ is strictly increasing. $\square$

**Remark 3** *Expression (29) implies the existence of $\omega \in \Omega$ with $z_\pi(\omega) > \dfrac{1}{1-\mu_0}$. Since $\lambda^* > 0$ (see (31)), we have that*

$$(1+\lambda^*)\, z_\pi(\omega) > \frac{1}{1-\mu_0}$$

*must hold for some $\omega \in \Omega$. Henceforth we will fix*

$$\omega_0 = Max \left\{ \omega \in \Omega;\ \ (1+\lambda^*)\, z_\pi(\omega) > \frac{1}{1-\mu_0} \right\}. \tag{32}$$

*Similarly, if $\mathbb{P}(z^* > 0) < 1$, Proposition 5 guarantees the existence of*

$$\omega_1 = Min\ \{\omega \in \Omega;\ z^*(\omega) = 0\}, \tag{33}$$

*and we will define $\omega_1 = n+1$ if $\mathbb{P}(z^* > 0) = 1$.* $\square$

[2] Remark 2 shows that (17) is unbounded and cannot be solved. However, we will give a concrete sequence of portfolios whose $(risk,\ return)$ tends to $(-\infty,\ \infty)$, $VaR$ and $CVaR$ being the used measures of risk.

Next let us show that $\{0, 1, ..., \omega_0\}$ and $\{\omega_1, ..., n\}$ are disjoint, along with the expression of $y^*$ and $z^*$ in these subsets of $\Omega$.

**Theorem 6** *a) If* $\mathbb{P}(z^* > 0) < 1$ *then*

$$\begin{cases} z^* = 0, & \omega \geq \omega_1 \\ z^* > 0, & \omega < \omega_1 \end{cases} \tag{34}$$

*b)* $y^*(\omega) = 0$ *for every* $\omega \leq \omega_0$.

*c) If* $\mathbb{P}(z^* > 0) < 1$ *then* $y^*(\omega) = 0$ *for every* $\omega \geq \omega_1$.

*d)* $\omega_0 < n$, *and* $\omega_0 + 1 < \omega_1$ *if* $\mathbb{P}(z^* > 0) < 1$.

*e)* $z^*(\omega) = \dfrac{1}{1 - \mu_0}$ *for every* $\omega \leq \omega_0$.

**Proof.** *a)* It trivially follows from Theorem 5 and (33).

*b)* Since $z_\pi$ is decreasing we have that

$$(1 + \lambda^*) z_\pi(\omega) > \frac{1}{1 - \mu_0} \geq z^*(\omega)$$

for $\omega \leq \omega_0$, and the eighth and ninth conditions in (22) imply that $y^*(\omega) = 0$.

*c)* Statement *a)* implies $z^*(\omega) = 0$ and (3), along with the eighth and ninth conditions in (22), show that $y^*(\omega) = 0$. Similar arguments show that $\omega_0 < n$.

*d)* $\omega_0 + 1 \geq \omega_1$ and Statements *b)* and *c)* would lead to $y^* = 0$, in contradiction with (30) and (15).

*e)* Theorem 5 and (21) imply that one only has to prove $z^*(\omega_0) = \dfrac{1}{1 - \mu_0}$. If $z^*(\omega_0) < \dfrac{1}{1 - \mu_0}$ then $\omega_0 \in \Omega_0$ in (22) due to Statement *d)*. Hence, according to Statement *b)*,

$$y_0(\omega_0) = y_0(\omega_0) + y^*(\omega_0) = \alpha.$$

On the other hand $\omega_0 + 1 \notin \Omega_2$ in (22) due to Statement *d)*. Hence,

$$y_0(\omega_0 + 1) \leq y_0(\omega_0 + 1) + y^*(\omega_0 + 1) = \alpha - \alpha_1 \leq \alpha.$$

We have a contradiction because $y_0$ is strictly increasing. □

Solution $y^*$ is already known in $\{0, 1, ..., \omega_0\}$ and $\{\omega_1, ..., n\}$, but we still need more details so as to compute $y^*$ within the interval $\omega_0 < \omega < \omega_1$. Next let us give a proposition focusing on this point.

**Remark 4** *Notice that*

$$\left(1+\lambda^*\right) z_\pi\left(\omega_0+1\right) \leq \frac{1}{1-\mu_0} \tag{35}$$

*due to (32).* □

**Theorem 7** *a) If (35) is a strict inequality then*

$$y^* = \begin{cases} 0, & \omega \leq \omega_0 \\ \frac{C + \sum_{\omega=\omega_0+1}^{\omega_1-1} y_0(\omega) z_\pi(\omega) \mathbf{P}(\omega)}{\sum_{\omega=\omega_0+1}^{\omega_1-1} z_\pi(\omega) \mathbf{P}(\omega)} - y_0, & \omega_0 < \omega < \omega_1 \\ 0, & \omega \geq \omega_1 \end{cases}. \tag{36}$$

*b) If (35) becomes a equality then there exist $\alpha \geq \tilde{\alpha} > 0$ such that*

$$y^* = \begin{cases} 0, & \omega \leq \omega_0 \\ \tilde{\alpha} - y_0, & \omega = \omega_0 + 1 \\ \alpha - y_0, & \omega_0 + 1 < \omega < \omega_1 \\ 0, & \omega \geq \omega_1 \end{cases}. \tag{37}$$

*Moreover,*

$$\begin{aligned} &\tilde{\alpha} z_\pi\left(\omega_0+1\right) \mathbb{P}\left(\omega_0+1\right) + \alpha\left(\sum_{\omega_0+2<\omega\leq\omega_1-1} z_\pi(\omega) \mathbb{P}(\omega)\right) = \\ &C + \sum_{\omega_0+1\leq\omega\leq\omega_1-1} y_0(\omega) z_\pi(\omega) \mathbb{P}(\omega). \end{aligned} \tag{38}$$

*$\tilde{\alpha} = y_0\left(\omega_0+1\right)$ if $z^*\left(\omega_0+1\right) < \dfrac{1}{1-\mu_0}$. Finally, if $z^*\left(\omega_0+1\right) = \dfrac{1}{1-\mu_0}$ then $0 < \tilde{\alpha} \leq \alpha$ may be arbitrary as far as $y^* \geq 0$ and the eighth condition in (22) and (38) hold.*

**Proof.** *a)* If (35) is a strict inequality then

$$\left(1+\lambda^*\right) z_\pi(\omega) < \frac{1}{1-\mu_0} \tag{39}$$

whenever $\omega_0 + 1 \leq \omega \leq \omega_1 - 1$ because $z_\pi$ is decreasing. Hence, the ninth expression in (22) implies that

$$z^*(\omega) < \frac{1}{1-\mu_0} \tag{40}$$

whenever $\omega_0+1 \leq \omega \leq \omega_1-1$. (34) implies that $z^*(\omega)>0$ for $\omega_0+1 \leq \omega \leq \omega_1-1$. Then, "the interval" $\{\omega_0+1,...,\omega_1-1\}$ is included in the set $\Omega_0$ of (22), which implies the existence of $\alpha \in \mathbb{R}$ such that $y^*=\alpha-y_0$ whenever $\omega_0+1 \leq \omega \leq \omega_1-1$. Since Theorem 6 implies that $y^*=0$ outside this interval, (30) leads to

$$\alpha \sum_{\omega=\omega_0+1}^{\omega_1-1} z_\pi(\omega)\, \mathbb{P}(\omega) - \sum_{\omega=\omega_0+1}^{\omega_1-1} y_0(\omega)\, z_\pi(\omega)\, \mathbb{P}(\omega) = C,$$

and (36) becomes obvious.

*b)* If (35) becomes a equality then (39) and (40) still hold for $\omega_0+1 \leq \omega \leq \omega_1-1$ and $\omega_0+1<\omega$. As in *a*), $y^*=\alpha-y_0$ for $\omega_0+1 \leq \omega \leq \omega_1-1$ and $\omega_0+1<\omega$. As in *a*), $z^*(\omega)>0$ for $\omega_0+1 \leq \omega \leq \omega_1-1$, so $\omega_0+1$ does not belong to the set $\Omega_2$ of (22) and $\alpha_2(\omega_0+1)=0$. Whence

$$y^*(\omega_0+1)=\alpha-\alpha_1(\omega_0+1)-y_0(\omega_0+1).$$

(37) trivially follows if one takes $\tilde{\alpha}=\alpha-\alpha_1(\omega_0+1)$ which is strictly positive because otherwise $y^*(\omega_0+1)$ would be strictly negative, in contradiction with the constraints of (16). Furthermore, (38) trivially follows from (30), $\tilde{\alpha}=y_0(\omega_0+1)$ if $z^*(\omega_0+1)<\dfrac{1}{1-\mu_0}$ due to the eighth condition in (22), and finally, we only must guarantee the fulfillment of (22) if $z^*(\omega_0+1)=\dfrac{1}{1-\mu_0}$. □

**Remark 5** *Let us assume that there are no short selling restrictions, i.e., let us deal with (17) rather than (16). Remark* 2 *shows that there are good deals. Furthermore, (23) shows that the riskiness also may become minus infinite if it is given by the $VaR$. In other words, one can construct sequences of portfolios such that $VaR$ and $CVaR$ tend to minus infinite while expected returns tend to plus infinite (Proposition* 1*). Hence, let us give an effective construction of such a sequence.*

*Consider $m \in \mathbb{N}$, along with an approximation of (17) given by Problem*

$$\begin{cases} Min\ CVaR_{\mu_0}\left(y+y_0-\mathbb{E}\left(yz_\pi\right)\right) \\ \mathbb{E}\left(yz_\pi\right) \leq C \\ y \geq -m \end{cases} \tag{41}$$

*Then, due to (4), it is easy to see that the change of variable $x_m = y + m$ leads to*

$$\begin{cases} Min\ CVaR_{\mu_0}\left(x_m + y_0 - \mathbb{E}\left(y z_\pi\right)\right) \\ \mathbb{E}\left(x_m z_\pi\right) \leq C + m \\ x_m \geq 0 \end{cases}, \tag{42}$$

*analogous to (16). Thus, (41) is bounded and achieves its optimal value (Theorem 4). Consider the sequence $(y_m^*)_{m=1}^{\infty} = (x_m^* - m)_{m=1}^{\infty}$ of solutions of (41), $(x_m^*)_{m=1}^{\infty}$ denoting the solutions of (42). It is easy to see that $(y_m^*)_{m=1}^{\infty}$ is "a good deal" (i.e., risk $= -\infty$, return $= +\infty$). Furthermore, every $x_m^*$ may be computed with Theorem 7. In practice one can compute $y_m^*$ for several values of $m \in \mathbb{N}$ and then stop once the objective value of (41) is "negative enough" and the expected return of $y_m^*$ is "positive enough".* □

**Remark 6** *Notice that (35) will often be a strict inequality, so (36) will frequently hold. In such a case $y^*$ may be given by*

$$y^* = \begin{cases} 0, & y_0 \leq y_0\left(\omega_0\right) \\ k - y_0, & y_0(\omega_0) < y_0(\omega) < y_0(\omega_1) \\ 0, & y_0(\omega) \geq y_0(\omega_1) \end{cases}, \tag{43}$$

*where*

$$k = \frac{C + \sum_{\omega=\omega_0+1}^{\omega_1-1} y_0\left(\omega\right) z_\pi\left(\omega\right) \mathbb{P}\left(\omega\right)}{\sum_{\omega=\omega_0+1}^{\omega_1-1} z_\pi\left(\omega\right) \mathbb{P}\left(\omega\right)}.$$

*Besides (32) shows that*

$$z_\pi\left(\omega_0\right) > \frac{1}{\left(1-\mu_0\right)\left(1+\lambda^*\right)} \geq \frac{1}{1-\mu_0} \to \infty$$

*if $\mu_0 \to 1$. Thus, if the level of confidence $\mu_0$ is large enough and Assumption 2 still holds then $\omega_0$ will become close to zero or zero, and (43) will be quite close to a "put option".*[3]

*There are several classical strategies providing "classical portfolio insurance". Maybe the most popular one is the purchase of an appropriate European put option. Theorem 7 and (43) highlight that for high levels of confidence the use of portfolio insurance strategies may be adequate to control the investor's risk. It is consistent with some empirical findings*

[3] Actually, $y^*$ is not a put option because it vanishes at $\omega = 0$, but $y^*$ equals a put option for $\omega$ not very close to zero.

*of recent literature. For instance, the test implemented by Annaert et al. (2009) seems to reveal that some put option-linked portfolio insurance strategies are not outperformed by other hedging methods. The authors use stochastic dominance criteria and $VaR$ and $CVaR$ in their empirical test.* $\square$

# 5 Sensitivity

This section is devoted to quantifying the effect on $CVaR_{\mu_0}(y^* + y_0 - \mathbb{E}(y^* z_\pi))$ of measurement errors and changes in the pricing model. To this purpose we will draw on the classical "Envelope Theorem" of Mathematical Programming. Mainly, this theorem states that the optimal value sensitivity (partial derivative) with respect to every variable involved in the model equals a partial derivative of the Lagrangian Function. A complete study about this topic may found, amongst others, in Balbás *et al.* (2005).

Define $CVaR^*_{\mu_0}(C, y_0, z_\pi)$ as the optimal value of (16) that depends on $(C, y_0, z_\pi)$.[4]

**Theorem 8** *Function $CVaR^*_{\mu_0}$ is Fréchet differentiable and*

$$\frac{\partial CVaR^*_{\mu_0}}{\partial C} = -\lambda^*, \ \frac{\partial CVaR^*_{\mu_0}}{\partial y_0} = -z^* \ \textit{and} \ \frac{\partial CVaR^*_{\mu_0}}{\partial z_\pi} = (1+\lambda^*)\, y^*.^5$$

**Proof.** The Envelope Theorem of Mathematical Programming implies that $CVaR^*_{\mu_0}$ is Fréchet differentiable if so is the Lagrangian Function at the $(primal, dual) - solution$, and both differentials coincide. The Lagrangian Function of (16) is (see Balbás *et al,* 2010*c*, for a general Lagrangian Function of optimization problems involving risk measures)

$$\mathcal{L}(y^*, \lambda^*, z^*, C, y_0, z_\pi) = -\lambda^* C - \mathbb{E}(y_0 z^*) + \mathbb{E}(y^*(1+\lambda^*) z_\pi - z^*),$$

and the conclusion of the theorem trivially follows. $\square$

[4] Theorem 4 guarantees that (16) has a minimum value if $C > 0$, $y_0 \in L^2$ is bounded from below, and $z_\pi$ satisfies (3) and (4).

[5] Notice that this result remains true if we consider the general setting of Section 2, *i.e.*, beyond the discrete time framework. The reason is that the proof does not draw on the findings of Section 4.

**Remark 7** *Theorem* 8 *allows us to give an approximation of the optimal risk level variation* $\nabla\left(CVaR^*_{\mu_0}\right)$ *with respect to modifications of the parameters. In particular,*

$$\nabla\left(CVaR^*_{\mu_0}\right) \approx -\lambda^*\nabla\left(C\right) - \mathbb{E}\left(z^*\nabla\left(y_0\right)\right) + \left(1+\lambda^*\right)\mathbb{E}\left(y^*\nabla\left(z_\pi\right)\right). \tag{44}$$

*In the discrete time framework of the latter section we know that* $y^*$ *vanishes outside* $\omega_0 < \omega < \omega_1$ *(Theorem* 7*) so there is no sensitivity with respect to errors of the* $SDF$ *estimates unless they significantly affect the central values of* $\Omega$*. Besides, the sensitivity with respect the initial portfolio* $y_0$ *becomes important if errors arise for small values of* $\omega \in \Omega$ *since in such a case* $z^*\left(\omega\right) = \dfrac{1}{1-\mu_0}$ *(Theorem* 6d*), while this sensitivity is negligible for the highest values of* $\omega \in \Omega$*.* □

**Remark 8** *In the particular case of the binomial model, (25) and (26) obviously lead to*

$$\begin{array}{l}\mathbb{E}\left(z^*\nabla\left(y_0\right)\right) \approx W\left[\sum\limits_{\omega=0}^{n}\omega u^{\omega-1}d^{n-\omega}z^*\left(\omega\right)\mathbb{P}\left(\omega\right)\right]\nabla\left(u\right)\\ +W\left[\sum\limits_{\omega=0}^{n}\left(n-\omega\right)u^{\omega}d^{n-\omega-1}z^*\left(\omega\right)\mathbb{P}\left(\omega\right)\right]\nabla\left(d\right)\end{array} \tag{45}$$

*and*

$$\begin{array}{l}\mathbb{E}\left(y^*\nabla\left(z_\pi\right)\right) \approx \sum\limits_{\omega=0}^{n}\left(n-\omega\right)\left(\frac{R-d}{R_{y_0}-d}\right)^{\omega}\left(\frac{u-R}{u-R_{y_0}}\right)^{n-\omega-1}\frac{R-R_{y_0}}{\left(u-R_{y_0}\right)^2}y^*\left(\omega\right)\mathbb{P}\left(\omega\right)\\ +\sum\limits_{\omega=0}^{n}\omega\left(\frac{R-d}{R_{y_0}-d}\right)^{\omega-1}\left(\frac{u-R}{u-R_{y_0}}\right)^{n-\omega}\frac{R-R_{y_0}}{\left(d-R_{y_0}\right)^2}y^*\left(\omega\right)\mathbb{P}\left(\omega\right).\end{array} \tag{46}$$

*Therefore, (44), (45) and (46) will give the variation* $\nabla\left(CVaR^*_{\mu_0}\right)$ *of the optimal risk level with respect to the parameters* $u$ *and* $d$*. Similarly, bearing in mind (25) and (26) one can compute closed formulas of the sensitivity with respect to the "risk-free rate"* $R$ *and the risky expected return* $R_{y_0}$*. Finally, if we take the binomial model as an approximation of the Geometrical Brownian Motion and therefore* $u = e^{\sigma\nabla t}$ *and* $d = e^{-\sigma\nabla t}$*,* $\sigma$ *denoting the risky asset volatility and* $\nabla t$ *denoting time between consecutive trading dates, then (44), (45) and (46) trivially lead to expressions providing us with the sensitivity of* $CVaR^*_{\mu_0}$ *with respect to the volatility of the underlying asset. Once again the comments above about the relative importance of* $y^*$ *(which vanishes for small and big values of* $\omega \in \Omega$*) and* $z^*$ *(which only vanishes for big values of* $\omega \in \Omega$*) still apply.* □

**Remark 9** *Consider the general risk adverse setting generating an efficient* $y_0$ *in a return/variance framework. The* $SDF$ *satisfies (27) and (28), so suppose that we can assume* *the fulfillment of* $\nabla(z_\pi) \approx \eta_2 \nabla(y_0)$*. Then, (44) implies*

$$\nabla\left(CVaR^*_{\mu_0}\right) \approx -\lambda^* \nabla(C) - \mathbb{E}\left(\nabla(y_0)\left(\eta_2\left(1+\lambda^*\right)y^* + z^*\right)\right).$$

*Notice that the relative importance of* $y^*$ *and* $z^*$ *applies in this case as well.* □

# 6 Conclusions

The paper has dealt with a complete arbitrage free pricing model and a general risk measure such that good deals and a non risk-free shadow riskless assets do exist. We have pointed out that this situation often arise when dealing with classical models in Financial Economics.

The main contribution of the paper is the effective construction of good deals and shadow riskless assets in a general Discrete Time Framework for both the Value at Risk and the Conditional Value at Risk. We have shown that good deals and shadow riskless assets may be closely related since the only difference between then is a short position in a risk-free asset when building good deals. Moreover, both strategies are close to classical portfolio insurance strategies, which seems to be consistent with recent findings of some empirical literature reflecting the effectiveness of combinations of European puts in practical hedging problems.

Finally, we have measured the sensitivity of our solutions with respect to important parameters affecting the pricing model, such as the pricing rule (or the stochastic discount factor) or the manager´s random final wealth. This analysis may be quite useful in practice since it provides the effect of measurement errors. □

**Acknowledgments.** This research was partially supported by "*Comunidad Autónoma de Madrid*" (Spain), Grant $S2009/ESP-1594$, and "*MICIN*" (Spain), Grant $ECO2009-14457-C04$.

The authors thank the participants in the *International Workshop on Appled Probability* (Madrid, June, 2010) for their useful comments and ideas.

The usual caveat applies. □

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